# AN ASSORTMENT OF COMBINATORIAL GAMES

**Adjacent Litton.** Play begins on an empty $n \times n$ board. A move is made by placing pieces on empty squares of the board. All the pieces placed on a given turn must be in the same column or row, and they must all be adjacent. The player who fills the board wins.

**Aliquot.** Play starts with a positive integer $n$. A move is made by subtracting from the current number $m$ a divisor $d$ of $m$ with $d < m$. Thus $m$ is replaced by $m - d$. The player who is given the number 1 has no legal move, and loses.

**Bridgit.** An $n \times (n+1)$ array of black dots is interlaced with an $(n+1) \times n$ array of circles. Alice moves by connecting two rectilinearly adjacent circles with a line segment. Bob moves by connecting two rectilinearly adjacent dots. A move is prohibited if it would cross a segment already drawn by the opponent. Alice wins by creating a path from the bottom row of circles to the top row, while Bob wins by creating a path from the leftmost column of dots to the rightmost column. A move is not required to be connected to prior moves.

**Brussels Sprouts.** This is similar to Sprouts, but starts with several crosses instead of spots. A move consists of drawing a curve from the end of an arm of a cross to another, and then a crossbar is marked in the middle of the curve. Only one curve can emanate from an end of a cross. Thus the crossbar is a cross that is already half used. A curve may not cross itself, cross a curve already drawn, nor may it pass through an existing cross. The first player unable to move loses.

**Card Switching.** *First form.* From a deck of 52 playing cards, remove the spades and diamonds. Lay the spades in a row in increasing order, A, 2, . . . , K. Under this row, make a second row of diamonds, in reverse order, K, Q, . . . , 2, A. A player moves by switching the places of two of the diamonds. The spades never move. Once a diamond lies below a spade of the same value, it moves no more. The game ends when the diamonds are in increasing order, and the player to make the last move wins.
*Second form.* The ace through 9 of spades are arranged in a row, A, 2, . . . , 9. Alice arranges the ace through 9 of diamonds in any desired order under the row of spades. The game then continues as before, with Bob choosing who should make the first interchange.

**Chameleon.** This is played with the same equipment as Hex: a rhombus made of hexagons, and pieces of two colors. Alice tries to make a monochromatic chain linking

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the bottom and top of the board, while Bob tries to link the left and right sides. Players may place pieces of either color, one piece per move. If a move results in paths for both players, then the mover wins.

**Clobber.** Red and black checkers are placed on an $m \times n$ board, with the colors alternating. Alice moves only red checkers, and Bob moves only black checkers. A move consists of moving a checker one unit rectilinearly onto a checker of the opposite color. This latter checker is 'clobbered', and is removed from the board. A player who is unable to clobber loses.

**Cram.** Players alternately place dominoes on an $m \times n$ board. Each domino occupies two adjacent squares (horizontally or vertially). The first player who cannot make a legal move loses.

**Crosscram.** Players alternately place dominoes on an $m \times n$ board, with each domino occupying two adjacent squares, but Alice's dominoes are oriented left to right, while Bob's are up-down. The first player who cannot make a legal move loses.

**Crossed Dots.** Play begins with 15 dots in a row. A move is made by marking a cross centered on an unmarked dot. A player wins when his move creates a string of three or more crosses in a row. Alice's crosses are considered to be indistinguishable from those of Bob, so the three crosses of the winner were not necessarily all made by that player. See also New Crossed Dots.

**Daisy.** The $n$ petals of a daisy are arranged in a circle. Each player in turn picks one petal or two adjacent petals. Petals separated by a gap of removed petals are not considered to be adjacent. The player who takes the last petal wins. This can also be played in misère form.

**Dawson's Kayles.** This game starts with a row of $n$ pins. A move is made by knocking over two adjacent pins. The first player unable to move loses. Thus from $D_n$ (a single row of $n$ pins) one can move (for $n \geq 2$) either to $D_{n-2}$ or (for $n \geq 4$) to $D_k + D_{n-k-2}$ for some $k$, $1 \leq k \leq n-3$. There is no legal move in $D_1$.

**The Division Game.** Two players agree on a positive integer $n$, and construct a list of all the positive divisors of $n$, including 1. A move is made by removing a divisor and all its divisors. The player who who removes $n$ loses.

