Math 425 Sect 6 **Fall 2015**

Second Hour Exam

NAME ______________________________

Answers may be expressed in terms of factorials, but not in terms of binomial or multinomial coefficients. Products do not need to be multiplied out, and fractions do not need to be expressed in lowest terms.

3 pts **1.** Let X and Y be discrete random variables with expectations $E[X$ and $E[Y]$, respectively. Under what circumstances is it true that $E[X+Y] = E[X] + E[Y]$?

2. Let X be an exponential random variable with parameter $\lambda > 0$. Assume that h is a positive number with the property that $P(X > h) = 1/2$. (This is called the *half-life*.)

7 pts (a) How are λ and h related?

3 pts (b) $P(X > 3h) =$

3. Let X be a uniform random variable on the interval $[0, 1]$, and put $Y = -\log(1 - X)$.

7 pts (a) For $a \geq 0$, derive a formula for $P(Y \leq a)$.

3 pts (b) Deduce a formula for $f_Y(y)$.

4. The joint density function of the random variables X and Y is

$$f(x,y) = \begin{cases} x+y & (0 \le x \le 1, 0 \le y \le 1). \\ 0 & (\text{otherwise}). \end{cases}$$

6 pts. (a) Find the probability density function $f_X(x)$ of the random variable X.

6 pts. (b) Let $F(x,y)$ be the joint cumulative distribution function of X and Y. Find $F(1/2, 1/2)$.

6 pts. (c) Are X and Y independent? Explain.

5. The weather on a given day is classified as either Good of Bad, and the probability of Good weather is $p = 0.6$. Moreover, the weather on a given day is independent of the weather on all other days. Let $X_i = 1$ if the weather changes from day i to day $i+1$, and set $X_i = 0$ otherwise.

5 pts. (a) Find $P(X_1 = 1)$.

5 pts. (b) Find $P(X_1 = X_2 = 1)$.

3 pts. (c) Are X_1 and X_2 independent? (Demonstrate that your answer is correct.)

3 pts. (d) In an 11 day period, what is the expected number of changes of weather?

6. Consider a traffic light that is alternately green for 30 seconds, and then red for 30 seconds. We ignore delay associated with the need to decelerate or accelerate, and model this as follows: Our probability space is $S = [0, 1]$, for $A \subseteq S$ we put $P(A) = \int_A 1\, dx$, and for $\omega \in S$ we set $X(\omega) = 0$ for $0 \le \omega < 1/2$, and $X(\omega) = 1 - \omega$ for $1/2 \le \omega \le 1$.

7 pts (a) Derive a formula (or several formulas in different ranges) for $F_X(a) = P(X \le a)$.

5 pts (b) Derive a formula (or formulas) for $P(X > a)$.

5 pts (c) Compute

$$\int_0^\infty P(X > a)\, da =$$

5 pts (d) Compute

$$\int_0^1 X(\omega)\, d\omega =$$

7. Let X and Y be continuous random variables with densities $f_X(x)$ and $f_Y(y)$, respectively.

7 pts (a) Express the density $f_{X+Y}(a)$ as an integral involving $f_X(x)$ and $f_Y(y)$ for suitable x and y.

$$f_{X+Y}(a) = \int_{-\infty}^{\infty}$$

7 pts (b) Suppose now that $f_X(x) = 0$ if $x < 0$ and that $f_Y(y) = 0$ if $y < 0$. The interval of integration can now be restricted: For $a > 0$ we have

$$f_{X+Y}(a) = \int$$

7 pts (c) Now suppose that X and Y are exponential random variables with parameter $\lambda = 1$. Derive a formula for $f_{X+Y}(a)$.