Gelfond's Problem

Let $n = \sum_i d_i 2^i$ be the binary expansion of n, so that each d_i is either 0 or 1. Then $s(n) = \sum_i d_i$ is the sum of the binary digits of n. Since $s(2n+1) = s(2n)+1$, it follows that $\big|\sum_{0\le n\le N}(-1)^{s(n)}\big| \le 1$ for all N. Our object now is to show that $\sum_{p\le x}(-1)^{s(p)} = o(\pi(x))$. We begin with some basic observations of harmonic analysis, and then address the generating function of $(-1)^{s(n)}$.

Lemma 18.1 *Let M and N be integers with $N \ge 2$. Then there exist weights $w(n)$ such that $w(n) \ge 1$ for $M+1 \le n \le M+N$, $w(n) \ge 0$ for all other n, and*

$$W(\alpha) = \sum_{n=-\infty}^{\infty} w(n)e(n\alpha)$$

has the properties that $\max_\alpha |W(\alpha)| = W(0) \ll N$ and $W(\alpha) = 0$ if $\|\alpha\| \ge 1/N$.

Suppose that $M+1 \le n \le M+N$. Then

$$1 \le w(n) = \int_0^1 W(\alpha)e(-n\alpha)\,d\alpha = \int_{-1/N}^{1/N} W(\alpha)e(-n\alpha)\,d\alpha \le \frac{2}{N}\max_\alpha |W(\alpha)|\,.$$

Thus our bound for $\max|W(\alpha)|$ is optimal, apart from constants. If sharp constants were required, then we would appeal to Theorem E.4, but for our present purposes we have no need for such sophistication.

Proof We recall that if $f(x) = \max(0, 1-|x|)$, then $\widehat{f}(t) = \big(\frac{\sin \pi t}{\pi t}\big)^2$. If N is even, put $K = M + N/2$. If N is odd, put $K = M + (N+1)/2$. Thus in either case, K is an integer. After several changes of variable we deduce that if $g(x) = N\max(0, 1-N|x+\alpha|)e(K(x+\alpha))$, then

$$\widehat{g}(t) = \Big(\frac{\sin \pi(t-K)/N}{\pi(t-K)/N}\Big)^2 e(t\alpha)\,.$$

By the Poisson summation formula in the form of Theorem D.3, we find that $\sum_n g(n) = \sum_k \widehat{g}(k)$. Thus

$$W(\alpha) = \frac{\pi^2}{4}\sum_{n=-\infty}^{\infty}\Big(\frac{\sin \pi(n-K)/N}{\pi(n-K)/N}\Big)^2 e(n\alpha) = \frac{\pi^2}{4}N\max(0, 1-N\|\alpha\|)e(K\alpha)$$

has the required properties. $\square$

Lemma 18.2 *Let M be an integer, $M \ge 2$, put $f(x) = \min(M, 1/\|x\|)$, and set*

$$g(x) = \sum_{m=-M}^{M} \widehat{f}(m)(1 - |m|/M)e(mx)\,.$$

Then $\widehat{f}(m) \ll \log M$ uniformly in m, and $f(x) \ll g(x)$ uniformly in x.

Proof For the first assertion we note that

$$|\widehat{f}(m)| \le \int_0^1 |f(x)|\,dx = 2 + 2\log M/2 \ll \log M\,.$$

For the second part, let

$$\Delta_M(x) = \frac{1}{M}\Big(\frac{\sin \pi M x}{\sin \pi x}\Big)^2 = \sum_{m=-M}^{M}(1 - |m|/M)e(mx)$$

be Fejér's kernel. Since $\Delta_M(x)$ is monotonically decreasing for $0 \le x \le 1/M$, it follows that

$$\int_0^{1/(2M)} \Delta_M(x)\,dx \ge \frac{1}{2M}\Delta_M(1/(2M)) = \frac{1}{2M^2\sin^2 \pi/(2M)} \ge \frac{2}{\pi^2}$$

since $\sin\delta \le \delta$. But $f(x)$ is monotonically decreasing in the interval $[-1/M, 1/2]$, so

$$\begin{aligned} g(x) = \int_0^1 \Delta_M(u)f(x-u)\,du &\ge \int_0^{1/(2M)} \Delta_M(u)f(x-u)\,du \\ &\ge f(x)\int_0^{1/(2M)} \Delta_M(u)\,du \gg f(x)\,. \end{aligned}$$

The bound therefore also holds for $1/2 \le x \le 1$, since f and g are both even functions. □

The function $(-1)^{s(n)}$ has a power series generating function

$$P(z) = \sum_{n=0}^{\infty}(-1)^{s(n)}z^n = \prod_{j=0}^{\infty}\big(1 - z^{2^j}\big) \qquad (18.\mathrm{xx}+1)$$

for $|z| < 1$, but we find it easier to work on the circle with a truncated sum, so we set

$$T_J(\theta) = \sum_{0 \le n < 2^J}(-1)^{s(n)}e(n\theta) = \prod_{j=0}^{J-1}\big(1 - e(2^j\theta)\big)\,. \qquad (18.\mathrm{xx}+2)$$

In order to derive an upper bound for $\max|T_J(\theta)|$, we show that $|1-e(\theta)|$ and $|1-e(2\theta)|$ cannot both be near 2 for the same θ.

Lemma 18.3 *Let* $f(\theta) = |1 - e(\theta)||1 - e(2\theta)|$. *Then*

$$\max_\theta f(\theta) = \frac{16}{3\sqrt{3}}\,.$$

Proof As $f(\theta)$ is an even function with period 1, we may restrict our attention to $0 \le \theta \le 1/2$. In this interval,

$$f(\theta) = 4\sin\pi\theta\sin 2\pi\theta\,.$$

Since $f(0) = f(1/2) = 0$, f attains its maximum at an interior critical point. But

$$f'(\theta) = 4\pi\cos\pi\theta\sin 2\pi\theta + 8\pi\sin\pi\theta\cos 2\pi\theta = 8\pi\sin\pi\theta\big(3\cos^2\pi\theta - 1\big)\,.$$

This vanishes at $\theta = 0$, where the minimum is attained, and also when $\cos\pi\theta = 1/\sqrt{3}$. For this θ we have $\sin\pi\theta = \sqrt{2/3}$ and $\sin 2\pi\theta = 2\sqrt{2}/3$, so the maximum value is $4\sqrt{2/3}2\sqrt{2}/3 = 16/(3\sqrt{3})$. □

Lemma 18.4 *Let* $\alpha = 0.188722\ldots$ *be determined by the equation* $4^\alpha = 3\sqrt{3}/4$, *and let* $T_J(\theta)$ *be defined as in* (18.xx+2). *Then*

$$T_J(\theta) \ll 2^{(1-\alpha)J}$$

uniformly in θ.

