Sep, 2012

by Gilad Pagi, adapted from solutions by Dondi Ellis

April 28, 2015

Morning

1. (a) For any two points $x, y \in X$ there are open sets U, V in X separating x, y. So the induced topology on $\{x, y\}$ is discrete by observing the induced open sets by $X, U, V, \emptyset$. For a single point the claim is obvious.

   (b) The finite complement topology on $\mathbb{R}$ gives a counter example. Any intersection of open sets in not empty.

   (c) Assume for the sake of contradiction that any nbhd of $x \in X$ contains a point other than x. Let $x_n \neq x$ be a point in $U_n = \{y | \, d(x, y) < 1/n\}$. Note that every nbhd of x contains infinitely many points of $\{x_n\}_{\mathbb{Z}}$. However, the topology of $\{x_n\}_{\mathbb{Z}} \cup \{x\}$ must be discrete, and in particular there is a nbhd of x in X that does not contain any of the x_n's. Contradiction.

2. (This solution is essentially different from Dondi's, but approved by him):

   In our solution for Jan 2013, morning, Q2, we gave a criterion for every topological manifold M of dimension > 1, which is that the relative homology $H_*(M, M \setminus \{x\})$ is composed of $\{0\}$ and $\mathbb{Z}$. Suppose M is contractible. So by the LES of of the homology of the pair we get $H_1(M \setminus \{x\}) \cong H_2(M, M \setminus \{x\})$. We conclude that $H_1(M \setminus \{x\})$ is either $\{0\}$ or $\mathbb{Z}$, and if further $M \setminus \{x\}$ is path connected, then $\pi_1^{ab}(M \setminus \{x\})$ is $\{0\}$ or $\mathbb{Z}$ by Hurewicz theorem.

   Suppose that X is a manifold. Observe that it is contractible to (0,0). Moreover, observe that $X \setminus \{(0, 0)\}$ is path connected. So we get that $\pi_1^{ab}(X \setminus \{(0, 0)\})$ is $\{0\}$ or $\mathbb{Z}$. Our goal is to prove that $|\pi_1(X \setminus \{(0, 0)\})| = n$ which contradicts the supposition.

   Notice that $\mathbb{C}^2 \setminus \{0, 0, \}$ is a covering space of $X \setminus \{(0, 0)\}$ with n preimages per every point. Also notice that $\mathbb{C}^2 \setminus \{0, 0, \}$ is simply connected, which makes it the universal covering space of $X \setminus \{(0, 0)\}$. So we can conclude that $|\pi_1(X \setminus \{(0, 0)\})| = n$.

3. Take a compact disk D in $\mathbb{R}^2$. For X, remove the center point, for Y remove a boundary point. We get that for both, the 1-pt-compactification is D but Y is simply connected while X not.

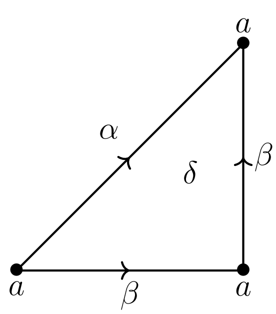
4. For a PID (like $\mathbb{Z}$) every submodule of a free module is free. So the statement is true for $k = n$ since $H_n = ker(d_n) < K_n$. For $k = 0$, $H_0 = \mathbb{Z}^r$ where r is the number of path connected component (true for every locally path connected space). However, for the rest of the H_i's we can take $\mathbb{R}P^n$ to get $\mathbb{Z}/(2)$ at H_i for $0 < i < n$ odd, and the suspension of $\mathbb{R}P^n$ will give us $\mathbb{Z}/(2)$ at H_i for $0 < i < n$ even, since $\widetilde{H}_n(X) = \widetilde{H}_{n+1}(SX)$.

5. (a) Let A be closed in $A \subset X \times Y$. WTS that $p(A)$ is closed in X. Suppose not, then $p(A)^c$ is not open in X and hence there is a point $a \in X, a \notin p(A)$ which is a boundary point for $p(A)$, that is, every nbhd U of a in X meets $p(A)$. That is, $U \times Y$ meets A in $X \times Y$. However, since A is closed there, it contains all of its boundary points and therefor for every $y \in Y$, there is a nbhd $U_{a,y} \times V_y$ for (a, y) in $X \times Y$ s.t. $U_{a,y}, V_y$ open in X, Y respectively and $U_{a,y} \times V_y$ does not meet A. Now $\{V_y\}$ is an open cover, so exists a finite sub cover $\{V_{y_i}\}$. So define $U = \cap U_{a,y_i}$, a nbhd of a in X . So $\cup U \times V_{y_i}$ contains $U \times Y$ but do not meet A. contradiction.

   (b) Take $\mathbb{R} \times \mathbb{R}$ and the regions bounded by the graph of $1/x$, namely $A = \{(x, y) | \, x \geq 0, y \geq 1/x\}$. This is closed by construction, but $p(A) = (0, \infty)$ which is not closed.

