1 Reminders

- Boundary of $\partial E$, is the set off all point which are neither interior nor exterior = $\mathbb{E}\setminus E^o = \text{ Every nbhd meets both } E \text{ and } E^c$.

- $E = (\text{disjoint}) E^o \cup \partial E = E \cup \{\text{limit points} = E'\} = (\text{disjoint union of})$:

$$E = \left\{ E^o \cup \begin{cases} E \cup \partial E \setminus E^o \cup \partial E \end{cases} \right\} \bigcup \begin{cases} \mathbb{E} \setminus E \cup \partial E \cap E' \cup \partial E \setminus E' \cup \partial E \setminus E \end{cases}$$

- $\partial E = \mathbb{E} \setminus E^o$

Theorem 1. • CTS bijection from compact to Hausdorff is homeomorphism.

- CTS bijection $\mathbb{R} \to \mathbb{R}$ is homeo.

Theorem 2. Tietze extension Let $f : K \to [a, b]$ CTS, $K$ closed in $\mathbb{R}^d$, then exists a CTS extension to $\tilde{f} : \mathbb{R}^d \to [a, b]$.

Cor/Lemma 1. $(y_n)$ Cauchy sequence in a metric space $X$, $y' \in X$ then $d(y_n, y')$ is bounded.

1 var Riemann integrability

- on $[a, b]$ $f(x)$ is integrable $\iff$ bounded and CTS almost everywhere.

- $\int_0^\infty e^{-x^2} = \sqrt{\pi}$

Gamma function

- $\Gamma(s) := \int_0^\infty e^{-t^{s-1}} dt$ when defined($s > 0$).
2 Main Results - complex Analysis

2.1 Function and series

- $e^z = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ..., $ radius $\infty$.
- $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} = -i \sin(iz)$
- $\cos(z) = \frac{e^{iz} + e^{-iz}}{2} = \cosh(iz)$

Stereographic projection

- $F : S^2 \rightarrow \mathbb{C}, (A, B, C) \mapsto \frac{A+iB}{1-iC}, x+iy \mapsto \frac{1}{x^2+y^2}(2x, 2y, r^2 - 1)$
- Circles in $\mathbb{C}$ mapped to circles in $S^2$. Lines mapped to circles in $S^2$ through the north pole.

Differentiability

- $f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h)-f(z)}{h}$ When the limit exists.
- $f(z)$ is Holomorphic in a region $\Omega$ (open connected set) if it is differentiable at any $z \in \Omega$.
- $f(z) = u(x,y) + iv(x,y)$ differentiable at $z_0 \Rightarrow$ Cauchy-Riemann equations hold at $z_0$: $u_x = v_y, u_y = -v_x$.
  - $f'(z_0) = u_x + iv_x = v_y - iv_y$.
- Conversely, CR on a region $\Omega + u, v \in C^1(\Omega) \Rightarrow f(z)$ holo’ in $\Omega$.

Power Series

- For $S(z) = \sum_0^\infty a_n z^n$ exist unique radius $R \in [0, \infty]$ s.t.
  - $S$ converges absolutely in $|z| < R$, diverges in $|z| > R$.
  - $S$ converges uniformly in $|z| < R - \epsilon$.
  - $1/R = \lim sup |a_n|^{1/n}$.
  - $S$ is holo’ in $|z| < R$ and $S' = \sum a_n z^{n-1}$ with the same $R$. e.i. $\infty$-times differentiable.
- By using later Cauchy’s formulas for $a_n$, the radius is up to the nearest singularity.
- $\log(x)$ power series is valid up to 0.

