Jan, 2013

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May 2, 2015

Morning

1. (a) Note that a union of connected subspaces with a point in common is connected. Denote two distinct points in $X \times Y$ as $(x, y), (w, z)$. One can observe that those two are in the same component by observing the union of $\{x\} \times Y \cup X \times \{y\}$ with $\{w\} \times Y \cup X \times \{z\}$ all have points in common $((x, y), (w, z), (w, y), (x, z))$. This can be used, together with induction, to prove that any cartesian product like the above is connected.

   (b) Any two points in $\mathbb{R}^\infty$ are in the same connected subspace homeomorphic to $\mathbb{R}^N$ (connected since it is a finite cartesian product of $\mathbb{R}$).

2. If M is a topological n manifold then $H_*(M \setminus \{x\}, M) = H_*(U \setminus \{x\}, U)$ by excision with the set $M \setminus U$ where U is a nbhd of x homeomorphic to $\mathbb{R}^n$. Therefore we get $H_*(M \setminus \{x\}, M) = H_*(\mathbb{R}^n \setminus \{x\}, \mathbb{R}^n)$ which is $H_*(S^n)$ for $n > 1$ by expending to long exact sequence of a pair and using the contractibility of $\mathbb{R}^n$ and that $\mathbb{R}^n \setminus \{x\}$ is a def.retract of S^{n-1}. This gives us a necessary condition for being an n-manifold for $n > 1$.

   Notice that the contraction of $\mathbb{R}^n$ to $\{x\}$, namely, $F(x, t) = xt$, factor through the relation $\sim$. So we conclude that $\mathbb{R}^n/\sim$ is contractible as well. Moreover, the def.retract $\mathbb{R}^n \setminus \{0\}$ to S^{n-1} also factor through $\sim$ which leads us to conclude that $S^{n-1}/\sim$ is a def.retract of $\mathbb{R}^n/\sim \setminus\{0\}$.

   Notice that $S^{n-1}/\sim$ homeomorphic to $\mathbb{R}P^{n-1}$ and thus for $n \geq 3$ we get $\mathbb{Z}/(2)$ as part of the homology complex. By contractibility, $H_i(\mathbb{R}^n/\sim \setminus\{0\}, \mathbb{R}^n/\sim)$ is $\mathbb{Z}/(2)$ for at least one choice of i. However, in order for $\mathbb{R}^n/\sim$ to be a topological manifold, the creterion above prohibits from $\mathbb{Z}/(2)$ to be a part of that homology sequence. Therefore, for $n \geq 3$, $\mathbb{R}^n/\sim$ is not a topological manifold.

   For $n = 1$ we get a space which is homeomorphic to $[0, \infty)$ and a nbhd of 0 is not homeomorphic to an open subset of any $\mathbb{R}^m$.

   Left to check for $n = 2$. $\mathbb{R}^n/\sim$ is homeomorphic to the upper half-plane with the non-negative x-axis $\{(x, y) : y \geq 0\} \setminus (-\infty, 0) \times \{0\}$, which is homeomorphic to the entire plane by $z \mapsto z^2$. So the answer is YES only for $n = 2$.

3. Consider the map $f : \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto x^2 + 2x + 3y^2 - 6y + z^2 + 4z = (x + 1)^2 + 3(y - 1)^2 + (z+2)^2 - 8.$ which is a smooth map. By the preimage theorem, for every $c \in \mathbb{R}$ which is a regular value of f, we get a 2-manifold. Notice that that manifold will be the empty set for $c < -8$. The differential of f in an arbitrary point is $(2(x + 1), 6(y - 1), 2(z + 2))$, and therefore there is only one critical point, $(-1, 1, -2)$ that correspond to a single critical value $c = -8$. Therefore, for $c > -8$ we get a non empty 2-manifold, for $c < -8$ we get the empty set and for $c = -8$ it is clear from f that the pre-image is a single point, which is not a 2-manifold.

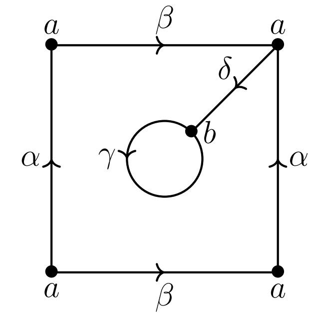
4. The space is path connected therefore we can calculate π_1 without keeping track of the base point.

   The space is clearly homotpy-equivalent to $T \vee S^1 \vee T$, where each gluing point has a contractible nbhd. Therefore $\pi_1 \cong \mathbb{Z}^2 * \mathbb{Z} * \mathbb{Z}^2$.

5. Enough to show that $f : X \to f(X)$ is an open map. Let U be open in X and assume, for the sake of contradiction, that $f(U)$ is not open in $f(X)$, meaning, consist of a boundary point $a \in f(U)$ and a sequence $\{a_i\}$ that converges to a but $a_i \notin f(U)$. Such sequence can be derived, for example, by observing a nbhds of radius $1/n$. Note that $\{a_i\}$ must contain infinitely many points by construction. Observer the set of pre-images under (the bijection) f, and denote them as $b_i = f^{-1}(a_i)$ and $b = f^{-1}(a)$. Note that $b \in U$, $b_i \notin U$. By properness, the set $\{b\} \cup \{b_i\}$ is compact ,and therefore, limit point compact. We conclude that the (infinite) set has a limit point $z \in X$. By continuity of f, and the fact that $a_i \to a$, we conclude that $z = b$. However, b is in U an therefore must have a nbhd in which no b_i are in. This is a contradiction since b is a limit point of the b_i's. Therefore, $f(U)$ must be open in $f(X)$.