**Dodgem.** To prepare for play, two white counters are placed in the first column of a $3 \times 3$ board, in the top two rows. Also two black counters are placed in the bottom row, in columns 2 and 3. Alice moves by sliding a white counter one unit, up or down or right, but only into an unoccupied location. When Alice has a white piece in the third column, she may slide it to the right, off the board. Bob moves by sliding a black counter one unit left or right or up, but only into an unoccupied location. When Bob has a piece in the top row, he may slide it up, off the board. The first player to get both pieces off the board wins.

**Dominono.** On an $n \times n$ board, players alternately mark **X**'s and **O**'s, as in tic-tac-toe. A player whose move is adjacent to a prior move (creating a domino of his symbol) loses.

**Dots and Boxes.** This is played on an $m \times n$ array of dots. A move consists of joining two rectilinearly adjacent dots. If the move completes one or more $1 \times 1$ squares, then the player moving enters his initial in those squares, and makes an additional mark. This continues repeatedly, until the player moving makes a mark that does not complete a $1 \times 1$ square. When the board is full, the player whose initial appears in the greater number of squares wins. The game is drawn if the score is tied.

**Dr. Nim.** From a pile of $n$ stones, players alternately remove 1, 2, or 3 stones. The player who takes the last stone wins. This can also be played in misère form.

**Dwarfs and Giant.** Alice has three three dwarfs, which initially are placed in locations 1, 2, and 3 on the board provided. Bob chooses the initial location of the giant, from among locations 5, 6, 7, 8. The dwarfs move first. Dwarfs and the giant move from one node to an adjacent empty node when the nodes are joined by an edge. The dwarfs can only move sideways or forwards, while the giant can move in any direction. Alice wins if the dwarfs trap the giant, and Bob wins if the giant gets behind the dwarfs or if the same position is repeated three times.

**Faux Nim.** To prepare for play, a number of chips are placed on an $m \times n$ board. A move is made by removing one or more chips from a single row. Also, after the chips have been removed, the player moving may also add chips to unoccupied squares on the same row, but only in locations that lie to the right of one of the locations from which a chip was removed. The player who takes the last chip wins.

**Fox and Geese.** This is played on a checker board, using only the dark squares. The geese are represented by four black checkers, which start on the bottom row of the board. The fox is represented by a single red checker, which is initially in the middle of the top row. The geese move as checkers do: always forwards and never into an occupied square. The fox moves forwards and backwards one square at a time, but never into an occupied square. Neither the fox nor the geese can jump over another piece. The object of the geese is to trap the fox so that the fox cannot move. The object of the fox is to get to the bottom of the board. The fox moves first. In one variant, the fox starts in any location of the same color, and the geese move first.

**Graphs.** Play starts with several nodes drawn in the plane. A move consists of connecting two nodes, subject to the following conditions: (i) Two nodes are connected by at most one edge (no double edges); (ii) The curve drawn must not end at the same node that it begins at; (iii) At most two edges may emanate from a node; (iv) The curve drawn must not cross previously drawn curves. The last player to move wins.

**Guiles.** This game is played with heaps of chips. There are two types of moves. Either a heap of size 1 or 2 is removed entirely, or, in a heap of size at least 4, two chips are removed and the remaining chips are divided into two heaps of positive size. There is no legal move on a heap of 3 chips.

**Grundy.** Play begins with several piles of stones of various sizes. A move is made by breaking a pile into two piles of unequal sizes. The first player unable to make a legal move loses.

**Hex.** Played on an $n \times n$ diamond-shaped board made of regular hexagons. Alice places white pieces in unoccupied cells, and Bob black pieces. Each tries to connect their two opposite sides with pieces of their own color. Invented by Piet Hein. Because the first player seems to have such a great advantage, a number of variants have been proposed: (1) The first player is not allowed to move in the short diagonal (assuming that $n$ is odd) on the first move; (2) The game is played on an $(n + 1) \times n$ board, with the first player trying to connect the more distant opposite sides; (3) The first player indicates a cell, and the second player gets to choose whether to allow the first player to move there, or else the second player moves there; (4) On the first move, one stone is placed, and on all subsequent moves, two stones are placed. This would need to be played on a much larger board, say $25 \times 25$.