Integrability

- A smooth curve $z(t)$ has param $z(t) : [a, b] \rightarrow \mathbb{C}$ differentiable (including one sided derivatives), $z'(t)$ exists,CTS, and non zero.
- Piecewise Smooth (p-s) exists finite intervals $a = a_0 < a_1 < ... < a_n = b$ dividing $[a, b]$ on each the CTS curve is smooth.
- For $f(z)$ CTS on (p-s) $\gamma$, define $\int_\gamma f(z)dz = \sum_{a_i}^{a_i+1} f(z(t))z'(t)dt$
- $length(\gamma) = \int_0^b |z'(t)|dt$
- \(|\gamma f(z)dz| \leq \int_0^1 |f(z(t))||z'(t)|dt \leq \sup_{\gamma} |f(z)| \cdot \text{len}(\gamma)\)
- \(f_n(z) \to f(z)\) uniformly Then \(\int_\gamma f_n \to \int_\gamma f\).
- \(f: \Omega \to \mathbb{C}\) (open set). \(F(z)\) is the Primitive of \(f(z)\) on \(\omega\) if \(F'(z) = f(z)\) on \(\Omega\).
  - Primitive of \(z^n\) is \(\frac{z^{n+1}}{n+1}\) on \(\mathbb{C}\) for \(n \neq 1\).
  - For \(\gamma \subset \Omega\) from \(w_1\) to \(w_2\), \(\int_\gamma f(z) = F(w_2) - F(w_1)\)
- **Goursat** \(\Omega\) open set, \(T\) closed triangle in \(\Omega\) including interior, \(f\) holo on \(\Omega\) then \(\frac{1}{\text{area}(T)} \int_T f = 0\).
- **Cauchy** \(D\) open disk, \(f\) holo on \(D\), \(\gamma \subset D\) closed then \(\frac{1}{2\pi i} \int_\gamma f = 0\). Can be extended to disk minus a ray, or any region with \(z_0\) for with any line \(z \to z_0\) is in \(D\).
- **Cauchy Integral Formula** \(D\) open disk, \(C = \partial D\) oriented positively, \(f\) holo on \(\bar{D}\), then for \(z \in D\):
  \(f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} dw\), (Later) \(f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}} dw\)
- \(f(z)\) holomorphic in open set \(\Omega\) Then \(f\) has a power series expansion for any \(z_0 \in \Omega\) for \(D_R(z_0) \subset \Omega\) with \(a_n = \frac{f^{(n)}(z_0)}{n!}\), and \(f(z)\) \(n\) times diff.
- **Cauchy Inequalities** \(D(z_0, R)\) open disk, \(C = \partial D\) oriented positively, \(f\) holo on \(\bar{D}\), then for \(z_0\):
  \(|f^{(n)}(z_0)| \leq \frac{n!|f(z_0)|}{R^n}\)
- Let \(\Omega\) be a region and \(z_n \to z_0\) is a sequence of distinct points with a limit point in \(\Omega\). If \(f\) is holomorphic on \(\Omega\) and \(f(z_n) = f(z_0) = 0\) then \(f = 0\).
- **Mean Value** \(f\) holo in \(D_R(z_0)\) then \(f(z_0) = \frac{1}{2\pi} \int_{\partial D} f(z) dz, 0 < r < R\).
- **Maximum modulus** \(f\) holo in a region \(\Omega\), non constant, then \(|f|\) has no local maximum.
  - If further, \(\Omega\) is bounded and \(f\) is CTS on \(\bar{\Omega}\) then \(\sup_{\Omega} |f| \leq \max_{\partial \Omega} |f|\). (Note that \(f\) is CTS on a compact set and thus attains its maximum.
- **Winding number** \(n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}\)
- \(\gamma \sim 0 \pmod{\Omega}\) if \(\forall z \notin \Omega, n(\gamma, z) = 0\)
- **Jordan’s Curve**: the complement of any simple p-s (not closed) curve is a region. The complement of a closed simple p-s curve is a disjoin union of the interior \((n(\gamma, z) = 1, \text{ simply connected bounded region})\) and the exterior \((n(\gamma, z) = 1, \text{ unbounded region})\).
- **General Cauchy** \(\Omega\) open, \(f\) holo in \(\Omega\), \(\gamma \subset \Omega\), \(\gamma \sim 0 \pmod{\Omega}\) then \(\int_\gamma f dz = 0\).
- A region \(\Omega\) is **Simply Connected** if \(\mathbb{C}\cup\{\infty\}\) \(\Omega\) is connected. Equivalently \(\iff\) of any Cycle(sum of closed curves) \(\gamma \in \Omega, \gamma \sim 0 \pmod{\Omega}\)
  - In a simply connected region, \(f\) holo, \(\gamma\) is a Cycle (sum of closed curves) Then \(\int_\gamma f = 0\).

**Zeros and Poles**
- \(\Omega\) region, \(f \neq 0\) holo:
  - The zeros of \(f\) are isolated. For each zero \(a \exists! k \in \mathbb{N}\) s.t. \(0 = f(a) = f^{(1)}(a) = \ldots = f^{(k-1)}(a), f^{(k)}(a) \neq 0\).
  - So \(f = (z-a)^k g(z), g(a) \neq 0\) g holo in \(\Omega\).
- **Laurent Expansion** \(f(z) = \sum_{-\infty}^{\infty} a_n(z-z_0)^n\) is a holo function in \(r < |z-z_0| < R\), \(1/r = \limsup |a_{-n}|^{1/n}\).
  - Converges uniformly far from the edges. \(a_n = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{n+1}} dz\) for \(r < \rho < R\).
- \(f\) holo in an annulus has a Unique Laurent Expansion.
- \(f\) has a **pole** at \(z_0\) if \(1/f\), with 0 at \(z_0\) is holo in a nbhd of \(z_0\).
• $f$ with a pole, so in a nbhd of $z_0$ exists unique $n \in \mathbb{N}$ and a non vanishing holo $g$ s.t. $f = (z-z_0)^{-n}g(z)$.