The bound $T_J(\theta) \ll 2^J$ would be trivial, so α reflects the margin by which our bound is non-trivial.

Proof By lemma 18.3 we know that

$$|1 - e(2^{2j}\theta)||1 - e(2^{2j+1}\theta)| \le 2^{2(1-\alpha)}\,.$$

We multiply these inequalities together for $0 \le j \le J/2 - 1$. If J is odd, then there is one further factor, which we bound trivially: $|1 - e(2^{J-1}\theta)| \le 2$. □

Since the functions $\log|1 - e(2^j\theta)|$ move on widely different scales, we expect them to be nearly independent, and so we expect that $\log|T_J(\theta)|$ should be distributed as if it were a sum of J independent random variables. As

$$\int_0^1 \log|1 - e(\theta)|\,d\theta = 0 \tag{18.xx+3}$$

and

$$\int_0^1 \big(\log|1 - e(\theta)|\big)^2\,d\theta = \frac{\pi^2}{12}, \tag{18.xx+4}$$

we expect that

$$\exp\big(-C\sqrt{J}\big) \le |T_J(\theta)| \le \exp\big(C\sqrt{J}\big) \tag{18.xx+5}$$

for most θ, if C is a large positive constant. On the other hand, by Parseval's identity it is trivial that

$$\int_0^1 |T_J(\theta)|^2 \, d\theta = 2^J, \tag{18.xx+6}$$

which is to say that $\|T_J\|_2 = 2^{J/2}$. This is much larger than the order of magnitude in (18.xx+5), so we infer that the large value of the 2-norm is due to a small set of θ for which $|T_J(\theta)|$ is exceptionally large. If this is the case, then we would expect $\|T_J\|_1$ to be smaller than the root-mean-square, $\|T_J\|_2$. The next lemma helps us to show that this is the case.

Lemma 18.5 *Let $g(\theta) = \sin \pi\theta \sin 2\pi\theta$, and put*

$$h(\theta) = |g(\theta)| + |g(\theta + 1/4)| + |g(\theta + 1/2)| + |g(\theta + 3/4)| \,.$$

Then

$$\max_\theta h(\theta) = h(1/8) = \sqrt{2 + \sqrt{2}}\,.$$

Proof Clearly $h(\theta)$ has period $1/4$. Since $g(\theta)$ is even, it follows that $h(\theta)$ is also even. Hence we may restrict our attention to $0 \le \theta \le 1/8$. In this interval, $g(\theta) \ge 0$, $g(\theta+1/4) \ge 0$, $g(\theta + 1/2) \le 0$, and $g(\theta + 3/4) \le 0$, so

$$\begin{aligned} h(\theta) &= \frac{1}{2}(2 - \sqrt{2}) \sin^2 \pi\theta \cos \pi\theta + \frac{1}{2} \sin \pi\theta \cos^2 \pi\theta - \frac{1}{4}\sqrt{2} \cos^3 \pi\theta \\ &= \frac{1}{2}(1 + \sqrt{2}) \cos \pi\theta + \frac{1}{2} \sin \pi\theta + \frac{1}{2}(\sqrt{2} - 1) \cos 3\pi\theta + \frac{1}{2} \sin 3\pi\theta \,. \end{aligned} \tag{18.xx+7}$$

Consequently,

$$\begin{aligned} h'(\theta) &= -\frac{\pi}{2}(1 + \sqrt{2}) \sin \pi\theta + \frac{\pi}{2} \cos \pi\theta - \frac{3\pi}{2}(\sqrt{2} - 1) \sin 3\pi\theta + \frac{3\pi}{2} \cos 3\pi\theta, \\ h''(\theta) &= -\frac{\pi^2}{2}(1 + \sqrt{2}) \cos \pi\theta - \frac{\pi^2}{2} \sin \pi\theta - \frac{9\pi^2}{2}(\sqrt{2} - 1) \cos 3\pi\theta - \frac{9\pi^2}{2} \sin 3\pi\theta \,. \end{aligned}$$

In this last formula all terms are ≤ 0, so it is clear that $h''(\theta) \le 0$. But

$$\begin{aligned} h'(1/8) = &-\frac{\pi}{2}(1 + \sqrt{2})\sqrt{\frac{1 - 1/\sqrt{2}}{2}} + \frac{\pi}{2}\sqrt{\frac{1 + 1/\sqrt{2}}{2}} \\ &- \frac{3\pi}{2}(\sqrt{2} - 1)\sqrt{\frac{1 + 1/\sqrt{2}}{2}} + \frac{3\pi}{2}\sqrt{\frac{1 - 1/\sqrt{2}}{2}} \\ = &\ \frac{\pi}{4}(2 - \sqrt{2})\sqrt{2 - \sqrt{2}} + \frac{\pi}{4}(4 - 3\sqrt{2})\sqrt{2 + \sqrt{2}} \\ = &\ 0 \end{aligned}$$

because $\sqrt{2-\sqrt{2}} = (\sqrt{2}-1)\sqrt{2+\sqrt{2}}$ and $(\sqrt{2}-1)(2-\sqrt{2}) = -(4-3\sqrt{2})$. Hence the maximum is attained at $\theta = 1/8$, and from (18.xx+7) we see that

$$h(1/8) = \frac{1}{2}(3-\sqrt{2})\sqrt{\frac{1+1/\sqrt{2}}{2}} + \frac{1}{2}\sqrt{2}\sqrt{\frac{1-1/\sqrt{2}}{2}} = \sqrt{2+\sqrt{2}}\,.$$

