Impartial Games

A combinatorial game is impartial if, in any position, the moves available to Alice are exactly the same as for Bob, and the won positions (i.e., those positions in which the game is won) are the same for both players. The player whose move takes the game to a won position is the winner. If a game is not impartial, then it is said to be partisan. Some impartial games, such as Nim, have only one won position, while others, such as Crossed Dots, have many. Given an impartial game, one can adjust the rules slightly, causing no material difference to the game, so that no moves are possible after the game reaches a won state. Thus one may say that the game is won when there are no more legal moves. For example, in Crossed Dots, the game is won when a move creates a string of three or more consecutive crosses. This would normally happen when there still remain uncrossed dots, and thus it might seem that further moves can be made. To make this game to conform to our system, we simply stipulate that no move may be made after the chain of crosses is created. Similarly, the misère form of an impartial game can be made impartial with only small alterations. To see why this is so, suppose we have a particular impartial game in mind, let \( W \) denote the collection of its won positions, and let \( W' \) denote the set of those positions from which the only possible moves are to positions in \( W \). Now define a new game, in which the moves are the same except that a move to a position in \( W \) is forbidden, and the won positions are the positions in \( W' \). This is the misère form of the original game. In Nim, \( W \) consists of the position with all heaps empty, and \( W' \) is the position with only one stick remaining.

A somewhat bizarre question arises at this juncture. We have been in the habit of thinking of the difference between a game and its misère is just to "nip the definition of winning. (Of course, the strategy for winning the misère game may be entirely different from that for winning the original game.) If you "nip it a second time, then you are back to the original game. However, in the above construct, the misère game is a slightly abridged version of the original game, and a second "nip will abridge it further. And yet this should be equivalent to the original game. It would seem that some explanation is called for, at this point.

We can depict an impartial game graphically, by associating a vertex with each position in the game, and drawing an edge from \( v_1 \) to \( v_2 \) indicate that when one is in position \( v_1 \) one can move to \( v_2 \) in a single move. The won positions correspond to vertices from which no edge leads further. Note that this is a directed graph, since we may move from \( v_1 \) to \( v_2 \), but usually not from \( v_2 \) to \( v_1 \). Indeed, if both moves are possible, then we have a problem, because the game might never terminate. We say that a game is finite if for each position of the game, there is a finite upper bound for the number of moves that the game can continue from that position. We also say that a game is bounded if there is a finite upper bound for the length of the game that applies to all positions. For example, Tic Tac Toe is a bounded game, while Dr. Nim is finite, but not bounded. This can be expressed in more quantitative terms, as follows. Let \( P_n \) denote the set of those positions from which
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the game can last as many as $n$ moves, but no more. Thus $P_0$ is the set of won positions, and $P_1$ is the set called $W'$ in our remarks about misère play. A game is finite if every position is in $P_n$ for some $n$, and it is bounded if in addition the sets $P_n$ are empty for all sufficiently large $n$. If the graph of a game contains a directed cycle, then the game can last forever. We do play games with this property, such as Card Switching, Poker Nim, and Northcott’s Nim. In these games, we observe that while the game could last forever, in fact one player is in control, and can force a win. For the present we ignore games of this type, and concentrate on finite impartial games. For a finite impartial game, the game graph is a directed acyclic graph.

