The Central Limit Theorem

This is a supplement to §8.3 of the Ross text. We assume that the independent summands X_i have a moment generating function

$$M(t) = E\big[e^{tX_i}\big]$$

that has a continuous second derivative for t in an interval $(-\delta, \delta)$. In proving the Central Limit Theorem we may (by a simple change of variable) assume that $E[X_i] = \mu = 0$, and that $\mathrm{Var}(X_i) = 1$. Let

$$Y_n = \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}\,.$$

Thus $E[Y_n] = 0$, and $\mathrm{Var}(Y_n) = 1$. In order to show that $F_{Y_n}(a) \to \Phi(a)$ for all a, by Example 7.7d and Lemma 8.3.1 it suffices to show that $M_{Y_n}(t) \to \exp(t^2/2)$ for all t. The moment generating function of $X_i/\sqrt{n}$ is

$$E\big[\exp(tX_i/\sqrt{n})\big] = M(t/\sqrt{n})\,.$$

From the discussion on p. 338 we know that if X and Y are independent random variables, then $M_{X+Y}(t) = M_X(t)M_Y(t)$. Hence

$$M_{Y_n}(t) = M(t/\sqrt{n})^n,$$

and our object is to determine the limit of this as $n \to \infty$. As we continue, our object is to present a simpler alternative to the calculations found on p. 371.

Our task is rather similar to that of showing that

$$\left(1 + \frac{c}{n}\right)^n \to e^c$$

as $n \to \infty$. By taking logarithms we see that the above is equivalent to

$$n \log(1 + c/n) \to c$$

as $n \to \infty$. We write the left hand side as

$$\frac{\log(1 + c/n)}{1/n}$$

and replace $1/n$ by δ, and see that it suffices to show that

$$\frac{\log(1 + c\delta)}{\delta} \to c$$

as $\delta \to 0$. This is a limit of the form 0/0, so we apply L'Hôpital's rule. In our present situation we take logarithms, and see that it is enough to show that

$$n \log M(t/\sqrt{n}) = \frac{\log M(t/\sqrt{n})}{1/n} \to t^2/2$$

as $n \to \infty$. We set $\delta = 1/\sqrt{n}$, so the above follows if

$$\frac{\log M(t\delta)}{\delta^2} \to t^2/2$$

as $\delta \to 0$. Now $M(0) = 1$, so this is a limit of the form $0/0$. By L'Hôpital's rule, the above is true if

$$\frac{\frac{M'(t\delta)}{M(t\delta)}t}{2\delta} \to t^2/2\,.$$

Since $\lim 1/M(t\delta) = 1$, and since the limit of a product is the product of the limits, it suffices now to show that

$$\lim_{\delta\to 0} \frac{M'(t\delta)}{\delta} = t\,.$$

Since $M'(0) = 0$, this is again a limit of the form $0/0$. By a second application of L'Hôpital's rule, we see that the above holds if

$$\lim_{\delta\to 0} \frac{M''(t\delta)t}{1} = t\,.$$

But this is clear since $M''(0) = 1$ and M'' is continuous at 0.

In using Lemma 3.1, we are limited to distributions for which the moment generating function exists. At a more advanced level, instead of the moment generating function, we would use the *characteric function*

$$\varphi_X(t) = \int_{-\infty}^{\infty} e^{-2\pi i t x} f(x)\, dx\,.$$

We mention a number of advantages of using $\varphi_X(t)$.

1. $\varphi_X(t)$ exists for all t and random variables X.
2. $\varphi_X(0) = 1$, and $|\varphi_X(t)| \le 1$ for all t.
3. $\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t)$ if X and Y are independent.
4. Let Y_n be a sequence of random variables with cumulative distribution functions $F_{Y_n}(x)$, and let Z be a further random variable with cumulative distribution function $F_Z(x)$. The following assertions are equivalent:
   (a) $F_{Y_n}(a)$ tends to $F_Z(a)$ at all points of continuity of F_Z;
   (b) $\varphi_{Y_n}(t) \to \varphi_Z(t)$ for all real t.

By using characteristic functions in the same way that we used moment generating functions, we can prove the Central Limit Theorem without imposing extraneous hypotheses.