The Ace of Hearts Method

1. History. At the ICM in Helsinki in 1978, Alf van der Poorten idly watched John Conway lose repeatedly at backgammon. When play was terminated, the great JHC owed his opponent 370 Finnish Marks. John pulled a 1000 markka note from his pocket, but no one present could make the necessary change. Another onlooker observed, "This calls for the ace of hearts method." Someone produced a coin, and flipped it. John handed the 1000 markka note to his opponent. The coin was flipped again, and the opponent handed the bill back. The coin was flipped a third time, John pocketed the 1000 markka note, and they all departed happily. Alf was left to ponder what it was he had just witnessed.

2. The problem. Devise a simple procedure, based on a coin that comes up heads with probability $1/2$, that produces a positive outcome with probability exactly α, for any preassigned α, $0 \le \alpha \le 1$.

3. The procedure. Make a sequence of double-or-nothing bets. Suppose that A owes α to B where $0 \le \alpha < 1$, and but the amount actually paid will either be 0 or 1. If $\alpha \ge 1/2$, then payment is made immediately, with the result that B owes $1-\alpha$ to A; note that $0 \le 1-\alpha \le 1/2$. The person holding the bill calls the flip of a coin. If he calls correctly, then he owes nothing. If he calls incorrectly, then his debt is doubled. If the resulting debt is $> 1/2$, then the debt is paid and the other party has a complementary debt. The cycle is repeated until the person holding the money calls the toss correctly.

In the historical case recalled above, Conway evidently lost the first bet, which meant that he owed 740 markka. Since this is more than half of 1000, he handed over the bill. His opponent then owed him 260 markka. The opponent lost the next bet, with the result that he owed John 520 markka. He handed the bill back; John then owed 480 markka. John won the next bet, which settled the issue.

4. Analysis. Your friendly professional probabilist will tell you that a double-or-nothing bet is fair; hence the debt is paid with probability α. We amateurs, however, feel the need for something a little more detailed. In the discussion that follows, suppose that $0.b_1\,b_2\,b_3\ldots$ is the binary expansion of α, which is to say that $\alpha = b_1/2 + b_2/4 + b_3/8 + \cdots$ with each $b_i = 0$ or 1. Suppose that the toss was incorrectly called on the first $k-1$ tosses, and that the money has exchanged hands, if necessary, so that the participants are ready for the k^{th} toss. We claim:

If $b_k = 0$, then A holds the money, and owes

$$2^{k-1}\alpha - \left[2^{k-1}\alpha\right] = \left\{2^{k-1}\alpha\right\} = \frac{b_k}{2} + \frac{b_{k+1}}{2^2} + \cdots .$$

If $b_k = 1$, then B holds the money, and owes

$$\left[2^{k-1}\alpha\right] + 1 - 2^{k-1}\alpha = 1 - \left\{2^{k-1}\alpha\right\} = \frac{1-b_k}{2} + \frac{1-b_{k+1}}{2^2} + \cdots .$$

Here $[u]$ denotes the integer part of u, which is to say that $[u]$ is the largest integer note exceeding u. In other words, $[u]$ is the unique integer such that $[u] \le u < [u] + 1$. The *fractional part* of u is $\{u\} = u - [u]$.

To prove the claim we argue by induction on k. Clearly the claim is correct when $k = 1$. Suppose the claim is correct for k. If $b_k = 0$, and A calls the toss incorrectly, then he owes twice as much, namely the amount

$$\frac{b_{k+1}}{2} + \frac{b_{k+2}}{2^2} + \cdots .$$

This number lies in the interval $[0, 1/2)$ or in $[1/2, 1)$ according as $b_{k+1} = 0$ or 1. Thus if $b_{k+1} = 0$, then A retains the money, and is ready for round $k + 1$. If $b_{k+1} = 1$, then A hands the bill to B, and B owes

$$1 - \Big(\frac{b_{k+1}}{2} + \frac{b_{k+2}}{2^2} + \cdots\Big) = \frac{1 - b_{k+1}}{2} + \frac{1 - b_{k+2}}{2^2} + \cdots ,$$

which is in accordance with the claim. Now suppose that $b_k = 1$, and that B calls the toss incorrectly. Then B owes twice as much, which is

$$\frac{1 - b_{k+1}}{2} + \frac{1 - b_{k+2}}{2^2} + \cdots .$$

This lies in $[0, 1/2)$ or in $[1/2, 1)$ according as $b_{k=1} = 1$ or $b_{k=1} = 0$. Thus if $b_{k+1} = 1$, then we are ready for round $k + 1$, while if $b_{k+1} = 0$, then B hands the bill back to A, and A owes B the amount

$$1 - \Big(\frac{1 - b_{k+1}}{2} + \frac{1 - b_{k+2}}{2^2} + \cdots\Big) = \frac{b_{k+1}}{2} + \frac{b_{k+2}}{2^2} + \cdots .$$

Thus the claim is established for $k + 1$, and the induction is complete.

5. Probability. On the basis of the above insights, it is now an easy exercise to determine the probability that B ends up with the money. Let E denote this event. Our sample space consists of the outcomes of the bets, each one of which may be won (W) or lost (L); we continue until a bet is won. Thus the space is $W,\ LW,\ LLW,\ \ldots,\ L^{k-1}W, \ldots$, and finally the unlikely event L^∞. Since the sample space is partitioned into these various cases, we see that

$$\begin{aligned} P(E) &= P(E \cap W) + P(E \cap LW) + P(E \cap LLW) + \cdots \\ &= P(E|W)P(W) + P(E|LW)P(LW) + P(E|LLW)P(LLW) + \cdots . \end{aligned}$$

As $P\big(L^{k-1}W\big) = 2^{-k}$, the above is

$$= \frac{P(E|W)}{2} + \frac{P(E|LW)}{2^2} + \frac{P(E|LLW)}{2^3} + \cdots .$$

In the preceding paragraph we found that B wins the money, say on round k, precisely when $b_k = 1$. That is, $P\big(E|L^{k-1}W\big) = b_k$. Thus the above sum is exactly

$$\frac{b_1}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \cdots = \alpha\,.$$

6. Efficiency. Let X denote the number of tosses needed to decide the issue. Then X is a geometric random variable with parameter $p = 1/2$, so the expected value of X is $E[X] = 2$. It seems remarkable that any desired threshold α can be fairly measured in so few steps.

7. Alternative procedures. A die with 100 numbered faces and the property that it comes up on each face with probability $1/100$ would seem to be hard to construct. In any case, while it would deal with cents of a dollar, it would not accurately measure $\alpha = 1/3$, much less $\alpha = 1/\sqrt{2}$. One could ask a 'random' number generator to produce a number β, uniformly distributed in $[0,1]$, and then A pays B if $0 \le \beta \le \alpha$. In the absence of a random number generator, one could flip a coin to determine the number β through its binary expansion, say $\beta = 0.d_1\,d_2\,\ldots = d_1/2 + d_2/2^2 + \cdots$. One would continue until one can distinguish which is the larger of α and β. Suppose $b_i = d_i$ for $1 \le i < k$. If $b_k = 0$ and $d_k = 1$, then $\alpha < \beta$, and A keeps the money. If $b_k = 1$ and $d_k = 0$, then $\beta < \alpha$, and A pays B. This, of course, is equivalent to the ace of hearts method, but lacks the immediacy and charm.

8. Final question. What on earth does any of this have to do with the ace of hearts?