In a finite impartial game, we can label the vertices of the game graph $N$ or $P$ to indicate that the position is a win for the next or previous player. This is done sequentially, so that when we examine a vertex, all its successors have already been labeled. The sets $P_n$ are useful for organizing this. The vertices in $P_0$ are all labeled $P$. From a vertex in $P_1$ one can move only to a vertex or vertices in $P_0$, so the vertices in $P_1$ are all labeled $N$. Suppose that the vertices in $P_0 \cup P_1 \cup \cdots \cup P_{n-1}$ have all been labeled, and that $v \in P_n$. There is certainly a move from $v$ to some vertex in $P_{n-1}$, since this begins a chain of $n$ moves leading to the end of the game, but there may also be moves to vertices lying in earlier sets. For example, in Dr. Nim, $P_n = \{n\}$, and while one can move from $n$ to $n-1$, one can also move from $n$ to $n-2$, or from $n$ to $n-3$. In any case, any vertex that one can move to from $v$ is closer to the end of the game than $v$, and hence is in $P_0 \cup P_1 \cup \cdots \cup P_{n-1}$. These vertices have all been labeled, so we know how to label $v$: If all vertices that can be reached in a single move from $v$ are labeled $N$, then we assign the value $P$ to $v$, while if from $v$ one can reach in a single move a vertex labeled $P$, then we give $v$ the value $N$. Once all vertices in $P_n$ have been labeled according to this rule, then we continue with $P_{n+1}$, and so on.

We now consider Nim. For this purpose it is useful to start by defining the Nim sum of two numbers. If $a$ and $b$ are natural numbers, which is to say members of the set

$$\mathbb{N} = \{0, 1, 2, \ldots\},$$

then we denote their Nim sum by $a \mathbin{\ddagger} b$. When the Nim sum is the form of addition being used, we refer to the numbers involved as nimbers. To define the Nim sum, $c = a \mathbin{\ddagger} b$, we write $a$, $b$, and $c$ in binary:

$$a = \alpha_0 + \alpha_1 2 + \alpha_2 2^2 + \cdots,$$
$$b = \beta_0 + \beta_1 2 + \beta_2 2^2 + \cdots,$$
$$c = \gamma_0 + \gamma_1 2 + \gamma_2 2^2 + \cdots.$$

Here each $\alpha_i$, $\beta_i$, and $\gamma_i$ is either 0 or 1. To define $c$ we need to set the values of the $\gamma_i$, and this is done by the formula $\gamma_i \equiv \alpha_i + \beta_i \pmod{2}$. Thus $\gamma_i = 0$ if $\alpha_i = \beta_i$, and otherwise $\gamma_i = 1$. The interesting characteristic here is that while we are adding in binary, there is no carrying. Indeed, in an individual column, the calculation is made as a congruence modulo 2: $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1$, and $1 + 1 = 0$.  

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We now prove two simple results concerning Nim sums, which are useful in analyzing the Nim game.

**Theorem 1.** Suppose that \(a_1, \ldots, a_k\) are natural numbers, and put

\[
 s = a_1 \dagger a_2 \dagger \cdots \dagger a_k.
\]

We alter this sum by changing one of the summands, say \(a_i\) is replaced by \(a'_i\), where \(a'_i \neq a_i\), and we form a new sum,

\[
 s' = a_1 \dagger \cdots \dagger a_{i-1} \dagger a'_i \dagger a_{i+1} \dagger \cdots \dagger a_k.
\]

Then \(s' \neq s\).
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Proof. Since $a'_i \neq a_i$, the binary expansions of $a'_i$ and $a_i$ are different. The binary expansion is a sum of distinct powers of 2. Suppose that $2^j$ is used in the expansion of one of $a_i, a'_i$, but not the other. In our computation of the Nim sum, in the column corresponding to $2^j$, the outcome for $s'$ is the reverse of that for $s$. That is, $2^j$ occurs in the binary expansion of one of $s, s'$, but not the other. Hence $s \neq s'$.

Theorem 2. Suppose that $a_1, \ldots, a_k$ are natural numbers, and let $s$ denote their Nim sum, as in (1). If $s > 0$, then there is an $i$, $1 \leq i \leq k$, and an $a'_i$, $0 \leq a'_i < a_i$, such that $s' = 0$, where $s'$ is defined as in (2).