• **Residue** for an order $n$ pole $Res(f, z_0) = \frac{1}{2\pi i} \oint_{|z-z_0|<\rho} f(z) = a_{-1} = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n f(z)$.

• $f$ with $z_0$ as an isolated singularity, then
  - Removable $\iff$ $f$ is bounded in a nbhd of $z_0 \iff \lim_{z \to z_0}$ exists an finite.
  - pole $\iff 1/f$ has a zero there $\iff |f| \to \infty$.
  - Essential $\iff \lim_{z \to z_0}$ do not exist.

• **Casorati-Weierstrass** The image of $f$ holo in $D_r(z_0) \setminus \{z_0\}$ with essential singularity in $z_0$ is dense in $\mathbb{C}$. (**Picard**: the image is $\mathbb{C}$ or minus one point).

• A **Meromorphic** function is holo in a region $\Omega$ except isolated poles.

• **Residue theorem** $f$ meromorphic with $\{z_1, z_2, \ldots\}$ poles, $\gamma \sim 0$ (mod $\Omega$) cycle in a region $\Omega$, Then $\oint_{\gamma} f = \sum n(\gamma, z_j) Res(f, z_j) 2\pi i$.

• **Morera** $f$ CTS in an open disk and for every triangle there $\oint_{\gamma} f = 0$ then $f$ is holo in the open disk.

• $f_n$ holo on the open set $\omega$ and converge uniformly to $f$ on every compact subset of $\Omega$. Then $f$ is holo and $f_n' \to f'$ also uniformly on $\omega$.

**Argument Principle**: Let

- $\gamma$ be a simple curve, $n(\gamma, z) = 0 \lor 1$ for $\mathbb{C}\setminus\gamma$. denote $\Omega = \{z \mid n(\gamma, z) = 1\}$
- Let $f$ be meromorphic in an open set containing $\gamma \cup \omega$, with no zeros or poles on $\gamma$.
- THEN $\frac{1}{2\pi i} \oint_{\gamma} \frac{f'}{f} = \sum_{\gamma} n(\gamma, f) - P(\gamma(f))$ i.e. the number of zeros minus the number of poles in $\Omega$, counting multiplicities.
- THEN $n(f(\gamma), 0) = 0$.

- **Rouche** $\Omega, \gamma$ as before, $f, g$ holo in an open set containing both, and $|f| > |g|$ on $\gamma$. THEN $f, f + g$ has the same number of zeros.

• **Open Mapping** $f$ holo non const in region is an open map.

• **Inverse** $f : \Omega \to \Omega'$ (region to region) holo and bijective then so is the inverse $f^{-1}$, and $f^{-1}(z)' = \frac{f'(f^{-1}(z))}{f(f^{-1}(z))}$

• $f$ holo $z_0 \in \Omega$ then is locally 1-1 $\iff f'(z_0) \neq 0$.

• **Liouville**: $f$ entire and bounded is constant.

**Conformal Mappings**

- $f : \Omega \to \mathbb{C}$ is **Conformal** if $\Omega$ is a region and $f'(z) \neq 0$ there.

- Regions $U, V$ are **conformally equivalent** if exists $f : U \to V$ conformal and bijective.

- Such $f$ is locally 1-1, preserves angles and if it is globally 1-1 then the inverse is conformal as well.

- $\gamma$ closed p-s simple curve (p-s Jordan curve), with interior $\Omega$. $f$ holo on $\Omega$, $f|_{\gamma}$ is 1-1 and denote $\Gamma = f|_{\gamma}(\gamma)$. THEN $\Gamma$ is Jordan, $f(\Omega)$ is the interior of $\Gamma$ and $f|_{\Gamma}$ is a conformal bijection.

- **Fractional Linear Transformation** $\frac{az + b}{cz + d}$ with $ad - bc \neq 0$
  - Bijection from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.
  - Maps lines/circles to lines/circles.

- let $\mathbb{D}$ be the open unit disc and $Aut(\mathbb{D})$ be the conformal bijections $\mathbb{D} \to \mathbb{D}$.  

Schwartz’s Lemma $f : \mathbb{D} \to \mathbb{D}$ holo, $f(0)=0$. THEN $|f(z)| \leq |z|$, $|f'(0)| \leq 1$ AND
- if for some $z_0 \neq 0$ we have $|f(z_0)| = |z_0|$, then $f$ is a rotation, $f = e^{i\theta}z$.
- if $|f'(0)| = 1$ the $f$ is a rotation.
- Corollary $f \in \text{Aut}(\mathbb{D})$ fixing 0 must be a rotation.

