Jan, 2012

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May 1, 2015

Morning

1. Since U is open then U^c is closed so $A = \bar{U} \cap U^c$ is closed as well. Therefor $A = \bar{A}$. Further, the interior of A composed of points which has a nbhd contained in A. However, by construction, A contains all the limit points of U not in U. As such, all the points of A are limit points of U and thus every nbhd of $a \in A$ meets both A and U so it cannot be an interior point.

2. (a) The spaces are path connected so $\pi_1(S^1 \times \mathbb{R}P^2) = \pi_1(S^1) \times \pi_1(\mathbb{R}P^2) = \mathbb{Z} \times \mathbb{Z}/(2)$. By Hurewicz, $H_1 = \pi_1^{ab} = \pi_1$ since the above group is abelian.

   (b) An isomorphism $h_* : \mathbb{Z} \times \mathbb{Z}/(2) \to \mathbb{Z} \times \mathbb{Z}/(2)$ must send a generator of each component to a generator in the range of the same order. After inspection, one can see that the only option other than the identity is $f : (a, b) \mapsto (-a, b)$. However, that would imply that $(a, 0)$ and $(-a, 0)$ have the same image under the covering map p_*, so $p_*(2a, 0) \mapsto 0$, in contradiction to p_* being injective.

3. First notice that the homology of K contains non trivial members H_i for $0 \leq i \leq 3$. By Kunneth formulas, we can see that $H_3(S^1 \times S^2) = H_2(S^1 \times S^2) = 0$. In addition, The map g takes $\mathbb{Z}/(2) = \pi_1(\mathbb{R}P^2 \times S^3) \to \pi_1(S^1 \times S^3) = \mathbb{Z}$, Together, we conclude that $g \circ f$ imposes zero map on π_1 and the homology.

   Furthermore Since $\mathbb{R} \times S^4$ is the universal cover for $S^1 \times S^4$ and since $g \circ f$ is the zero map on the π_1's, we can lift $g \circ f$ to $\widetilde{h} : K \to \mathbb{R} \times S^4$ which is contractible as long as the map is not surjective on the S^4 part. proving that will conclude the problem.

   By the **Simplicial approximation theorem** we can conclude that the map f is in the same homotopy class as a simplicial map from K (of dimension 3) to S^4 of dimension 4, so such map is not surjective.

4. (a) Suppose X is disconnected, so we have a surjective CTS map $g : X \to \{0, 1\}$ where point of the same component are mapped to the same image. Since $f^{-1}(y)$ is connected, then all those preimages are at the same component of X and thus mapped to the same image under g. Therefore, by the universal property of the quotient maps we get a surjective CTS map $Y \to \{0, 1\}$. Contradiction!

(b) $[0,1) \cup (1,2] \hookrightarrow [0,2]$ is a CTS map and every preimage is a single point or the empty set.

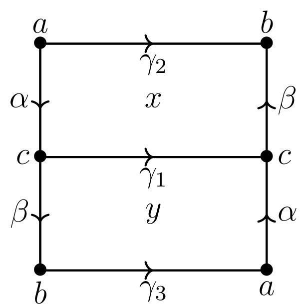
5. (a) The image of a loop in $X = \cup X_k$ is compact. X_k is an open cover for X which mean that the loop must be contained in a finite sub-cover. Meaning, it must be in some X_m and therefor null homotopic.

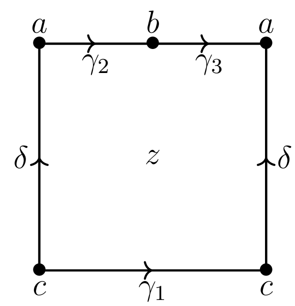
   (b) consider the map $[0,1) \to S^1$. It is surjectvice and notice that $[0,a]$ is mapped to a closed set in S^1 (the closed arc of length $2\pi a$). Now, consider the images of $[0, 1-1/n]$ in S^1: each is a simply connected space but the union is the image of $[0,1)$ which is the entire S^1.

Afternoon

1. One can endow a CW-complex that consist of 3 points, six 1-cells, and 3 2-cells, as will be done in the following paragraph. However, in retrospective, we get that the space has the same homology as S^1. Indeed, the Moebius band and the annulus are each homotopy equivalent to the circle, and those homotopies respect the "glueing" function on the boundary.

   Endow the following CW-Structure on the union of the Moebius band and the annulus (which has the same CW-structure as a cylinder). The left square is the Meobius strip, and the right one is the annulus. We have three 0-cells: a, b, c, six 1-cell: $\alpha, \beta, \gamma_1, \gamma_2, \gamma_3, \delta$, and three 2-cells: x, y, z. Notice that all the degrees in the boundary maps are either 0 or ± 1, so we can calculate the H_i's strictly by calculating the rank of each map.