Proof. Let $2^j$ be the largest power of 2 that occurs in the binary expansion of $s$. There exists an $i$ such that $2^j$ occurs in the binary expansion of $a_i$; indeed, the number of such $i$ is odd. Choose such an $i$, and set $a'_i = a_i + s$. Then

$$s' = a_1 + \cdots + a_{i-1} + a'_i + a_{i+1} + \cdots + a_k$$
$$= a_1 + \cdots + a_{i-1} + (a_i + s) + a_{i+1} + \cdots + a_k$$
$$= s + s = 0.$$ 

It remains to show that $a'_i < a_i$. Since $2^j$ occurs in the binary expansions of both $s$ and $a_i$, it follows that $2^j$ is not in the binary expansion of $a'_i$. There may be higher powers of 2 in the binary expansion of $a_i$, say $2^{j'}$ with $j' > j$, but such powers of 2 will also occur in the binary expansion of $a'_i$, since $2^{j'}$ does not occur in the binary expansion of $s$. When the binary expansions of $a_i$ and of $a'_i$ are compared, from the leading digits downward, we find that all digits are the same, up to the position corresponding to $2^j$, where $a_i$ has a 1 and $a'_i$ has a 0. Thus $a'_i < a_i$.

In Nim, we have heaps of sticks, say of sizes $a_1, \ldots, a_k$. A move is made by choosing a heap, and removing one or more sticks from the heap, up to and including all the sticks. From the above results it is immediate that a position in Nim is a $P$ position if and only if the Nim sum of the heap sizes is 0, since if $s = 0$, then by Theorem 1 any move takes us to a position with $s > 0$, while if $s > 0$, then by Theorem 2 there is a move to a position with $s = 0$.

As far as misère Nim is concerned, we find that the $N$ and $P$ positions are exactly the same as in Nim, except that when all heaps have size 1, the position is $P$ or $N$ according as the number of heaps is odd or even, respectively.

Although our analysis of Nim is now complete, we now establish an interesting strengthening of Theorem 2, which will play an important role in the further development of our theory.

Theorem 3. Let $a_1, \ldots, a_k$ be natural numbers, and let $s$ denote their Nim sum, as in (1), and suppose that $s > 0$. If $0 \leq s' < s$, then there is an $i$, and a number $a'_i$ with $0 \leq a'_i < a_i$ such that the altered sum in (2) has the value $s'$.
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The special case $s' = 0$ is Theorem 2.

Proof. Compare the binary expansions of $s$ and of $s'$, working from the highest terms to the lower. Since $s' < s$, the first difference encountered in the binary digits is place where $s$ has the digit 1 but $s'$ has the corresponding digit 0. Suppose that this place corresponds to $2^j$. Choose an $i$ so that $a_i$ has a 1 in this place. There certainly is such an $i$, because the number of such $i$ is odd. Put $a'_i = a_i + s + s'$. Then

$$a'_i = a_i + \cdots + a_{i-1} + a'_i + a_{i+1} + \cdots + a_k$$

$$= a_1 + \cdots + a_{i-1} * a'_i * a_{i+1} + \cdots + a_k$$

$$= a_1 + \cdots + a_{i-1} * (a_i + s + s') + a_{i+1} + \cdots + a_k$$

$$= a_1 + \cdots + a_k + s + s'$$

$$= s + s + s' = s'.$$

It remains to show that $a'_i < a_i$. Consider a position corresponding to $2^j'$ with $j' > j$. The numbers $s$ and $s'$ have the same binary digit in this place, and hence $a'_i$ and $a_i$ have the same binary digit in this place. However, in the place corresponding to $2^j$, the numbers $s$ and $a_i$ both have $2^j$ in their expansions, while $s'$ does not, so $a'_i$ does not have a $2^j$ in its expansion. Thus when we compare the binary expansions of $a_i$ and of $a'_i$, starting with the largest terms and proceeding to smaller ones, the first difference encountered is a $2^j$ in $a_i$ that is missing in $a'_i$. Thus $a'_i < a_i$.