$\text{Aut}(\mathbb{D}) = \left\{ f(z) = e^{i\theta} \frac{az+b}{cz+d} \mid a \in \mathbb{D}, \theta \in [0, 2\pi) \right\}$

Let $\mathbb{H}$ be the upper half plane $\{x+iy \mid y < 0\}$. $\text{Aut}(\mathbb{H}) = \left\{ f(z) = \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbb{R}, ad - bc \neq 0 \right\}$

Equicontinuity: Left $\mathcal{F}$ be a family of functions from a fixed region $\Omega$ to a metric space $S$. Then $\mathcal{F}$ is Equicontinuous on a subset $E \subset \Omega$ if $\forall \epsilon > 0, \exists \delta$ s.t. $|z - w| < \delta$ implies $d(f(z), f(w)) < \epsilon$ for the entire family.

A family $\mathcal{F}$ as above is Normal if every sequence in $\mathcal{F}$ has a subsequence which converges uniformly on every compact subset of $\Omega$. (limit need not be in $\mathcal{F}$)

Arzela-Ascoli A family $\mathcal{F}$ of CTS functions on $\Omega$ to a metric space $S$ is normal iff both
- $\mathcal{F}$ is equicontinuous on every compact subset of $\Omega$
- for any $z \in \Omega$ the values $f(z)$, $f \in \mathcal{F}$ lie in a compact subset of $S$.

Montel’s: let $\mathcal{F}$ be a family of holomorphic functions on $\Omega$ which are uniformly bounded on any compact subset $B \subset \Omega$: exists $M < 0$ s.t. $|f(z)| < M \forall z \in B, \forall f \in \mathcal{F}$. THEN $\mathcal{F}$ is equicontinuous on every compact subset $\subset$ AND normal.

Riemann Mapping: Let $\Omega$ be simply connected proper region. Given $z_0 \in \Omega$, there exists a unique (up to rotation) conformal bijection $f : \Omega \to \mathbb{D}$ with $f(z_0) = 0$.

3 Measure Function

3.1 Notation

Definition 1 (Indicator Function). For $S \subset A, \mathbb{1}_S : A \to \{0,1\}$ is defined by $\mathbb{1}_S(a) = \begin{cases} 1 & a \in S \\ 0 & a \notin S \end{cases}$

Definition 2 (liminf lim sup). (Note: inf on less members is getting bigger. limit of monotonously increasing sequence.

$sup$ on less members is getting smaller. limit of monotonously decreasing sequence).

$$\lim \inf x_n = \sup \left( \inf_{k \geq 1} x_n \right)$$

$$\lim \sup x_n = \inf \left( \sup_{k \geq 1} x_n \right)$$

Cor/Lemma 2. (Countable sets)

- A countable $\iff A = \phi$ or $\exists f : \mathbb{N} \to A$.
- A countable and infinite $\rightarrow \exists f : \mathbb{N} \to A$ bijection.
- finite Cartesian product of countable is countable.
- countable union of countable is countable.
3.2 Measure function

Definition 3. Measure function $m: \{\text{some sets of } \mathbb{R}^d\} \to [0, +\infty]$, must have to following:

1. $m(\emptyset) = 0$
2. $m(A_1 \cup A_2 \cup \ldots) = \sum m(A_i)$, where $A_i$-disjoint.
3. $m$ is invariant under translation and rotation.
4. $m([a_1, b_1] \times \cdots \times [a_n, b_n]) = (b_1 - a_1) \cdot \ldots \cdot (b_n - a_n)$

Definition 4. For a box $B$ in $\mathbb{R}^d$, $|B|$ is defined as the product of the lengths of its intervals.

Definition 5. For a subset $E$ in $\mathbb{R}^d$, $m_{\text{pixel}} \overset{\text{def}}{=} \lim_{n \to \infty} \frac{\#(E \cap N_{-1/n}^d)}{N^d}$

Theorem 3. $B^{\text{box}} \subset \mathbb{R}^d$, $m_{\text{pixel}}(B) = |B|$

Cor/Lemma 3. $E_1, E_2, \ldots, E_k$ are disjoint pixel-measurable sets. So $m_{\text{pixel}}(\bigcup_1^k E_i) = \sum_1^k m_{\text{pixel}}(E_i)$

Cor/Lemma 4. Examples:

- $m_{\text{pixel}}(Q) = +\infty$
- $m_{\text{pixel}}(Q + \sqrt{2}) = 0$

Cor/Lemma 5. $E = B_1 \cup \ldots \cup B_d$ disjoint union of boxes. So $E$ is pixel-measurable and $m_{\text{pixel}}(E) = \sum |B_i|$. This sum is independent of the choice of boxes.

Definition 6. $E$ is elementary if $E$ is a finite union of boxes (not necessarily disjoint).

IMT lemma 1.1.2 $\implies$ $E$ can be written as a disjoint union of boxes so it is pixel-measurable, and we write the measure as $m_{\text{elem}}(E)$.