□

It is convenient to observe that

$$T_J(\theta) = T_{J-j}(\theta)T_j(2^{J-j}\theta)\,. \tag{18.xx+8}$$

Lemma 18.6 *Let* $\beta = 0.057111674\ldots$ *be determined by the relation* $4^\beta = 2/\sqrt{2+\sqrt{2}}$. *Then*

$$\sum_{a=1}^{2^J} |T_J(\theta + a/2^J)| \ll 2^{(3/2-\beta)J}$$

uniformly in θ*.*

By integrating this bound over $0 \le \theta \le 1/2^J$, it is immediate that

$$\int_0^1 |T_J(\theta)|\, d\theta \ll 2^{(1/2-\beta)J}\,. \tag{18.xx+9}$$

Thus β reflects the margin by which we can say that $\|T_J\|_1$ is smaller than $\|T_J\|_2$.

Proof Let $S_J(\theta)$ denote the sum to be bounded. By taking $j = J-2$ in (18.xx+8), we find that

$$S_J(\theta) = \sum_{a=1}^{2^J} |T_2(\theta + a/2^J)||T_{J-2}(4\theta + a/2^{J-2})|\,.$$

Here the second factor has period 2^{J-2} with respect to a, so the above is

$$= 4\sum_{a=1}^{2^{J-2}} |T_{J-2}(4\theta + a/2^{J-2})| h(\theta + a/2^J)$$

in the notation of Lemma 18.5. Hence by that lemma it is immediate that

$$S_J(\theta) \le 4\sqrt{2+\sqrt{2}}S_{J-2}(4\theta)\,.$$

We apply this $K = [J/2]$ times to see that

$$S_J(\theta) \le \Big(4\sqrt{2+\sqrt{2}}\Big)^K S_{J-2K}(2^{2K}\theta)\,.$$

But $S_0(\theta) \ll 1$, $S_1(\theta) \ll 1$, and $2(2+\sqrt{2})^{1/4} = 2^{3/2-\beta}$, so we have the stated result. □

By applying (18.xx+8) and then Lemma 18.6 with J replaced by $J-j$, we deduce that

$$\sum_{c=1}^{2^{J-j}} |T_J(\theta + c/2^{J-j})| \ll |T_j(2^{J-j}\theta)| 2^{(3/2-\beta)(J-j)}\,. \tag{18.xx+10}$$

In Chapter 16 we saw that θ is irrational, then the numbers $n\theta$ are uniformly distributed modulo 1. In general, as we let n run from 1 to N, we expect that $n\theta$ will fall into a short interval I approximately the expected number of times. However, it can sometimes happen that a short interval is hit far more times than expected. We now show that this can only happen when θ has a rational approximation a/q that is exceptionally good, and with q unusually small.

Lemma 18.7 *Let θ be a given real number. Suppose that $\delta_1 \le \delta_2/12$, that $N \ge 3/\delta_2$, and that $n\theta \in I = [\phi - \delta_1, \phi + \delta_1]$ (mod 1) for at least $\delta_2 N$ of the integers $n \in [1, N]$. Then there is an integer q with $1 \le q \le 9/\delta_2$, such that*

$$\|q\theta\| \le \frac{3\delta_1}{\delta_2 N} \tag{18.xx+11}$$

By Dirichlet's theorem we know that there is a $q \le N$ such that $\|q\theta\| \le 1/N$, but the q described above gives a better approximation, and is quite small as well.

Proof Among the positive integers $q \le N$, let q be the one for which $\|q\theta\|$ is minimal. For $0 \le n \le N$, arrange the numbers $\{n\theta\}$ in increasing order, and consider the minimal gap between consecutive terms, say $\{n_1\theta\} \le \{n_2\theta\}$. Then $\|(n_1 - n_2)\theta\|$ is the length of this gap. But $0 < |n_1 - n_2| \le N$, and $\|q\theta\|$ is minimal, so we see that of all the gaps between the numbers $\{n\theta\}$, the gap between $\{q\theta\}$ and 0 (or 1) is minimal. With $n\theta \in I$ for at least $\delta_2 N$ values of N, we have $\ge \delta_2 N - 1$ gaps, each of length at least $\|q\theta\|$. Hence $\|q\theta\|(\delta_2 N - 1) \le 2\delta_1$. This implies (18.xx+11), since $\delta_2 N \ge 3$.

We divide the interval $[1, N]$ into $\le N/q + 1$ intervals of length $\le q$. For a given n_0, we consider those n, $n_0 \le n < n_0 + q$ such that $n\theta \in I$ (mod 1). We put $\delta = \theta - a/q$, so that

$$n\theta = n_0\theta + (n - n_0)a/q + (n - n_0)\delta\,.$$

By (18.xx+11) we know that $|\delta| \le 3\delta_1/(\delta_2 qN)$. Hence $|(n - n_0)\delta| \le 3\delta_1/(\delta_2 N \le 1/(4N)$, since $\delta_1 \le \delta_2/12$. Thus if $n\theta \in I$, then $(n - n_0)a/q \in J = [\phi - n_0\theta - \delta_1 - 1/(4N), \phi - n_0\theta + \delta + 1/(4N)]$. Since the numbers $(n - n_0)a/q$ are in arithmetic progression with common difference $1/q$, the number of n, $n_0 \le n < n_0 + q$, for which $(n - n_0)a/q \in J$ is $\le 1 + q(2\delta_1 + 1/(2N)) \le 2\delta_1 q + 3/2$ since $q \le N$. Consequently, the total number of $n \le N$ for which $n\theta \in I$ is

$$\le (N/q + 1)(2\delta_1 q + 3/2) = 2\delta_1 N + 3N/(2q) + 2\delta_1 q + 3/2\,.$$