Five Nimber Exercises

1. Suppose that $a$ is a natural number. Explain why it is always the case that $a \nplus a = 0$.

2. Show that if $a$ and $b$ are less than $2^k$, then so also is $a \nplus b$.

3. Show that if $a < 2^k$, then $a \nplus 2^k = a + 2^k$.

4. Let $a = \sum_{j \in A} 2^j$ and $b = \sum_{k \in B} 2^k$ be the binary expansions of $a$ and $b$. Show that if $A$ and $B$ are disjoint (i.e., $A \cap B = \emptyset$), then $a \nplus b = a + b$. In particular, the Nim sum of the numbers $2^j$ for $j \in A$ is equal to their ordinary sum, namely $a$.

5. Show that $a \nplus b \leq a + b$, and that the difference $(a + b) - (a \nplus b)$ is even.

In order to extend our analysis from Nim to other impartial games, we introduce a

Definition. Let $S$ be a set of natural numbers, which is to say that $S \subseteq \mathbb{N} = \{0, 1, 2, \ldots\}$. The minimal excluded number of $S$, written $\text{mex}(S)$, is the least natural number not in $S$.

For example, $\text{mex}\{0, 1, 7, 10\} = 2$, $\text{mex}\{3, 7, 11\} = 0$ and $\text{mex}\{0, 1, 2, 3, 5, 7, 9\} = 4$. Suppose that $(a_1, a_2, \ldots, a_k)$ is a position in Nim, with value $s$ as given by (1). Consider

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the mex of the values \( s' \) that can be reached in a single move. Here \( s' \) is defined by (2). If \( s = 0 \), then by Theorem 1 all values \( s' \) are non-zero, so their mex is 0. If \( s > 0 \), then by Theorem 3, the numbers 0, 1, \ldots, \( s-1 \) all occur as \( s' \) for some move. However, by Theorem 1 again the number \( s \) does not occur as an \( s' \). Thus the mex of the values \( s' \) is \( s \) in this case also. Thus we have a

**Corollary.** Let \((a_1, a_2, \ldots, a_k)\) be a position in Nim with value \( s \) as defined by (1). The mex of the values \( s' \) that can be reached in a single move is \( s \).

Once we realize that values of positions in Nim have this mex property, it takes only a moment of thought to discover that similar values could be computed for positions in any finite impartial game. Recall that \( \mathcal{P}_0 \) is the set of those positions in which the game has been won. Since no move is possible, we are taking the mex of the empty set, and the least value not seen is 0. Hence \( g(v) = 0 \) for all \( v \in \mathcal{P}_0 \). The set \( \mathcal{P}_1 \) is the collection of those positions from which the only possible move is to a position in \( \mathcal{P}_0 \). Since the only value seen is 0, \( g(v) = 1 \) for all \( v \in \mathcal{P}_1 \). The set \( \mathcal{P}_2 \) is the set of positions from which the game can last as many as 2 moves, but not more. Thus if \( v \in \mathcal{P}_2 \), there will be at least one move from \( v \) to a position in \( \mathcal{P}_1 \), but there may also be a move to a position in \( \mathcal{P}_0 \). If there is no possible move from \( v \) to a position in \( \mathcal{P}_0 \), then \( g(v) = 0 \), while if there is such a move, then \( g(v) = 2 \). If \( v \in \mathcal{P}_3 \), then there is at least one move from \( v \) to a position \( v' \in \mathcal{P}_2 \), but there may also be moves to positions in \( \mathcal{P}_1 \) and/or \( \mathcal{P}_0 \). Since the \( g \)-function has already been computed for all positions in \( \mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_2 \), we can define \( g(v) \) to be the mex of the values \( g(v') \) for all positions \( v' \) that can be reached in a single move from \( v \).

Once this has been done for all positions in \( \mathcal{P}_3 \), we continue with positions in \( \mathcal{P}_4 \), and so on. If \( g(v) = 0 \), then 0 is not seen among the numbers \( g(v') \), so \( g(v') > 0 \) at the next stage, regardless of the move selected. On the other hand, if \( g(v) > 0 \), then there are positions \( v' \) such that \( g(v') \) takes the values 0, 1, \ldots, \( g(v) - 1 \). In particular, there is a \( v' \) such that \( g(v') = 0 \). By these observations we deduce that a position \( v \) is a \( P \) position if \( g(v) = 0 \), and that \( v \) is an \( N \) position if \( g(v) > 0 \). We compute the \( g \)-values inductively up the game graph, in much the same way that we compute \( N \) and \( P \) values, and we know that the \( N \) and \( P \) information is contained in the \( g \)-values. We shall see shortly that the \( g \)-values contain invaluable additional information. Indeed, we shall learn to think of a position \( v \) as a ‘bogus Nim heap’ of size \( g(v) \).