**Kayles.** Play begins with a row of $n$ sticks. A player moves by removing one or two adjacent sticks. Sticks that are separated by a gap of removed sticks are not considered to be adjacent. The player who removes the last stick wins.

**Lasker's Nim.** This game starts with several heaps of chips. A move is made in one of two ways. Either one or more (possibly all) chips are removed from one heap, or one heap is split into two heaps, without removing any chip.

**Litton.** On an $n \times n$ board, players alternate in placing chips. On any given move, all chips placed must be on the same row or the same column, in unoccupied squares. The player whose move fills the board wins.

**MAL.** (meaning Misère Adjacent Litton) Play begins on an empty $n \times n$ board. A move is made by placing pieces on empty squares of the board. All the pieces placed on a given turn must be in the same column or row, and they must all be adjacent. The player who fills the board loses.

**Mort.** Play begins with nine cards face up on a table; each card is marked with a number, 1, 2, ..., 9. A player moves by claiming one of the cards as his own. A player has won when three cards in his collection sum to 15.

**Mother's Nightmare.** Cook has baked a number of cookies, each in a the shape of a game board. A move is made by eating an entire column or row of a cookie. If this is an interior column or row, then the two remaining portions are considered to be separate cookies.

**Multi Nim.** Suppose that a positive integer $k$ has been agreed upon. Play begins with several piles of sticks. The number of sticks in each pile may vary from pile to pile. A move is made by removing one or more sticks from no more than $k$ piles. The player to take the last stick wins. When $k = 1$, this is Nim.

**New Crossed Dots.** As a game of Crossed Dots is played, the game breaks up into separate components. Some components have crosses at both ends, while there may be one or two components with a cross only at one end. Components of this latter type can be converted to the former type by adding two additional dots and then a cross:

$$\underbrace{\bullet \quad \bullet \quad \cdots \quad \bullet}_{n \text{ dots}} \quad X \qquad = \qquad X \quad \underbrace{\bullet \quad \bullet \quad \cdots \quad \bullet}_{n+2 \text{ dots}} \quad X$$

If a cross is placed next to an existing cross, or with only one dot between, then that player immediately loses. Thus we may say that the game ends when it is impossible to place a cross that has at least two dots between it and its neighboring crosses. Thus in New Crossed dots, we start with a single heap. A move is made by selecting a heap, removing one member from that heap, and separating the remaining members of the heap into two heaps each one of size at least two. New Crossed Dots with a heap of size $n$ is equivalent to the original Crossed Dots with $n - 4$ dots.

**Nim.** The game begins with several piles of stones. A move consists of removing one or more stones, all from the same pile. The player who removes the last stone wins. Can also be played in misère form.

**Nimble.** Play begins with several chips on a $1 \times n$ board. A move is made by sliding a chip any number of units to the left, subject to the condition that it does not land on or cross over another chip. The first player unable to move loses.

**Nim String.** This game is the same as Dots and Boxes, except that initials are not marked, since the winner is the last player to complete a legal move. Note that when the last box is completed, the player moving should make a further mark, and being unable to do so, loses.

**Non-diagonal Gomoku.** On a very large game board, Alice marks an **O** in a square, and then Bob marks an **X** on an empty square. They alternate making their marks, until one of them has made a column or row of 5 of their marks. The player achieving this wins.

**Northcott's Nim.** To prepare for play, one white chip and one black chip is placed in each row of an $m \times n$ board. A move is made by taking a chip of your color and placing it in an unoccupied square in the same row, without jumping the black chip in the row. A player loses when he cannot make a legal move, because all his pieces are pinned against the side of the board.

**Numerical Hex.** On a $6 \times 6$ board, a move is made by writing an integer (positive, negative or zero) in an empty square. No integer may be used more than once. When the board is full, in each row the square that contains the largest number in that row is colored black. Alice wins if she can draw a curve from the top of the board to the bottom that lies entirely in blackened squares, and Bob wins if she cannot. Alice can pass from one square to another if they share a vertex.

**Numerical Tic-Tac-Toe.** Play starts with an empty $3 \times 3$ board. Alice moves by writing one of the numbers $1, 3, 5, 7, 9$ in an empty square, and Bob moves by writing one of the numbers $2, 4, 6, 8$. The same number may not be written twice. A player wins by moving in such a way that three numbers in a row, column, or on a diagonal sum to 15.