   The three boundary maps are:

$$\partial_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\partial_1 \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$rank(\partial_2) = 3 \Rightarrow H_2 = \mathbb{Z}^{3-3} = 0.$

$rank(\partial_1) = 2 \Rightarrow H_1\mathbb{Z}^{(6-2)-3} = \mathbb{Z}, H_0 = \mathbb{Z}^{3-2} = \mathbb{Z}$

2. We shall prove that X is not locally compact. Obviously the cone is locally compact at every point different than the base. The question is how the base is behaving. Note that every nbhd of the base correspond to an open set in $\mathbb{R} \times I$ containing the entire $\mathbb{R} \times \{0\}$. The statement depends on whether we can show the existence of a set containing a nbhd of $\mathbb{R} \times \{0\}$ in $\mathbb{R} \times I$, where every cover of open sets containing $\mathbb{R} \times \{0\}$, admits a finite sub-cover.

   Let us assume, for the sake of contradiction, the existence of such set, $A \subset \mathbb{R} \times I$ and we shall find an open cover subjected to the above requirement that does not admit a finite sub-cover. Denote $f(x) = \sup(A \cap \{x\} \times I)$. Since the set A contains an open nbhd of $\mathbb{R} \times \{0\}$, $f(x) > 0$, so in fact we just defined a function $f : \mathbb{R} \to I$ where $f(x) > 0$ for all x.

   Let U be the open subset constructed by the following procedure, for each $m \in \{0, 1, 2, ...\}$, connect the points $(m, f(m)/2)$ and $(m+1, f(m+1)/2)$ with a line in $\mathbb{R} \times I$ and let U to be the interior of the region under the graph. We get an open set containing $\mathbb{R} \times \{0\}$, however do not cover A at any integer point. Now, For each $n \in \{1, 2, ...\}$ define U_n in the following fashion: $U_n = U \cup (n - 2/3, n + 2/3) \times I$. Also denote $U_0 = U \cup [0, 2/3) \times I$. Clearly $\{U_n\}$ covers A since they cover the entire plane $\mathbb{R} \times I$, however, by construction, removing a single U_j from the cover will throw the points in $\{j\} \times (f(j)/2, f(j))$ out of the cover, so it cannot cover A anymore.

3. (a) See May2014, Afternoon, Q4.

   (b) Since $S^2 \to S^2$ is a smooth compact 4-manifold, an immersion would be a covering space for $\mathbb{R}^4$. This is impossible since $\mathbb{R}^4$ is its own universal covering space.

   Now for $S^2 \times S^2$ we shall use the following maps. First, there is a natural embedding $g : S^2 \hookrightarrow \mathbb{R}^3$. Second there is a diffeomorphism $f : S^2 \times \mathbb{R} \cong S^2 \times (0, \infty) \cong S^2 \times \mathbb{R}^2 \setminus \{0\}$ where the last one is by $(e^{i\theta}, t) \mapsto te^{i\theta}$. So use (f, g) to get an embedding $S^2 \times S^2 \to \mathbb{R}^2 \setminus \{0\} \times \mathbb{R}^3 \cong \mathbb{R}^5 \setminus \{0\} \subset \mathbb{R}^5$.

4. (Not a full solution, yet...)

   closed (=compact in manifolds jargon) connected surfaces are classified to be either S^0 (simply connected) or M_g - the orientable compact surface of genus g, which has $\pi_1(M_g) = \mathbb{Z}^{2g}$.

   Let Y' be the covering space of Y which correspond to the group A, meaning the number of sheets in this covering is $[\pi_1(Y) : A]$. Since $f_*(\pi_1(X)) = A$, f is lifted to $X \xrightarrow{\tilde{f}} Y' \longrightarrow Y$.

   For the case where $Y = S^2$, the statement is trivial. In the other case, we need to find show that $H_2(Y') = \mathbb{Z}$. Then we need to find correlation between the degree of maps and the degree of a covering space. We we could show that the degree of $H_2(Y') \to H_2(Y)$ is

d, then we are done by commutativity.

Any ideas?

5. (a) See May2014, morning, Q1.

   (b) Consider $\mathbb{R}P^1$ as the equator of S^2 under the quotient of the antipodal map. Notice that the loop on the great are from east to west is a generator of $\pi_1(\mathbb{R}P^2)$ and of $\pi_1(\mathbb{R}P^1)$. However, that implies $f_* : \mathbb{Z}/(2) \to \mathbb{Z}, 1 \mapsto 1$ which is not a group map.

   (c) In W $a = (0,0)$ and $b = (1/\pi, 0)$ cannot be connected by a path. However they can be in $\mathbb{R}^2$. so the such path $\alpha : I \to \mathbb{R}^2$ and compose with f to get a map $I \to W$ which connects a and b. Contradiction.