The \( g \)-function is easily computed in Dr. Nim, and its generalizations. Consider the original form of Dr. Nim, in which we can take 1, 2, or 3 stones. Then \( g(0) = 0 \), \( g(1) = 1 \), \( g(2) = 2 \), \( g(3) = 3 \), \( g(4) = 0 \), \( g(5) = 1 \), \( g(6) = 2 \), \( g(7) = 3 \), and \( g(8) = 0 \). The calculation of \( g(9) \) is the same as the calculation of \( g(5) \), and so on, so the values are periodic with period 4. In generalized Dr. Nim with \( S = \{2, 3\} \), we compute that \( g(0) = 0 \), \( g(1) = 0 \), \( g(2) = 1 \), \( g(3) = 1 \), \( g(4) = 2 \), \( g(5) = 0 \), \( g(6) = 0 \), and \( g(7) = 1 \). The computation of \( g(8) \) is the same as the computation of \( g(3) \), so we deduce that the table of \( g \)-values has period 5.

We now turn our attention to sums of games. If \( G_1 \) and \( G_2 \) are two games, then \( G_1 + G_2 \) is defined as follows. A position in \( G_1 + G_2 \) is a pair \((v_1, v_2)\) of positions, where \( v_1 \) is a position in \( G_1 \) and \( v_2 \) is a position in \( G_2 \). A move in \( G_1 + G_2 \) is to make a move in \( G_1 \) or \( G_2 \), but not both. The won positions in \( G_1 + G_2 \) are those positions \((v_1, v_2)\) in which
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$v_1$ is a won position in $G_1$ and $v_2$ is a won position in $G_2$. For example, if $G_1$ is a game of Nim with a single pile, and $G_2$ is a game of Nim with a single pile, then $G_1 + G_2$ is a game of Nim with two piles. If $v_1$ is a $P$ position in $G_1$, and $v_2$ is a $P$ position in $G_2$, then the position $(v_1, v_2)$ in $G_1 + G_2$ is $P$ position, for when the next player moves, the state will become $(N, P)$ or $(P, N)$, and the previous player need only move in the same component to restore the state to $(P, P)$. Similarly, the combination $(N, P)$ is an $N$ position in $G_1 + G_2$, since the next player can put the game in the state $(P, P)$. What is not so clear is the combination $(N, N)$. If $v_1$ is a single Nim heap of size 2, and $v_2$ is a single Nim heap of size 3, then $(v_1, v_2)$ is the position $(2, 3)$ in a game of Nim with two piles. All three of these positions are of type $N$. On the other hand, if $v_1$ is a single Nim heap of size 2, and $v_2$ is a single Nim heap of size 2, and these are $N$ positions, but the combination $(2, 2)$ in $G_1 + G_2$ is of type $P$. Thus we see that knowing the $N$ or $P$ state of two games is not always sufficient to determine the state of their sum. However, if $G_1$ and $G_2$ are finite impartial games and we know $g(v_1)$ and $g(v_2)$, then we know where we stand: $(v_1, v_2)$ is a $P$ position if and only if $g(v_1) = g(v_2)$. More generally, if $G_1, G_2, \ldots, G_k$ are finite impartial games, then a position $(v_1, v_2, \ldots, v_k)$ in $G_1 + G_2 + \cdots + G_k$ is a $P$ position if and only if