Afternoon

1. By taking the usual CW-structure of the torus and applying the quotient relation we get the following:



   Then we get the following complex,

$$0 \longrightarrow \mathbb{Z}\delta \xrightarrow{\alpha+2\beta} \mathbb{Z}\alpha + \mathbb{Z}\beta \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

   which gives us $\mathbb{Z}$ at H_1, H_0 and 0 elsewhere. Note that the cokernel of the matrix (1,2) is $\mathbb{Z}$.

2. (a) We need to check when $(u, v) : \mathbb{R}^2 \to \mathbb{R}^2$ is a local diffeomorphism. Take the differential and compute the determinant at (a, b), we get $Det = u_x v_y - u_y v_x = (2a)(3a + 2b) - (-3b^2)(3b)$. The above smooth function is a local-diffeo whenever $Det \neq 0$.

   (b) We use the preimage theorem, which guarantees that the preimage $f^{-1}(0)$ of $f(x, y) = y^2 - x(x - 1)(x - c)$ is a smooth manifold whenever the value 0 is regular. The differential is $(-3x^2+2x+c(2x-1), 2y)$. When $y \neq 0$, the differential in onto. left to check the points where the preimage contains a point with 0 at the y coordinate. In this case, x is either $0, 1, c$. For $x = 0$ we get $c \neq 0$. For $x = 1$ we get $c \neq 1$. For $x = c$ we get $c - c^2 \neq 0$, i.e. again $c \neq 0, 1$. So for those numbers, the locus is a smooth manifold. For $c = 0, 1$ we need to check specifically.

For $c = 0$ we get $y^2 = x^2(x-1) \Rightarrow y = \pm x\sqrt{x-1}$ which consist of a point $(0, 0)$ and a curve, so cannot be a smooth manifold since the dimension does not agree.

For $c = 1$ we get $y^2 = x(x-1)^2 \Rightarrow y = \pm(x-1)\sqrt{x}$, which consist of two curves intersecting at (1,0) so it cannot be a smooth manifold.

3. (Comment: I assume that the condition $X = U \cup V$ was omitted in the question. Otherwise, X can have loops that cannot described by U, V, W_1, W_2).

   Open subset in $\mathbb{R}^n$ are locally path connected, there for connected implies path connected. Observe U, and pick two points in W_1, W_2. connect those points in U by a path α. Since U is open, we can cover α with open balls and get an open subset of U, namely Z, homotopy equivalent to the path, the connects W_1, W_2 in U. Define $V' = V \cup Z$, and notice that V' is homotopy equivalent to $V \vee S^1$. Now, use Van-Kampen on (all path-connected) $V', U, V' \cap U$ where the last one is homotopy-equivalent to $W_1 \vee W_2$, and we get the desired result.

4. (a) In a topological group, the map $f_g : x \mapsto xg$ is a homeomorphism. Thus, for an open set U the image under the quotient is $\{uH\}_{u \in U}$. The full preimage is $UH = \{Uh\}_{h \in H} = \cup_{h \in H} f_h(U)$ and therefore open.

   (b) The map: $Det : GL \to \mathbb{R} - \{0\} =$ is continous, and factor though the cosets so we get a continuous map $Det : GL/SL \to \mathbb{R} - \{0\}$. This is clearly surjective, and it is also injective since if $A \mapsto r, B \mapsto r \Rightarrow AB^{-1} \in SL$ having a 1 determinant, and therefore the cosets $ASL = BSL$ are equal. Left to show that this is an open map as well. Consider the following diagram:

$$\mathbb{R} - \{0\} \xrightarrow{f: a \mapsto diag\{a,1,1,\ldots 1\}} GL \xrightarrow{\qquad q \qquad} GL/SL \xrightarrow{\qquad det \qquad} \mathbb{R} - \{0\}$$

   The entire composition is the identity. Take an open $U \subset GL/SL$. Then $det(U) = f^{-1}q^{-1}(U)$ so it is open since both q, f are CTS.

5. (Not a full solution...yet)

   Observe that X is a $4n$ dimensional vector space, from which we remove $\binom{n}{2}$ vector spaces of co-dimension 4, denote as $P_1, P_2, \ldots P_{\binom{n}{2}}$. Each removal leaves us with a simply connected space so we get that $\pi_1(X) = 0$[1].

   As for, $\sim$, notice that the orbits under the relation contains full copies of $\cup P_i$, i.e. $x \in P_i, y \sim x \Rightarrow y \in P_j$ for some j. Therefore, we can see that $X/\sim$ is the quotient space X/S_n where S_n is the permutation group and the action of S_n on X is as described in

[1] source of simply connectedness of removing manfold of co-dimension 4?

the question. Notice that this action is properly discontinuous on X: for each $x \in X$ we can find a nbhd U s.t. for every $g \in S_n$ the subsets $U, g(U)$ are disjoint unless $g = e$. Therefore, X is a covering space for $X/\sim = X/S_n$. further, since $X/\sim$ is path connected and locally path connected and X is simply connected, then X is the universal covering of $X/\sim$ and $pi_1(X/\sim) = S_n$.