**Penny–Dime.** On the board provided, the penny starts at position 10 and the dime at position 0. Players alternate turns, with the penny moving first. A move is made by sliding the coin to an adjacent position along the indicated paths. A penny tries to capture the dime by landing on it. The dime tries to escape capture, and is not allowed to land on the penny. If the penny moves seven times without capturing the dime, then the dime wins.

**Piles.** Play begins with a single pile of stones. A move is made by dividing a pile of stones into two smaller piles. The first player who is unable to do this loses.

**Poison Cookie.** Play begins with an $m \times n$ array of cookies. All the cookies are quite tasty except for the poisonous one in the lower left-hand corner. A player moves by eating (removing) a cookie and also all cookies to the right and/or above the initial cookie. (That is, if the initial cookie is in location $(i_1, j_1)$, then a cookie in location $(i_2, j_2)$ is removed if $i_2 \geq i_1$ and $j_2 \geq j_1$.) The player who removes the poisoned cookie loses.

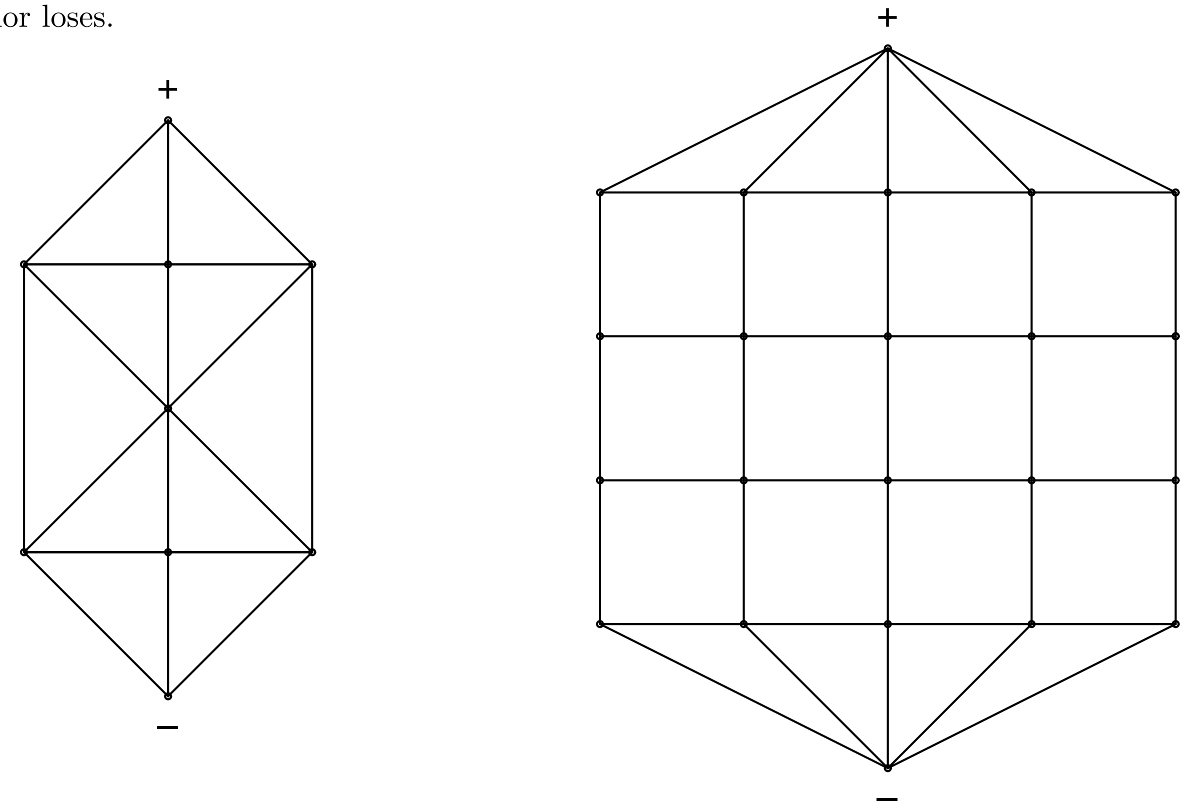
**Poker Nim.** To prepare for play, several piles of stones are formed. Moves are made in one of two ways. Either stones are taken from a single pile, or else some stones previously taken by that player are added to a single pile. A player wins by removing all stones from the board.