Since $q \le N$, the first and third terms on the right hand side sum to $\le 4\delta_1 N \le \delta_2 N/3$. The last term is $\le \delta_2 N/2$, since $N \ge 3/\delta_2$. Since the number of $n \le N$ for which $n\theta \in I$ is by hypothesis $\ge \delta_2 N$, we conclude that

$$\delta_2 N \le \frac{5}{6}\delta_2 N + \frac{3N}{2q},$$

which implies that $q \le 9/\delta_2$. $\square$

Theorem 18.8 (Mauduit–Rivat) *Let $s(n)$ denote the sum of the binary digits of n. Then*

$$\sum_{n\le N}(-1)^{s(n)}\Lambda(n) \ll N^{1-1/263}\,.$$

Proof For $N \ge 2$ we set $T(\theta) = \sum_{n\le N}(-1)^{s(n)}e(n\theta)$. If we choose J so that $2^{J-1} < N \le 2^J$, then T is a truncation of the sum T_J, and hence by (E.13) and (E.14) we know that

(18.xx+12) $$\|T\|_\infty \ll N^{1-\alpha}\log N$$

and that

(18.xx+13) $$\|T\|_1 \ll N^{1/2-\beta}\log N$$

where α and β are defined in Lemmas 18.4 and 18.6. We take $f(n) = (-1)^{s(n)}$ in Vaughan's identity in order to estimate $S = \sum_{n\le N} f(n)\Lambda(n)$. Our treatment of the Type I sums is very simple:

$$\sum_{\substack{n\le N\\ t\mid n}} f(n) = \frac{1}{t}\sum_{a=1}^{t} T(a/t)\,.$$

By the triangle inequality and (18.xx+12) it follows that

(18.xx+14) $$\sum_{t\le U}\Big|\sum_{r\le N/t} f(rt)\Big| \ll UN^{1-\alpha}\log N\,.$$

By a second appeal to (E.13) we see that

$$\max_{w}\Big|\sum_{w\le n\le N} f(n)e(n\theta)\Big| \ll N^{1-\alpha}\log N$$

uniformly in θ. Hence by the same reasoning,

$$\sum_{d\le V}\max_{w\ge 1}\Big|\sum_{w\le h\le N/d} f(dh)\Big| \ll VN^{1-\alpha}\log N\,.$$

Thus

(18.xx+15) $$S_3 \ll VN^{1-\alpha}(\log N)^2$$

in the notation of (18.6). We write $S_2 = \sum_{t\le U} + \sum_{U<t\le UV} = S_I + S_{II}$; then

(18.xx+16) $$S_I \ll UN^{1-\alpha}(\log NUV)^2$$

by (18.xx+14). We treat S_{II} and S_4 as Type II sums, and for that we show that if $|b_m| \le 1$ for all m, $|c_k| \le 1$ for all k, and $M \le K$, then

(18.xx+17)
$$\sum_{\substack{M<m\le 2M\\ K<k\le 2K\\ mk\le N}} b_m c_k f(mk) \ll K^{1+\varepsilon} M^{1-\beta/(3-4\beta)+\varepsilon} + K^{1-1/(10-8\beta)+\varepsilon} M^{1+(1-2\beta)/(10-8\beta)+\varepsilon}\,.$$

Here the second term is largest when $M \asymp K \asymp N^{1/2}$, at which point it is $\asymp N^{1-\beta/(10-8\beta)}$. Here $\beta/(10-8\beta) = 0.00598\ldots > 1/200$. The first term on the right above becomes larger as M becomes smaller (with $K \asymp N/M$), but we take $U = V = N^{\alpha(3-4\beta)/(3-3\beta)}$, and note that then $NU^{-\beta/(3-4\beta)} = UN^{1-\alpha} = N^{1-\alpha\beta/(3-3\beta)}$. Here $\alpha\beta/(3-3\beta) = 0.0038104 > 1/263$. To treat a block with $M > K$ we simply reverse the roles of m and k. For S_4 we take
$$b_m = \mu(m), \qquad c_k = \frac{1}{\log N}\sum_{\substack{d|k\\ d>V}} \Lambda(d)$$
or vice versa if $M > K$. Conditions such as $m > U$ and $k > V$ can be met by stipulating that $b_m = 0$ if $m \le U$ and $c_k = 0$ if $k \le V$. To treat S_{II} we take
$$b_m = \begin{cases} b(m)/\log N & (m \ge U),\\ 0 & (m < U),\end{cases} \qquad c_k = \begin{cases} 1 & (k \le UV),\\ 0 & (k > UV)\end{cases}$$
or vice versa.

By Cauchy's inequality, the left hand side of (18.xx+17) is
$$\le M^{1/2}\Big(\sum_{M<m\le 2M}\Big|\sum_{\substack{K<k\le 2K\\ mk\le N}} c_k f(mk)\Big|^2\Big)^{1/2}.$$

Thus to prove (18.xx+17), it suffices to show that

(18.xx+18)
$$\sum_{M<m\le 2M}\Big|\sum_{\substack{K<k\le 2K\\ mk\le N}} c_k f(mk)\Big|^2 \ll K^{2+\varepsilon} M^{1-2\beta/(3-4\beta)+\varepsilon} + K^{2-1/(5-4\beta)+\varepsilon} M^{1+(1-2\beta)/(5-4\beta)+\varepsilon}\,.$$

By van der Corput's lemma (Lemma 17.6) we see that
$$\Big|\sum_{\substack{K<k\le 2K\\ mk\le N}} c_k f(mk)\Big|^2 \le \frac{K+H-1}{H}\sum_{\substack{K<k\le 2K\\ mk\le N}} |c_k f(mk)|^2 + 2\Re\frac{K+H-1}{H}\sum_{h=1}^{H}(1-h/H)\sum_{\substack{K<k\le 2K-h\\ m(k+h)\le N}} c_{k+h}\overline{c_k} f(m(k+h))f(mk)\,.$$

Here H is a parameter to be chosen later, subject to $H \le K$. The first term on the right hand side above is $\ll K^2/H$. We sum the above over m to see that the left hand side of (18.xx+18) is

$$\ll \frac{K^2M}{H} + \frac{K}{H}\sum_{h=1}^{H}\sum_{K<k\le 2K}\Big|\sum_{\substack{M<m\le 2M\\ m(k+h)\le N}} f(m(k+h))f(mk)\Big|. \tag{18.xx+19}$$

