Linear Recurrences

Suppose that a sequence $\{u_n\}$ is generated by a recurrence of the form

$$u_{n+1} = au_n + bu_{n-1},$$

and that we wish to find a formula for the general term. The first step is to find sequences of this type in which the general term is simply a power of a fixed base: $u_n = \lambda^n$. Such a sequence will satisfy the recurrence if $\lambda^{n+1} = a\lambda^n + b\lambda^{n-1}$, which is to say if $\lambda^2 = a\lambda + b$. Usually, the polynomial $P(x) = x^2 - ax - b$ will have two distinct roots, say λ_1 and λ_2. The general solution of the linear recurrence is then given as a linear combination of the two basic solutions λ_1^n and λ_2^n:

$$u_n = c_1\lambda_1^n + c_2\lambda_2^n\,.$$

The values of the constants c_1 and c_2 are chosen so that the seuqence satisfies some prescribed initial conditions.

Example 1. The Fibonacci numbers F_n are defined by the relations $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$. The associated polynomial is $x^2 - x - 1$, which has the two roots $\lambda_1 = (1+\sqrt{5})/2$ and $\lambda_2 = (1-\sqrt{5})/2$. We note that $\lambda_1 = 1.618\ldots$ and $\lambda_2 = -0.618\ldots$. Thus $F_n = c_1\lambda_1^n + c_2\lambda_2^n$ for some choice of c_1 and c_2. These constants are to be chosen so that $c_1 + c_2 = 0$ and $c_1\lambda_1 + c_2\lambda_2 = 1$. Here we have two linear equations in two variables, and by elimination we discover that $c_1 = 1/\sqrt{5}$, $c_2 = -1/\sqrt{5}$. That is,

$$F_n = \frac{1}{\sqrt{5}}\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \frac{1}{\sqrt{5}}\Big(\frac{1-\sqrt{5}}{2}\Big)^n.$$

Since $|\lambda_2| < 1$, the contribution made by $-\lambda_2^n/\sqrt{5}$ tends to 0 exponentially. Since $\lambda_2 < 0$, this exponentially small contribution is also alternating in sign. Thus, for example,

$$\begin{aligned} \lambda_1^{20}/\sqrt{5} &= 6765.00002956\ldots & F_{20} &= 6765, \\ \lambda_1^{21}/\sqrt{5} &= 10945.99998173\ldots & F_{21} &= 10946. \end{aligned}$$

What we have discussed thus far is called a linear recurrence of order 2, but the theory of linear recurrences of higher order is the same. Sometimes it happens that the polynomial $P(x)$ does not have a full complement of distinct roots, but instead has some repeated roots. If λ is a double root of P, then in addition to λ^n as a basic solution of the recurrence, one also has $n\lambda^n$. Similarly, if λ^n is a triple root, then one has basic solutions λ^n, $n\lambda^n$, and $n^2\lambda^n$.

Example 2. Find a formula for u_n where $u_0 = 1$, $u_1 = 2$, and $u_{n+1} = 2u_n - u_{n-1}$. Here the associated polynomial is $x^2 - 2x + 1 = (x-1)^2$, so the general solution is of the form $u_n = c_1 \cdot 1^n + c_2 n \cdot 1^n = c_1 + c_2 n$. Thus $1 = u_0 = c_1$ and $2 = u_1 = c_1 + c_2$. Again we have two linear equations in two unknowns, which we solve to find that $u_n = n + 1$.