**Queen Cornering.** The first player places a chess queen anywhere on the top row or right hand column of an $m \times n$ board. After that, players alternately move the queen any number of squares South, West, or diagonally Southwest. A player wins by moving the queen to the lower left-hand corner of the board.

**Rock in a Hard Place.** This is played on an $m \times n$ board. The rock starts in the lower right-hand corner. A move is made by sliding the rock one square to the left, or one square up, or one square diagonally up and left. The player who moves the rock into the upper left-hand corner loses. The misère form of this is called Soft Rock.

**Rook on a 3D Board.** A rook starts at location $(i_0, j_0, k_0)$ where $i_0$, $j_0$, $k_0$ are nonnegative integers. A move is made by reducing one of the coordinates to a new nonnegative integral value. The player who moves the rook to $(0, 0, 0)$ wins.

**Salt & Pepper.** Play begins with an empty board made of small circles arranged around a large circle, as in Daisy. Alice moves by placing a black chip in an empty circle, and Bob moves similarly with white chips. The first player to place a chip next to one of the same color loses.




(a) (b)

**The Shannon Switching Game.** Play begins with a graph consisting of several nodes and edges joining some of them. One of the nodes is labeled +, and a different node is labeled −. The edges are drawn in pencil, which is to say that they are subject to removal. Mrs. Shortt is trying to make an electrical connection between + and −, while Mr. Cutt is trying to prevent this by cutting potential paths. Mrs. Shortt moves by overwriting a temporary edge in ink, making it permanent. Mr. Cutt moves by erasing a penciled edge. Mrs. Shortt wins by creating an inked path between + and −, while Mr. Cutt wins by separating the + and − nodes into two unconnected components of the graph. In (a) we have an example of the game. The example in (b) is equivalent to $5 \times 6$ Bridgit, so Bridgit is seen to be a special case of this game.

**Silver Coinage.** Players alternately name a positive integer, thought of as a unit of coinage, but an integer named must not be expressible as a nonnegative linear combination of the numbers previously named. The player who names the number 1 loses.

**Slither.** This is played on an $m \times n$ array of dots. Alice begins by joining two rectilinearly adjacent dots. Subsequently, the players move by extending one end or the other of the existing drawn curve by one rectilinear unit. The player whose move causes the curve to intersect itself loses.

**Soft Rock.** This is played on an $m \times n$ board. The rock starts in the lower right-hand corner. A move is made by sliding the rock one square to the left, or one square up, or one square diagonally up and left. The player who moves the rock into the upper left-hand corner wins. The misère form of this is called Rock in a Hard Place.

**The SOS Game.** Play begins with a row of $n$ empty squares. A player moves by writing either S or O in an empty square. A player wins by moving in such a way that the letters S, O, S appear in consecutive squares. If all squares are filled and this pattern does not appear, then the game is drawn.

**Sprouts.** Several spots are marked on a sheet of paper. A move consists of drawing a curve from from spot to another, and marking a new spot in the middle of the curve. Such a curve must not cross itself, and must not cross other curves already drawn, nor may it pass through an existing spot. No spot may have more than three curves emanating from it. The first player unable to make a legal move loses.

**Table Pennies.** Players alternately place a penny on a round or rectangular table. A penny must not protrude beyond the edge of the table. The first player who cannot place a penny loses.

**Trumps.** From a standard deck of 52 cards, the 36 cards of value 6 or higher are selected. The cards are shuffled, and then dealt to Alice and Bob, who lay out their hearts and diamonds in two columns each. The black cards are redistributed so that the values that Bob has in spades are identical to the values that Alice has in hearts, and the values that Bob has in clubs are identical to Alice's values in diamonds. Thus the players start with four columns of cards, face up, and with the same value distribution, albeit in different suits. Hearts are Alice's trump suit, and Spades are Bob's. To start play, Alice places one of her cards face up in the middle, which is an area called the 'discard pile'. Once the discard pile has been started, a move is made in one of two ways: (i) The top card in the discard pile is covered by a card of higher rank, which is to say greater value in the same suit, or one of the player's trump cards. On top of this card, the player who is moving places a second card from his collection. (ii) All cards in the discard pile are picked up and arranged in the player's four columns. In this case, the opponent chooses one of his cards, and places it in the middle to start a new discard pile. Note that when a player is covering the card offered by the opponent, the player can do this with one of his trump cards, even if he would be able to cover it with a card in the same suit offered. The object is to get rid of all your cards.