Let $n = \sum_j d_j 2^j$ be the binary expansion of n. We divide 2^J into n, so that $n = q2^J + r$. Then $r = \sum_{j<J} d_j 2^j$ and $s(n) = s(q) + s(r)$. Put $s_J(n) = \sum_{j<J} d_j = s(r) = s(n) - s(q)$. Thus if $q2^J \le m, n < (q+1)2^J$, then $s(m) - s_J(m) = s(q) = s(n) - s_J(n)$. Put $f_J(n) = (-1)^{s_J(n)}$. Then $f(m(k+h))f(mk) = f_J(m(k+h))f_J(mk)$ unless there is a multiple of 2^J between mk and $m(k+h)$. We choose J so that 2^J is large compared with MH, but small compared with MK. Suppose that $mk < q2^J \le m(k+h)$. Then $\{mk/2^J\} \ge 1 - mh/2^J \ge 1 - 2MH/2^J$. Thus

$$\begin{aligned}\sum_{\substack{M<m\le 2M\\ m(k+h)\le N}} f(mk)f(m(k+h)) = &\sum_{\substack{M<m\le 2M\\ m(k+h)\le N}} f_J(mk)f_J(m(k+h))\\ &+ O\Big(\sum_{\substack{M<m\le 2M\\ \{mk/2^J\}\ge 1-2MH/2^J}} 1\Big).\end{aligned} \tag{18.xx+20}$$

We group pairs m, k according to the value of mk to see that

$$\frac{K}{H}\sum_{h=1}^{H}\sum_{K<k\le 2K}\sum_{\substack{M<m\le 2M\\ \{mk/2^J\}\ge 1-2MH/2^J}} 1 \ll K^{1+\varepsilon}M^{\varepsilon}\sum_{\substack{n\le 4MK\\ \{n/2^J\}\ge 1-2MH/2^J}} 1$$

since $d(n) \ll (MK)^{\varepsilon}$. We divide the interval $0 < n \le 4MK$ into $\ll MK/2^J$ intervals of length 2^J. For n in an interval of length 2^J, the inequality $\{n/2^J\} \ge 1 - 4MH/2^J$ holds for $\ll MH$ values of n. Hence the above is

$$\ll (KM)^{2+\varepsilon}H/2^J\,. \tag{18.xx+21}$$

The function f_J is periodic with period 2^J, and so has a finite Fourier transform,

$$\widehat{f_J}(a) = \frac{1}{2^J}\sum_{n=1}^{2^J} f_J(n)e(-an/2^J) = \frac{1}{2^J}T_J(-a/2^J),$$

so that

$$f_J(n) = \sum_{a=1}^{2^J}\widehat{f_J}(a)e(an/2^J)\,.$$

Thus the first term on the right hand side of (18.20) is

$$\sum_{a=1}^{2^J}\sum_{b=1}^{2^J}\widehat{f_J}(a)\widehat{f_J}(b)\sum_{\substack{M<m\le 2M\\ m(k+h)\le N}} e((am(k+h)+bmk)/2^J)$$
$$\ll \sum_{a=1}^{2^J}\sum_{b=1}^{2^J}|\widehat{f_J}(a)\widehat{f_J}(b)|\min(M,1/\|(am(k+h)+bmk)/2^J\|)$$

by (16.5). To (18.xx+19) this contributes an amount

$$\ll \frac{K}{H}\sum_{h=1}^{H}\sum_{K<k\le 2K}\sum_{a=1}^{2^J}\sum_{b=1}^{2^J}|\widehat{f_J}(a)\widehat{f_J}(b)|\min(M,1/\|(am(k+h)+bmk)/2^J\|)\,.$$

Our estimate for this depends on the power of 2 dividing $a+b$. Write $a+b=c2^j$ with c odd. We may assume that a and b are odd, since $\widehat{f_J}(a)=0$ if a is even. Thus $1\le j\le J$, and the above is
(18.xx+22)
$$\ll \frac{K}{H}\sum_{a=1}^{2^J}\sum_{j=1}^{J}\sum_{\substack{c=1\\ 2\nmid c}}^{2^{J-j}}|\widehat{f_J}(a)\widehat{f_J}(c2^j-a)|\sum_{h=1}^{H}\sum_{K<k\le 2K}\min(M,1/\|ck/2^{J-j}+ah/2^J\|)\,.$$

Let $w_1(h)$ be weights that arise when Lemma 18.1 is applied to the interval $[1,H]$, and let $w_2(k)$ denote the weights when Lemma 18.1 is applied to the interval $[K,2K]$. Then

$$\sum_{h=1}^{H}\sum_{K<k\le 2K}\min(M,1/\|ck/2^{J-j}+ah/2^J\|)$$
$$\le \sum_{h=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}w_1(h)w_2(k)\min(M,1/\|ck/2^{J-j}+ah/2^J\|)\,.$$

Let $g(x)$ be defined as in Lemma 18.1. Then the above is

$$\ll \sum_{h=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}w_1(h)w_2(k)g(ck/2^{J-j}+ah/2^J)$$
$$=\sum_{m=-M}^{M}\widehat{g}(m)\sum_{h=-\infty}^{\infty}w_1(h)e(mah/2^J)\sum_{k=-\infty}^{\infty}w_2(k)e(mck/2^{J-j})$$
$$=\sum_{m=-M}^{M}\widehat{g}(m)W_1(ma/2^J)W_2(mc/2^{J-j})\,.$$

By Lemmas 18.1 and 18.2, this is

$$\ll HK(\log M)\Bigg(1+\sum_{\substack{m=1\\ \|ma/2^J\|<1/H\\ \|mc/2^{J-j}\|<1/K}}^{M} 1\Bigg)\,. \tag{18.xx+23}$$

By (18.xx+10) we see that

$$\sum_{c=1}^{2^{J-j}} |\widehat{f_J}(c2^j-a)| \ll |\widehat{f}_j(a)|2^{(1/2-\beta)(J-j)}\,. \tag{18.xx+24}$$