$$g(v_1) \oplus g(v_2) \oplus \cdots \oplus g(v_k) = 0.$$

To see why this is so, note that when the next player moves, say in component $i$, the summand $g(v_i)$ is replaced by $g(v'_i)$. Moreover, $g(v'_i) \neq g(v_i)$, since $g(v_i)$ is the mex of all possible $g(v'_i)$. Since $g(v'_i) \neq g(v_i)$, it follows by Theorem 1 that the Nim sum of the new set of $g$-values is not the same as it was before. It had been 0, but now it has some positive value, say $s$. By Theorem 2 we know that there is a suuond, say $g(v_j)$, which if it is reduced by the right amount, will cause the Nim sum to return to the value 0. Moreover, among the positions $v'_j$ that can be reached from $v_j$ there are positions for which the $g$-values are $0, 1, \ldots, g(v_j) - 1$. Hence there is a choice of $v'_j$ such that $g(v'_j)$ has the desired value, and by making that move the previous player can return the combined game to a state in which the Nim sum of the $g$-values is again 0. Note that we are using Theorems 1 and 2 in the same way that we used them to play Nim. Indeed, we play $G_1 + G_2 + \cdots + G_k$ as if we were playing Nim with the ‘bogus Nim heaps’ of sizes $g(v_1), g(v_2), \ldots, g(v_k)$.

To play a game successfully, it suffices to know its $P$ positions, or at least a sufficiently rich subset of the $P$ positions. We have achieved this for $G_1 + G_2 + \cdots + G_k$, but we can take this a step further: We can not only compute the $N$ and $P$ values in the sum of several impartial games (with known $g$-functions), we can also compute the $g$-values for the sum. In the special case in which the sum is Nim with $k$ heaps, this was essentially done in the Corollary to Theorem 3, which asserts essentially that the $g$-value of a position $(a_1, a_2, \ldots, a_k)$ is the Nim sum of the heap sizes. Since we are already thinking of the sum of impartial games as consisting of a game of Nim with ‘bogus Nim heaps’ of sizes $g(v_i)$, the next result should come as no surprise.

**Theorem 4.** Suppose that $G_1, \ldots, G_k$ are finite impartial games, and that $(v_1, \ldots, v_k)$ is a position in $G_1 + \cdots + G_k$. Then

$$g(v_1, v_2, \ldots, v_k) = g(v_1) \oplus g(v_2) \oplus \cdots \oplus g(v_k).$$

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Proof. Let $G = G_1 + \cdots + G_k$, and let $P_n$ denote the set of positions in $G$ from which the game could last $n$ moves, but not longer. If $(v_1, \ldots, v_k) \in P_0$, then $v_i$ is a won position in $G_i$, so that $g(v_i) = 0$ for all $i$. Since $g(v_1, \ldots, v_k) = 0$, we see that (3) holds when for positions $(v_1, \ldots, v_k) \in P_0$. Suppose that (3) has been proved for positions in $P_0 \cup P_1 \cup \cdots \cup P_{n-1}$, and that $v = (v_1, \ldots, v_k) \in P_n$. When a move is made from position $v$, we obtain a position $v' = (v_1, \ldots, v_{i-1}, v'_i, v_{i+1}, \ldots, v_k)$. Now $v'$ is closer to the end of the game than $v$, so that $v' \in P_0 \cup P_1 \cup \cdots \cup P_{n-1}$. Hence (3) holds for $v'$, which is to say that

$$g(v') = g(v_1) \oplus \cdots \oplus g(v_{i-1}) \oplus g(v'_i) \oplus g(v_{i+1}) \oplus \cdots \oplus g(v_k).$$