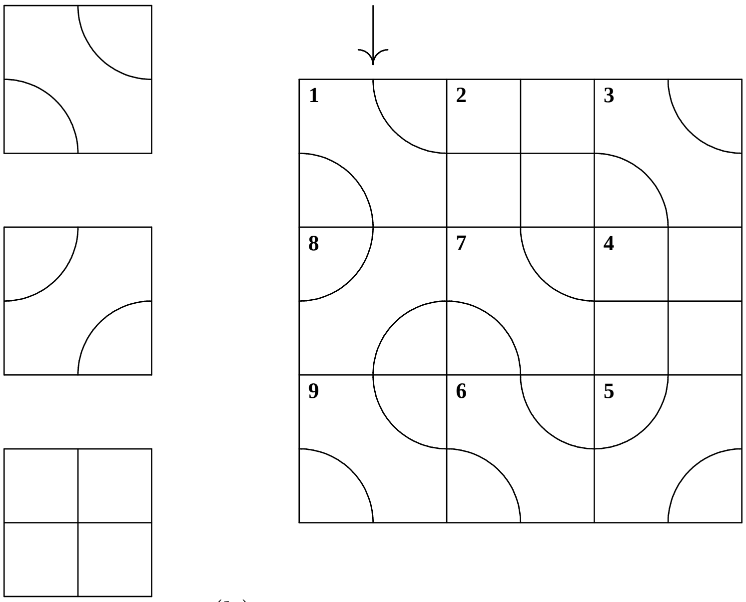
**Turning Turtles.** Play begins with a row of turtles, some upright, others upside down. A move consists of righting an upside down turtle. After this is done, the player moving may optionally turn over a turtle in any of the positions to the left of the turtle just overturned. This second part of the move may turn an upright turtle upside down, or set upright an upside down turtle. The player whose move results in all turtles being upside down wins.

**Two Nim.** Play starts with a single pile of stones. The first player may remove as many stones as he likes, but not the whole pile. After that, each player may remove at most twice as many stones as were removed on the previous move. The player who removes the last stone wins.

**Three–Four.** Starting from 0, players move by adding either 3 or 4 to the existing total. A player wins by bringing the total to 13. If a player brings the total to 14, then the game is a draw, and if a player brings the total to a number ¿ 14, then that player loses.

**Tic-Tac-Toe.** This is played on a $3 \times 3$ board. The first player makes an $X$ in a square. The second player marks an **O** in an unmarked square, and players continue to alternate, making their marks in unmarked squares. A player wins by making three of his marks in a row (up-down, left-right, or diagonally). If neither player has won when the board is full, then the game is drawn. Can also be played misère.

**Wiggle.** To prepare for play, an $m \times n$ rectangle is drawn, divided into $mn$ unit squares. In her first move, Alice draws in the top leftmost square one of the patterns displayed below in (a). This begins a (wiggling) curve that starts at the center of the top of this square. On each successive move, this curve is to be continued. The player whose move takes the curve back to the boundary of the big rectangle loses. Note that the drawing on each unit square depicts pieces of two separate curves. In the case of the piece whose curves cross, making a +, one curve runs from left to right while the other runs from top to bottom. These curves are considered to cross without touching, which is to say you cannot make a turn in the middle of the square to change directions. In (b) a completed game is depicted. The small numbers on the corners of the squares indicate the order in which the squares were played in. Alice lost on her fifth move.




(a) (b)

**Wythoff's Nim.** Play starts with two piles of stones. A move is made either by removing any positive number of stones from one pile, or by removing the same positive number of stones from both piles. The player who removes the last stone wins.

**Y.** This is played on one of the provided boards. As in Hex, players alternate in placing pieces of their own color in empty cells of the board. The player who creates a connected region that links all three edges of the board wins. A corner cell counts as a member of both edges.