Hence

$$\sum_{a=1}^{J} |\widehat{f_J}(a)| \sum_{c=1}^{2^{J-j}} |\widehat{f_J}(c2^j-a)| \ll 2^{(1/2-\beta)(J-j)} \sum_{a=1}^{2^J} |\widehat{f_J}(a)\widehat{f}_j(a)|\,. \tag{18.xx+25}$$

By (18.xx+8) we see that

$$\widehat{f_J}(a) = \frac{1}{2^J}T_J(a/2^J) = \frac{1}{2^J}T_{J-j}(a/2^J)T_j(a/2^j) = \frac{1}{2^{J-j}}T_{J-j}(a/2^J)\widehat{f}_j(a)\,. \tag{18.xx+26}$$

Write $a = a_0 + a_1 2^j$. Then the right hand side of (18.xx+25) is

$$= 2^{(-1/2-\beta)(J-j)} \sum_{a_0=1}^{2^j} |\widehat{f}_j(a_0)|^2 \sum_{a_1=1}^{2^{J-j}} |T_{J-j}(a_0+a_1/2^{J-j})|\,.$$

By Lemma 18.6 this is

$$\ll 2^{(1-2\beta)(J-j)} \sum_{a_0=1}^{j} |\widehat{f}_j(a_0)|^2\,.$$

Here the sum over a_0 is $=1$ by Parseval's identity, so we conclude that

$$\sum_{a=1}^{J} |\widehat{f_J}(a)| \sum_{c=1}^{2^{J-j}} |\widehat{f_J}(c2^j-a)| \ll 2^{(1-2\beta)(J-j)}\,. \tag{18.xx+27}$$

Hence the term $HK\log M$ in (18.xx+23), which reflects the mean value of $\min(M, 1/\|x\|)$, contributes to (18.xx+22) an amount that is

$$\ll K^2 2^{(1-2\beta)J}\log M\,. \tag{18.xx+28}$$

It remains to estimate

$$K^2(\log M)\sum_{a=1}^{2^J}\sum_{j=1}^{J}\sum_{\substack{c=1\\ 2\nmid c}}^{2^{J-j}} |\widehat{f_J}(a)\widehat{f_J}(c2^j-a)| \sum_{\substack{m=1\\ \|ma/2^J\|<1/H\\ \|mc/2^{J-j}\|<1/K}}^{M} 1\,. \tag{18.xx+29}$$

The way that we proceed depends on the size of 2^{J-j}. Suppose first that $2^{J-j} \ge K$. Since $M \le K$, the numbers $m = 1, 2, \ldots, M$ comprise at most one complete system of residues modulo 2^{J-j}, and hence the number of them for which $\|mc/2^{J-j}\| \le 1/K$ is $\ll 2^{J-j}/K$ since c is odd. By (18.xx+27), such a j contributes to (18.xx+29) an amount

$$\ll K 2^{(2-2\beta)(J-j)} \log M,$$

and the sum over such j contributes

$$\ll K 2^{(2-2\beta)J} \log M\,. \tag{18.xx+30}$$

Next suppose that $M < 2^{J-j} < K$. For $1 \le m \le M < 2^{J-j}$ we have $m \not\equiv 0 \pmod{2^{J-j}}$, and hence $\|mc/2^{J-j}\| \ge 1/2^{J-j} > 1/K$. Thus in (18.27), the sum over m is empty when $M < 2^{J-j} < K$. Finally, suppose that $2^{J-j} \le M$. Since $K \ge M$, the inequality $\|mc/2^{J-j}\| < 1/K$ holds only when m is a multiple of 2^{J-j}. Write $m = r2^{J-j}$. Then we have to estimate

$$K^2(\log M) \sum_{a=1}^{2^J} \sum_{\substack{j \\ 1 \le 2^{J-j} \le M}} \sum_{c=1}^{2^{J-j}} |\widehat{f_J}(a)\widehat{f_J}(c2^j - a)| \sum_{\substack{r=1 \\ \|ra/2^j\| < 1/H}}^{M/2^{J-j}} 1\,. \tag{18.xx+31}$$

To the extent possible, we argue as before. By (18.xx+24) the above is

$$\ll K^2(\log M) \sum_{a=1}^{2^J} \sum_{\substack{j \\ 1 \le 2^{J-j} \le M}} 2^{(1/2-\beta)(J-j)} |\widehat{f_J}(a)\widehat{f}_j(a)| \sum_{\substack{r=1 \\ \|ra/2^j\| < 1/H}}^{M/2^{J-j}} 1\,.$$

We appeal to (18.xx+26) and write $a = a_0 + a_1 2^j$ to see that the above is

$$= K^2(\log M) \sum_{\substack{j \\ 1 \le 2^{J-j} \le M}} 2^{(-1/2-\beta)(J-j)} \sum_{a_0=1}^{2^j} |\widehat{f}_j(a_0)|^2 \sum_{\substack{r=1 \\ \|ra/2^j\| < 1/H}}^{M/2^{J-j}} \sum_{a_1=1}^{2^{J-j}} |T_{J-j}(a_0/2^J + a_1/2^{J-j})|\,.$$

We apply Lemma 18.6 to the sum over a_1, and thus see that the above is

$$\ll K^2(\log M) \sum_{\substack{j \\ 1 \le 2^{J-j} \le M}} 2^{(1-2\beta)(J-j)} \sum_{a_0=1}^{2^j} |\widehat{f}_j(a_0)|^2 \sum_{\substack{r=1 \\ \|ra_0/2^j\| < 1/H}}^{M/2^{J-j}} 1\,. \tag{18.xx+32}$$

In general, we would expect the sum over r to be about $M/(H2^{J-j})$ in size. Let B be chosen later, $B \le H$. The a_0 for which the sum over r is $\le M/(B2^{J-j})$ contribute an amount $\ll M/(B2^{J-j})$, by Parseval's identity. Now consider those a_0 for which the sum over r lies between $2^i M/(B2^{J-j})$ and $2^{i+1}M/(B2^{J-j})$. This is far more solutions than we expect, and by Lemma 18.7 it follows that there is a $q \ll B/2^i$ such that $\|qa_0/2^j\| \ll B2^{J-j}/(2^iHM)$. Let h denote the integer nearest $qa_0/2^j$. Then $1 \le h \le q$, and

$$\Big|a_0 - \frac{2^j h}{q}\Big| \ll \frac{B2^J}{2^iHMq},$$

so for each h there are $\ll B2^J/(2^iHMq)$ such a_0. (There is no need to add 1 to this estimate, since the interval in which the a_0 lie has length $\gg 1$.) On summing over h and over $q \ll B/2^i$, we find that there are $\ll B^2 2^J/(2^{2i}HM)$ values a_0 in question. Since $\widehat{f}_j(a_0) \ll 2^{-\alpha j}$ by Lemma 18.4, we find that the contribution of such a_0 is $\ll B2^{(1-2\alpha)j}/(2^iH)$. We sum over i, and combine our estimates to see that

$$\sum_{a_0=1}^{2^j} |\widehat{f}_j(a_0)|^2 \sum_{\substack{r=1\\ \|ra_0/2^j\|<1/H}}^{M/2^{J-j}} 1 \ll \frac{M}{B2^{J-j}} + \frac{B2^{(1-2\alpha)j}}{H}.$$

To optimize this bound we take $B = M^{1/2}H^{1/2}2^{-J/2+\alpha j}$, and thus see that

$$\sum_{a_0=1}^{2^j} |\widehat{f}_j(a)|^2 \sum_{\substack{r=1\\ \|ra_0/2^j\|<1/H}}^{M/2^{J-j}} 1 \ll M^{1/2}H^{-1/2}2^{-J/2+(1-\alpha)j}. \tag{18.xx+33}$$

Hence the quantity (18.xx+32) is

$$\ll K^2M^{1/2}H^{-1/2}2^{(1/2-\alpha)J}(\log M) \sum_{\substack{j\\ 1\le 2^{J-j}\le M}} 2^{(\alpha-2\beta)(J-j)}.$$

But $\alpha - 2\beta > 0$, so the largest term occurs when $2^{J-j} \asymp M$, and hence the above is

$$\ll K^2M^{1/2+\alpha-2\beta}H^{-1/2}2^{(1/2-\alpha)J}\log M. \tag{18.xx+34}$$

On combining this with (18.xx+19), (18.xx+21), (18.xx+28), and (18.xx+30), we conclude that the left hand side of (18.xx+18) is

$$\begin{aligned}\ll &K^2MH^{-1} + (KM)^{2+\varepsilon}H2^{-J} + K^2 2^{(1-2\beta)J}\log M\\ &+ K2^{(2-2\beta)J}\log M + K^2M^{1/2+\alpha-2\beta}H^{-1/2}2^{(1/2-\alpha)J}\log M.\end{aligned}$$

Suppose that $2^J \asymp MHA$. Then (apart from the ε in the exponent), the first two terms are $\ll K^2M(1/H + 1/A)$. If A and H are allowed to vary in such a way that AH is held constant, then the third and fourth terms above are fixed, and the sum of the first two terms is minimized by taking $A = H$. Accordingly, we take J so that $2^J \asymp MH^2$. Thus the above is

$$\begin{aligned}\ll &K^{2+\varepsilon}M^{1+\varepsilon}H^{-1} + K^2M^{1-2\beta}H^{2-4\beta}\log M\\ &+ KM^{2-2\beta}H^{4-4\beta}\log M + K^2H^{1/2-2\alpha}M^{1-2\beta}\log M.\end{aligned}$$

Here the last term is smaller than the second one, so may be ignored. If $K^{1-4\beta/3} \le M \le K$, then we take $H = K^{1/(5-4\beta)}/M^{(1-2\beta)/(5-4\beta)}$. Then the first and third terms are roughly equal and the second term is smaller. In this range, all terms are

$$\ll K^{1-1/(5-4\beta)+\varepsilon} M^{1+(1-2\beta)/(5-4\beta)+\varepsilon}\,.$$

For $2 \le M \le K^{1-4\beta/3}$, we take $H = M^{2\beta/(3-4\beta)}$. Then the first and second terms are nearly equal, and the third one is smaller. In this range, all terms are

$$\ll K^{2+\varepsilon} M^{1-2\beta/(3-4\beta)+\varepsilon}\,.$$

Thus we have (18.xx+18), and the proof is complete. $\square$

We note that (18.xx+33) is worse than the trivial bound

$$\sum_{a_0=1}^{2^j} |\widehat{f}_j(a)|^2 \sum_{\substack{r=1 \\ \|ra_0/2^j\|<1/H}}^{M/2^{J-j}} 1 \ll M2^{-(J-j)} \sum_{a_0=1}^{2^j} |\widehat{f}_j(a)|^2 \ll M2^{-(J-j)}$$

when $2^{\alpha(J-j)} > M^\alpha H^{-1/2+2\alpha}$. Thus we could improve on (18.xx+34), but this would not lead to a stronger conclusion because the bound in (18.xx+34) makes a smaller contribution than the estimate (18.xx+28).

Exercises

1. Let $f(x)$ be defined as in Lemma 19.1. Show that $\widehat{f}(m) \ll \log 2M/|m|$ for $|m| \le M$, and that $\widehat{f}(m) \ll (M/|m|)^2$ for $|m| \ge M$.

2. Let $T_J(\theta)$ be defined as in (18.2).
(a) Show that $|T_J(1/3)| = 3^{J/2}$.
(b) Suppose that $4/3 = 4^{\alpha'}$, so that $\alpha' = 0.20752\ldots$. Show that Lemma 18.4 would be false if α is replaced by a number $> \alpha'$.

3. For $0 \le r \le 1$, let $f_r(\theta) = \log|1 - re(\theta)|$.
(a) Show that if $0 \le r < 1$, then
$$f_r(\theta) = -\sum_{n=1}^\infty \frac{r^n}{n} \cos 2\pi n\theta\,.$$
(b) Show that if $\theta \notin \mathbb{Z}$, then $\sum_{n=1}^\infty (\cos 2\pi n\theta)/n$ converges.
(c) By Abel's theorem (cf §5.2), deduce that
$$f_1(\theta) = -\sum_{n=1}^\infty \frac{\cos 2\pi n\theta}{n}$$

when $\theta \notin \mathbb{Z}$.
(d) Show that

$$f_1(\theta) - f_r(\theta) \ll \min\Big(\frac{1-r}{\|\theta\|}, \log\frac{1-r}{\|\theta\|}\Big).$$

(e) Deduce that $\|f_1 - f_r\|_1 \ll (1-r)\log(2/(1-r))$.
(f) Show that if $0 \le r < 1$, then

$$\widehat{f_r}(n) = \begin{cases} \dfrac{-r^{|n|}}{2|n|} & (n \ne 0), \\ 0 & (n = 0) \end{cases}$$

(g) By the inequality $|\widehat{f_r}(n) - \widehat{f_1}(n)| \le \|f_r - f_1\|_1$, deduce that

$$\widehat{f_1}(n) = \begin{cases} \dfrac{-1}{2n} & (n \ne 0), \\ 0 & (n = 0). \end{cases}$$

(h) Deduce (18.3).
(i) Deduce (18.4).

4. (a) Show that $|1 - e(\theta)| + |1 + e(\theta)| \le 2\sqrt{2}$ for all θ.
(b) Let $S_J(\theta)$ denote the sum in Lemma 18.6. Show that $S_J(\theta) \le 2\sqrt{2}S_{J-1}(2\theta)$.

5. For $1 \le n \le N$, let X_n denote independent random variables with $P(X_n = \pm 1) = 1/2$. For a generic point ω of our probability space, let $f_\omega(\theta) = \sum_{n=1}^N X_n e(n\theta)$ denote a random exponential polynomial.
(a) Show that

$$\int_0^1 |f_\omega(\theta)|^2\, d\theta = N$$

for all ω.
(b) Show that

$$\int_0^1 |f_\omega(\theta)|^4\, d\theta = \sum_{n=2}^{2N} \Big(\sum_{\substack{1\le m,k\le N \\ m+k=n}} X_m X_k\Big)^2.$$

(c) Show that the number of pairs (m, k) with $1 \le m, k \le N$ and $m + k = n$ is $\max(0, N - |N + 1 - n|)$.

(d) Show that if n is odd, $2 \le n \le 2N$, then

$$E\Big[\sum_{\substack{1\le m_1,k_1,m_2,k_2\le N \\ m_1+k_1=m_2+k_2=n}} X_{m_1}X_{k_1}X_{m_2}X_{k_2}\Big] = 2(N - |N + 1 - n|).$$

(e) Show that if n is even, $2 \le n \le 2N$, then

$$E\Big[\sum_{\substack{1\le m_1,k_1,m_2,k_2\le N\\ m_1+k_1=m_2+k_2=n}} X_{m_1}X_{k_1}X_{m_2}X_{k_2}\Big] = 2(N-|N+1-n|)-1\,.$$

(f) Show that

$$E\Big[\int_0^1 |f_\omega(\theta)|^4\,d\theta\Big] = 2N^2 - N\,.$$

(g) Deduce that

$$P\Big(\int_0^1 |f_\omega(\theta)|^4\,d\theta > 4N^2\Big) \le \frac{1}{2}\,.$$

(h) Show that

$$\int_0^1 |f|^2 \le \Big(\int_0^1 |f|\Big)^{2/3}\Big(\int_0^1 |f|^4\Big)^{1/3}$$

for all f.

(i) Show that

$$P\Big(\int_0^1 |f_\omega(\theta)|\,d\theta \ge \frac{\sqrt{N}}{2}\Big) \ge \frac{1}{2}\,.$$

With more work, it can be shown that $\int_0^1 |f_\omega(\theta)|^4\,d\theta$ is usually near its expectation, with the result that the probability considered in (i) above tends rapidly to 1 as N tends to infinity. Also, it is unlikely that $\|f_\omega\|_\infty$ would be much larger than $\sqrt{N\log N}$. Hence in Lemma 18.6 and Exercise 2(a) we see that the coefficients $(-1)^{s(n)}$ produce behavior that would be highly atypical for a random sequence.

6. Suppose that $0 < \delta_1 \le \delta_2/2$, that $N \ge 1/\delta_2$, that $1 \le q \le 1/(2\delta_2)$, choose a so that $(a,q)=1$, put $\theta = a/q + \delta_1/(\delta_2 qN)$, and set $I = [0, 2\delta_1]$.
   (a) Show that $\|q\theta\| = \delta_1/(\delta_2 N)$.
   (b) Show that $n\theta \in I \pmod 1$ for at least $\delta_2 N$ values of n, $1 \le n \le N$.

7. (a) Explain why $|T_J(\theta)|^4 = |T_{J-2}(\theta)|^4|T_2(2^{J-2}\theta)|^4$.
   (b) Explain why $|T_{J-1}(\theta)|^4 = |T_{J-2}(\theta)|^4|T_1(2^{J-2}\theta)|^4$.
   (c) Write $|T_2(\alpha)|^4 - 2|T_1(\alpha)|^4 - 16 = \sum_{n=0}^{6} c_n\cos 2\pi n\alpha$. Show that $c_0 = c_1 = 0$.
   (d) Explain why $\int_0^1 |T_{J-2}(\theta)|^4 e(k\theta)\,d\theta = 0$ if $|k| \ge 2^{J-1}-1$.
   (e) Put $u_J = \int_0^1 |T_J(\theta)|^4\,d\theta$. Show that $u_0 = 1$, $u_1 = 6$, and that

$$u_J = 2u_{J-1} + 16u_{J-2}$$

   for $J \ge 2$.
   (f) Show that

$$u_J = \frac{17+5\sqrt{17}}{34}(1+\sqrt{17})^J + \frac{17-5\sqrt{17}}{34}(1-\sqrt{17})^J\,.$$