The mex of numbers of this form is $g(v)$. Let $s = g(v_1) \oplus \cdots \oplus g(v_k)$. We shall show that sums of the form on the right above are never equal to $s$, but that all values smaller than $s$ can be expressed as such a sum, so that $s$ is the mex of these values, and hence (3) holds for $v$. Since $g(v_i)$ is the mex of numbers of the form $g(v'_i)$, it follows that $g(v_i) \neq g(v'_i)$. Hence by Theorem 1 it follows that the sum on the right above is not equal to $s$. If $s = 0$, then there is nothing further to be proved. Suppose that $0 \leq s' < s$. We apply Theorem 3 with $a_i = g(v_i)$. Thus there is an $i$, and an $a'_i$, $0 \leq a'_i < a_i$, such that the altered sum has the value $s'$. Since $g(v_i)$ is the mex of numbers of the form $g(v'_i)$, it follows that there is a choice of $v'_i$ such that $g(v'_i) = a'_i$. For this choice of $i$ and $v'_i$, the sum on the right above has the value $s'$. Since this can be done for every $s' < s$, it follows that $s$ is the mex of these numbers, and the proof is complete.

Let us summarize. We have shown that any position $v$ in a finite impartial game can be treated as if it were a single Nim heap. This theory was discovered by Roland Percival Sprague in 1935, and was rediscovered by Patrick Michael Grundy in 1939, and is known as the Sprague–Grundy theory. We continue with a few applications of this theory.

In Grundy’s Game, we start with a pile of $n$ stones. A move is made by breaking a pile of stones into two smaller piles of unequal sizes. The game is won when all piles are of size 1 or 2. Let $g(n)$ denote the Grundy value of a single pile of $n$ stones. Clearly $g(1) = g(2) = 0$. When $n = 3$, the only move is to $(2, 1)$. Since $g(2) \oplus g(1) = 0 \oplus 0 = 0$, it follows that $g(3) = \text{mex} \{0\} = 1$. From $n = 4$, the only move is to $(3, 1)$. Since $g(3) \oplus g(1) = 1 \oplus 0 = 1$, it follows that $g(4) = \text{mex} \{1\} = 0$. From a pile of 5 stones we can move to $(4, 1)$ and to $(3, 2)$. We note that $g(4) \oplus g(1) = 0 \oplus 0 = 0$ and that $g(3) \oplus g(2) = 1 \oplus 0 = 1$. Hence $g(5) = \text{mex} \{0, 1\} = 2$. The first few values are as follows:

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<td>4</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

The program Grundy allows you to view the first 10,000 Grundy values. It is conjectured that the values are eventually periodic. A tendency toward period 3 may be noted in the values.

Game Theory Results
Impartial Games

Kayles begins with a row of bowling pins. A move is made by knocking over a single pin or two adjacent pins. A player wins by knocking over the last pin. Let $K_n$ denote the initial position with $n$ pins. Clearly $g(0) = 0$. From $K_1$ we can reach $K_0$, so $g(1) = 1$. From $K_2$ we can reach $K_1$ and $K_0$, so $g(2) = 2$. From $K_3$ we can reach $K_2$, $K_1$ and $K_1 + K_1$, so $g(3) = 3$. In general, from $K_n$ we can reach $K_{n-1}$, $K_{n-2}$, and positions of the form $K_a + K_b$ where $a$ and $b$ are positive integers such that $a + b = n - 1$ or $n - 2$.

Table 3. Grundy values for Kayles

| $n$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|-----|
| $g(n)$ | 0  | 1  | 2  | 3  | 1  | 4  | 3  | 2  | 1  | 4  | 2  | 6  | 4  | 1  | 2  | 7  | 1  | 4  | 3  | 2  | 1  |     |

The program Kayles provides the first 10,000 values of $g(n)$.

Kayles’ Questions

1. In Kayles, $g(n) > 0$ for all $n > 0$. That is, Alice can always win. Why? That is, why is there always a 0 in the set being mexted?
2. Consider a game of Daisy starting with $n$ petals. After Alice’s first move, the game becomes Kayles—specifically, $K_{n-1}$ or $K_{n-2}$. Use this observation, and the Grundy values for Kayles, to calculate the first few Grundy values for Daisy.
3. The Grundy values for Kayles is eventually periodic. What is the period, and at what point does the periodic behavior begin. Can you prove that the periodic behavior observed will continue indefinitely? How far must you compute values before you can be certain that it will last forever?
4. A position in Kayles is depicted below. It is your turn to move. What do you do?

|     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |