An analytic proof of the binomial theorem

On pages 7 and 8, the text gives two proofs of the binomial theorem, one by induction, and the other by combinatorial reasoning. Here is a third proof, one that depends on calculus.

Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a polynomial of degree at most n. We give two proofs that

$$f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k . \tag{1}$$

First proof. Consider what happens when we differentiate the monomial $a_j z^j$ k times. If $j < k$, the result is 0. If $j \ge k$, then the result is $a_j j(j-1)\cdots(j-k+1)z^{j-k}$. We note that $j(j-1)\cdots(j-k+1) = j!/(j-k)! = k!\binom{j}{k}$. Thus

$$f^{(k)}(z) = k! \sum_{j=k}^{n} \binom{j}{k} a_j z^{j-k} .$$

In particular, on taking $z = 0$ we discover that $f^{(k)}(0) = k! a_k$. That is, $a_k = f^{(k)}(0)/k!$, and so we have (1).

Second proof. If f is a function with derivatives through the order $n+1$, Taylor's formula asserts that

$$f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} z^{n+1} .$$

When f is a polynomial of degree at most n, $f^{(n+1)}(z)$ is identically 0, so the remainder vanishes, and we have (1).

Now take $f(z) = (1+z)^n$. Then $f'(z) = n(1+z)^{n-1}$, $f''(z) = n(n-1)(1+z)^{n-2}$, and in general $f^{(k)}(z) = n(n-1)\cdots(n-k+1)(1+z)^{n-k}$. Hence $f^{(k)}(0) = n(n-1)\cdots(n-k+1) = n!/(n-k)!$ for $0 \le k \le n$, and so $a_k = f^{(k)}(0)/k! = \binom{n}{k}$. That is,

$$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k . \tag{2}$$

If we take $z = x/y$ and multiply by y^n we find that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{3}$$

We can do this only when $y \ne 0$, but the above is also true (obviously) when $y = 0$, so it holds for all x and y. Conversely, if we take $x = z$, $y = 1$ in (3) we obtain (2), so (2) and (3) are equivalent forms of the binomial theorem.

In combinatorics, a polynomial or power series generating function is often used to establish combinatorial identities. Thus the argument above is not an isolated curiosity, but is rather a first simple example of a host of arguments that proceed along these lines.