Afternoon

1. Notice that the space is a Torus minus a disk, where on the boundary we attach S^1 (to which the Mobius band is homotopy-equivalent). The attaching map sends the the generator of the Boundary of the missing disk to twice the generator of the attached S^1 since the boundary of the Mobius band wraps around its center loop twice. Endow the following CW-structure of the entire space. Let a, b be two points. Let α, β be loops via a, γ - loop via b δ path from a to b. Attach a single 2-cell η with the following attaching map: $\alpha, \beta, \delta, \gamma\gamma, -\delta, -\alpha, -\beta$. We get the following structure:



Now we can get the CW-Complex:

$$0 \longrightarrow \mathbb{Z}\eta \xrightarrow{\eta \mapsto 2\gamma} \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \oplus \mathbb{Z}\gamma \oplus \mathbb{Z}\delta \oplus \xrightarrow{\delta \mapsto b-a, else-0} \mathbb{Z}a \oplus \mathbb{Z}b \longrightarrow 0$$

Now we easily see that $H_2(X) = 0$ due to infectivity in the left map, and that $H_1(X) = \mathbb{Z}^2 \times \mathbb{Z}/(2)$ by direct calculation. $H_0 = \mathbb{Z}$ by the above sequence or by observing the X is path connected.

2. In the solutions for May 2014, afternoon, question 4 , we proved the following **Lemma**: Let $f : X \to Y$ be a surjective local homeomorphism, X, Y Hausdorff, X compact, Y locally compact. Then f is a covering map.

   So in order to prove the statement in question, suffices to prove surjectivity. If so, than Y is compact as continuous image of a compact space. There for Y is locally compact as well and we can apply the lemma.

   The map f is a open map by the following argument. For each $x \in X$ pick a nbhd U_x in X s.t. $f(U_x)$ is open in Y. So $f(X) = f(\cup U_x) = \cap f(U_x)$ is open in Y. Moreover, $f(X)$ is compact in Y. Being a Hausdorff space, $f(X)$ is closed as well. Since Y is connected, $f(X)$ must by Y.

3. (a) S^3 is simply connected, so there's only one covering map which is the trivial one. Also, see argumentation for (d).

   (b) Since covering maps commute with cross product use $[z \mapsto z^2] \times id_{S^1}$ to observe a non trivial covering.

   (c) use $[z \mapsto z^2] \times id_{S^2}$, similar to the above.

   (d) By the "Riemann-Hurewitz" formula, for a k-sheeted covering map, the Euler characteristic of the covering space is k times the Euler characteristic of the base space. That is, for X, a finite CW-complex, if X is a non trivial cover of itself, $\chi(X) = 0$. For $T\#T$ we have a hexagon structure with one 0-cell, 4 1-cells, and one 2-cell, making χ to be -2. Therefore such covering must be trivial.

4. Note that for each $x \in M$ exists nbhd U_x in M homeomorphic to $\mathbb{R}^n$, and therefore homeomorphic to the unit ball in $\mathbb{R}^n$.

   Case 1: M is a unit ball in $\mathbb{R}^n$. Then there is a homeomorphism taking any point to any point, while fixing the boundary. (deforming the radius, then rotating element in radius r

by $\theta(R-r)$).

Case 2: $y \in U_x$. Then use case 1 on U_x and attach back to $id : M \setminus U_x \to M \setminus U_x$. This is possible since the map in case 1 is the identity on the boundary. So we can use the pasting lemma to construct the homeomorphism.

Case 2.5: U_x and U_y have non trivial intersection, $z \in U_x \cap U_y$. Use case 2 to shift x to z and then z to y.

Case 3: x, y can be connected with a path. The path $\lambda : [0,1] \to M$ connects x and y is compact in M. For each $a \in \lambda$ take U_a and construct an open cover, for which there is a finite sub-cover. Join U_x, U_y to that sub cover and denote m as the minimal number of open sets in such sub-cover generated by this process (exists by partial order of inclusion and bounded below by 1). Now prove by induction on m, where the base case, $m = 1$ is proved in Case 2. For general m, observe the intersection of U_x with the rest of the open sets. The intersection cannot be empty since λ is connected. For some b in $U_a \cap U_x$, use the above argument to create homeomorphism that shifts x to b. Then we reduced the case to (at leasts) $m-1$ since U_x is no longer needed.

Case 4: x, y not on the same path-component. For each a in the path component of x , take U_a and observe $\cup U_a$. Do the same for each b in the path component of y. Since M is connected we cannot have all the path componnets of M to be open. In the case where $\cup U_a$ and $\cup U_b$ have intersection, we have a' and b' that correspond to case 2.5. So use case 3 to shift x to a' and y to b'. So for every path components with intersection the statement is true. We are left with path components without intersection, which are open in that case and thus correspond to one component with is M.

5. (a) The reverse direction is easy since X is homeomorphic to $X \times \{1\}$ - a subsapce in a Hausdorff CX. The forward direction requires a few cases: we wish to separate (x,t) from (y,s). If $t \neq s \neq 0$ then take $X \times (t-\epsilon, t+\epsilon)$ and $X \times (s-\epsilon, s+\epsilon)$ or close one edge if s or t is 1 or 0. (choose epsilon appropriately, i.e. $\epsilon = |t-s|/2$). In the case that $t = s$ it must not be equal to 0 because otherwise this is the same point in CX. For this case, use the appropriate nbhds of x, y in X to create the separation.

   (b) Take $X = [0,1] \coprod [2,3]$ which is not connected, however CX is path connected since every point exhibit a path to $X \times \{0\}$.