Cor/Lemma 6 (properties of $m_{\text{elem}}$): Let $E,F$ elementary sets, then:

- (Finite additivity) $m_{\text{elem}}(E \cup F) = m_{\text{elem}}(E) + m_{\text{elem}}(F)$, if disjoint.
- $m_{\text{elem}}(E \cup F) = m_{\text{elem}}(E) + m_{\text{elem}}(F \setminus E)$
- (Monotonicity) $m_{\text{elem}}(E) \geq m_{\text{elem}}(F)$ if $E \supset F$.
- (Finite sub-additivity) $m_{\text{elem}}(E \cup F) \leq m_{\text{elem}}(E) + m_{\text{elem}}(F)$

Cor/Lemma 7. $E$ elementary, $m_{\text{elem}}(E) = m_{\text{elem}}(E)$

Definition 7. $E$ bounded set.

- Jordan Inner Measure: $m_{\text{*,}(j)}(E) = \sup_{A_{\text{elem}} \subset E} m_{\text{elem}}(A)$
- Jordan Outer Measure: $m_{\text{*,}(j)}(E) = \inf_{B_{\text{elem}} \supset E} m_{\text{elem}}(B)$
- Note: $m_{\text{*,}(j)}(E) \subseteq m_{\text{*,}(j)}(E)$
- Note: If equal, $E$ is Jordan measurable.

Cor/Lemma 8 ("Cauchy" Criterion). $E$ bounded. $E$ is Jordan Measurable $\iff \forall \epsilon > 0, \exists A_{\text{elem}} \subset E \subset B_{\text{elem}}$ s.t. $m_{\text{elem}}(B) - m_{\text{elem}}(A) < \epsilon$ (i.e. $m_{\text{elem}}(B \setminus A) < \epsilon$).

Cor/Lemma 9. $E,F$ are Jordan Measurable, then so are $E \cup F, E \cap F, E \setminus F, E \Delta F$

- Jordan Measure holds for finite additivity, monotonicity, finite sub-additivity, invariant under translation.
- Let $B$ be a box in $\mathbb{R}^d$ and $f: B \to \mathbb{R}$ a cts function. so the set $\{(x, t) : x \in B; 0 \leq t \leq f(x)\} \subset \mathbb{R}^{d+1}$ is Jordan measurable. Also the set of the graph itself $\{(x, t) : x \in B; t = f(x)\} \subset \mathbb{R}^{d+1}$ in Jordan Measurable and has measure 0.
- Cross product of Jordan measurable sets is also Jordan measurable. The new measure is just multiplication.
Cor/Lemma 10. $E$ bounded:

- $m_*(J\cap E) = m_*(J\cap E^c)$
- $m_*(J\setminus E) = m_*(J\setminus E)$

Cor/Lemma 11. $\mathbb{Q}\cap [0,1]$ is not Jordan-Measurable.

Definition 8 (Cantor middle-thirds). Define $U_{1,1} = \left(\frac{1}{3}, \frac{2}{3}\right)$, leaving 2 intervals.

Definition 9. (Lebesgue Outer Measure of $E$): $m^*(E) = \inf_{F\in\mathcal{U}} \sum_{j=1}^{\infty} |B_j|$

Theorem 5. $E$ compact $\implies m^*(E) = m^*(J)(E)$

Cor/Lemma 13. $E$ Elementary $\implies m^*(E) = m_{\text{elem}}(E)$

Cor/Lemma 14. properties of $m^*$:

- $m_*(J\cap E) \leq m^*(E) \leq m_*(J\setminus E)$
- $E$ countable, or empty $\implies m^*(E) = 0$
- (monotonicity) $E \subset F \implies m^*(E) \leq m^*(F)$
- (countable sub-additivity) $m^*(\bigcup_{n} E_n) \leq \sum_{n} m^*(E_n)$

Cor/Lemma 15. (Banach-Tarski) Exists disjoint sets s.t. $m^*(E \cup F) < m^*(E) + m^*(F)$

Definition 10. $\text{dist}(E,F) = \inf \{ ||x - y||, x \in E, y \in F \}$

Cor/Lemma 16. Fix $\delta > 0$ and considering only boxes s.t. $\text{diam}(B_j) < \delta$ so $\inf_{E \in \mathcal{U}} \sum_{j} |B_j| = m^*(E)$

Theorem 6. Suppose $\text{dist}(E,F) > 0$ then $m^*(E \cup F) = m^*(E) + m^*(F)$

Definition 11. (almost disjoint) $B_1, B_2, \ldots$ are almost disjoint if their interiors are disjoint.

Cor/Lemma 17. $B_1, B_2, \ldots$ are almost disjoint, so $m^*(\bigcup B_j) = \sum_{j=1}^{\infty} |B_j| = m_*(\bigcup B_j)$.

- For $E$ with positive measure, for every $\epsilon$, exists an interval $I$ with $m(E \cap I) > (1 - \epsilon)m(I)$

3.3 From HW2

Given $E$ bounded, and closed elementary $F \supset \partial E$, WTS $E \setminus F$ is elementary: See hints in HW2. basically, we are $B_1F$ is a disjoint union of disconnected open sets, one of them is a subset of $E^c$, and both elementary.
3.4 Back to Lectures

**Definition 12 (closed dyadic box).** A closed dyadic box is of the form $[j_1/2^n, (j_1 + 1)/2^n] \times \ldots \times [j_r/2^n, (j_r + 1)/2^n], n, j_r \in \mathbb{Z}$. Let $Q$ be the set of closed dyadic boxes with side length $\leq 1$.

**Cor/Lemma 18.** $E$ open, then $E$ is almost disjoint union of boxes in $Q$, and so $m^*(E) = m^*_{\ast,J}(E)$

**Cor/Lemma 19.** $m^*(E) = \inf_{U \supseteq E} m^*(U)$, where $U$ is open

**Definition 13.** $E$ is (Lebesgue) **Measurable** if $\forall \epsilon > 0, \exists U \supseteq E$ s.t. $U$ is open, and $m^*(U \setminus E) < \epsilon$

**Cor/Lemma 20.** Properties:
- $E$ open (or closed), then measurable
- $m^*(E) = 0$ then measurable.
- $E$ compact, then measurable.
- $E_1, E_2, \ldots$ measurable then the countable union is measurable.
- $E$ measurable, then the compliment is as well.

**Definition 14.** $\sigma$-**algebra on a set $X$, is a collection of subsets $A$ where:**
- $\phi \in A$
- $A$ closed under countable union.
- $A$ closed under complement.

**Theorem 7.** The set of measurable subsets $\mathcal{L}$ is a $\sigma$-algebra

**Cor/Lemma 21.** $E$ meas, the exists closed set $F \subseteq E$ s.t. $m^*(E \setminus F) < \epsilon, \forall \epsilon > 0$

**Definition 15.** $E$ is a
- $F_\sigma$ set if $E = \bigcup_n F_n$ closed
- $G_\delta$ set if $E = \bigcap_n F_n$ open

**Cor/Lemma 22.** $E$ meas then,
- $E$ is a union of $F_\sigma$ set and a null set
- $E$ is $G_\delta$ set cut a null set.

**Cor/Lemma 23.** $m^*(E) = \inf_{U \supseteq E} \sup_{F \subseteq U} m^*_{\text{elem}}(F)$, where $U$ is open, $F$-elementary. If $m^*(E) < \infty$, then $\exists F_{\text{elem}}$ s.t. $m^*(E \setminus F) < \epsilon$.

**Cor/Lemma 24.** Criteria for measurability: the following are equivalent to measurability:
- $U \supseteq E$ open with $m^*(U \setminus E) < \epsilon$.
- $U$ open with $m^*(U \setminus E) < \epsilon$.
- $F \subseteq E$ closed with $m^*(E \setminus F) < \epsilon$.
- $F$ closed with $m^*(F \setminus E) < \epsilon$.
- $G$ measurable with $m^*(G \setminus E) < \epsilon$.

**Definition 16.** A function of $\sigma$-algebra is a **measure function** if the measure of the empty set is 0 and the function in countably additive.

**Cor/Lemma 25.** $m^*$ on $\mathcal{L}$ is a measure function.

**Theorem 8 (jordan measurable implies lebesgue).** If $E$ is Jordan meas then $E \in \mathcal{L}$:

> **Proof.** $E$ Jordan so exits elementary $A \subseteq E \subseteq B$ s.t. $m_J(B \setminus A) < \epsilon$. WLOG $A, B$ are open. So $m^*(B \setminus E) \leq m^*_{\ast,J}(B \setminus E) \leq m^*_{\ast,J}(B \setminus A) \leq \epsilon$. 

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Cor/Lemma 26. Limits properties, all sets are in $\mathcal{L}$:

- $E_1 \subset E_2 \subset \ldots \text{ then } m(\cup_{1}^{d}E_i) = \lim m(E_i)$.
- $E_1 \supset E_2 \supset \ldots \text{ then } m(\cap_{1}^{d}E_i) = \lim m(E_i)$, given the series is not $\infty$ from some point.
- **Fatou’s for sets** $E_1, E_2, E_3, \ldots$, $\liminf E_j := \cup_{k} \cap_{j=k} E_k$, so $m(\liminf E_j) \leq \liminf m(E_j)$.
- $E_1, E_2, E_3, \ldots,$ $\limsup E_j := \cap_{k} \cup_{j=k} E_k$, so $m(\limsup E_j) \geq \limsup m(E_j)$.
- When $\lim E_j$ exists, $E_j \subset F, m(F) < \infty$ (dominating set) then $m(E_j)$ exists and $m(\lim E_j) = \lim m(E_j)$.

Theorem 9. Not all sets measurable. **Vitali Set** is a set of representatives from $[0,1]$ when partitioning $\mathbb{R}$ to cosets of $\mathbb{Q}$. This set contains no measurable set of positive measure.

Cor/Lemma 27. $m^*(A) > 0$ $\iff$ A contains a non-meas set.

Definition 17. Riemann-Darboux Integration Vs. Lebesgue.

- For Riemann-Darboux:
  - $g, f : B_{\text{closed-box}} \rightarrow [0, \infty)$
  - $f$ is boxwise-constant (b.c.) $\iff$ $\#f(B) < \infty, f^{-1}(y)$ is elementary
  - $\iff f = \sum_{i=1}^{n} c_j 1_{E_j}, c_j \in [0, \infty), E_j$ disjoint boxes
  - $f$ b.c. so $\int_B f := \sum_{y \in f(B)} y \cdot m(f^{-1}(y))$
  - Lower Darboux $\int_B g = \sup_{0 \leq f \leq g} \int_B f$
  - Upper Darboux for bounded function $\int_B g = \inf_{g \leq f \leq g} \int_B f$
  - If upper-lower agree, $\int_B g$ is one of them.

- For Lebesgue:
  - $g, f : \mathbb{R}^d \rightarrow [0, +\infty]$.
  - $f$ is simple $\iff$ $\#f(\mathbb{R}^d) < \infty, f^{-1}(y)$ is measurable.
  - $\iff f = \sum_{i=1}^{n} c_j 1_{E_j}, c_j \in [0, +\infty), E_j$ disjoint measurable.
  - $f$ simple. so $\int_{\mathbb{R}^d} f := \sum_{y \in f(B)} y \cdot m(f^{-1}(y))$
  - Lower Lebesgue $\int_{\mathbb{R}^d} g = \sup_{0 \leq f \leq g} \int_{\mathbb{R}^d} f$ (NOTE: $f$ can be taken of finite values).
  - Upper Lebesgue $\int_{\mathbb{R}^d} g = \inf_{g \leq f \leq g} \int_{\mathbb{R}^d} f$

Cor/Lemma 28. $g$ Darboux integrable $\iff E_g := \{(x,t) \in B \times [0,\infty) | 0 \leq t \leq g(x)\}$ is Jordan Meas.

Cor/Lemma 29. Properties of Integrals:

- For $g : B_{\text{closed-box}} \rightarrow [0, \infty)$ bounded we can extend with 0 to get function on $\mathbb{R}^d$. Thus $\text{Darb}_{\mathbb{R}^d} g \leq \text{Leb}_{\mathbb{R}^d} g \leq \text{Darb}_{\mathbb{R}^d} g$.
- Riemann integrable so above are equalities.
- on simple functions: $\int f + g = \int f + \int g$
- $f \leq g$ then $\int f \leq \int g$
- for $c \in [0,\infty)$ $\int cg = c \int g$ and same for $\int$ Not for $c = \infty$.
- $\int f + g \leq \int f + \int g$
- $\int f + g \geq \int f + \int g$
- $\int_{E} g = m^*(E)$

Definition 18. Inner measure of a bounded set $E$, bounded by $B$: $m^*(E) = m(B) - m^*(B/E)$.  

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Theorem 10. \( f : \mathbb{R}^d \to [0, +\infty] \) is \textit{measurable} if one of the (equivalent) following:

1. \( f^{-1}(S) \) measurable for all \( S = (\lambda, +\infty] \).
2. \( f^{-1}(S) \) measurable for all \( S = [0, \lambda] \).
3. \( f^{-1}(S) \) measurable for all \( S = [0, \lambda) \).
4. \( f^{-1}(S) \) measurable for all \( S = [\lambda, +\infty] \).
5. \( f^{-1}(S) \) measurable for all \( S = \text{interval in } [0, +\infty] \).
6. \( f^{-1}(S) \) measurable for all \( S = \text{open in } [0, +\infty] \).
7. \( f^{-1}(S) \) measurable for all \( S = \text{closed in } [0, +\infty] \).
8. \( \exists f_n \text{ simple with } f_n \to f \text{ pointwise.} \)
9. like above, but in addition \( f_1 \leq f_2 \leq \ldots \) and all \( f_j \) bounded and \( m(f_j^{-1}(0, +\infty)) < \infty \)

Cor/Lemma 30. \( g, f_n \) measurable functions so:
- \( \sup f_n, \inf f_n, \limsup f_n, \liminf f_n \) if exists, are meas.
- \( f + g \) meas.
- CTS function is meas.

4 From HW5

Cor/Lemma 31. \( f : \mathbb{R}^d \to [0, +\infty] \) is bounded and measurable \( \iff \) it is a uniform limit of bounded simple functions.

5 Back to Lectures

Cor/Lemma 32. Let \( g \) be bounded, meas and \( m(\{g > 0\}) < \infty \). Then \( \int g = \int f \). And for two like those: \( \int g + f = \int f + \int g \)

Definition 19. for a measurable \( f : \mathbb{R}^d \to [0, +\infty] \) define \( \int f = \int f \)

Cor/Lemma 33.
- \( f : \mathbb{R}^d \to [0, +\infty] \Rightarrow \lim_{n \to \infty} \int \min(f, n) = \int f \).
- \( f, g \) means so (without boundedness or something like that) \( \int (f + g) = \int f + \int g \)

Cor/Lemma 34. \textbf{Markov/Chevishev Inequality} \( g \) meas., unsigned, \( \lambda \in (0, \infty), E = \{g \geq \lambda\} \Rightarrow m(E) \leq \frac{\int g}{\lambda} \)

counter examples
- Cantor closed uncountable set \( K \) has measure 0 but can be a CTS preimage of a set of measure 1.
- measurability is not a topological property - exists a non measurable set whose image is measurable under homeomorphism. (same for pre-image)

Definition 20. For \( f : \mathbb{R}^d \to \mathbb{R} \):
- \( f \) meas \( \iff \) \( f_+, f_- \) are meas.
- \( f \) (abs.) integrable \( \iff \) \( f_+, \int f_-, \int f < \infty \iff \int |f| < \infty \). Then \( \int f = \int f_+ - \int f_- \).

For \( f : \mathbb{R}^d \to \mathbb{C} \)
- \( f \) meas \( \iff \) \( \text{Re} f, \text{Im} f \) are meas \( (\Rightarrow |f| \) meas.\).
\begin{itemize}
  \item f (abs int) \iff \text{Ref, Im int} \iff \text{f meas and } \int |f| < \infty.
  \item Properties: f, g \mathbb{C}-valued, int. So:
    \begin{itemize}
    \item \(\int \overline{f} = \overline{\int f}\)
    \item \(c \in \mathbb{C} \Rightarrow \int cf = c \int f\)
    \item \(\int (f + g) = \int f + \int g\)
    \end{itemize}
\end{itemize}

Cor/Lemma 35. Triangle inequalities:
\begin{itemize}
  \item \(\int |f + g| \leq \int |f| + \int |g|\)
  \item \(\|f\| \leq \int |f|\).
\end{itemize}

Theorem 11. Littlewood’s 3 principles:
\begin{itemize}
  \item 1. E meas, \(m(E) < \infty, \epsilon > 0 \Rightarrow \exists F^{d_{\text{lem}}}, m(E \Delta F) < \epsilon\)
  \item 2a. f integrable, \(\exists h = \sum c_j \mathbb{1}_{B_j^{\text{box}}} \) (boxwise const function) s.t. \(\int |f - h| < \epsilon\).
  \item 2b. f integrable, \(\exists h \text{ CTS function s.t. } \{h(x) \neq 0\}\) is bounded and \(\int |f - h| < \epsilon\).
  \item 2c. h can be even be \(C^\infty\).
\end{itemize}

Egorov thm \(f_n^{\text{meas}} \to f\) pointwise (almost everywhere) on \(\mathbb{R}^d\), Then exists \(B^{\text{meas}}, m(B) < \epsilon\) s.t. \(f_n \to f\) uniformly on each bounded set of \(\mathbb{R}^d\).

Lusin’s thm Given f integrable, then exists \(B^{\text{meas}}, m(B) < \epsilon\) s.t. \(f|_{B^c}\) is CTS on \(B^c\).

revised Lusin we can choose \(B\) to be open, and then \(f|_{B^c}\) is CTS on \(\mathbb{R}^d\).

Let \(f\) be measurable and supported by a measurable set of finite measure(0 outside that set). So exists a meas set \(E\) of meas \(\epsilon\) s.t. \(f\) is locally bounded: Given \(R > 0\), exists \(M > 0\) s.t. \(|f| < M\) on \(B(0, R)\setminus E\).

6 General Measures

Definition 21. Let \(B\) be collection of subsets of \(X\). \(B\) is boolean algebra if:
\begin{itemize}
  \item \(\phi \in B\)
  \item \(E \in B \Rightarrow X\setminus E \in B\)
  \item \(E, F \in B \Rightarrow F \cup E \in B\)
\end{itemize}

properties:
\begin{itemize}
  \item Intersection of bool/\(\sigma\)-algebra’s is a bool/\(\sigma\)-algebra.
  \item Borel \(\sigma\)-algebra \(B[X] = \{\text{open \ sets}\}\).
  \item \(B[\mathbb{R}^d] \subset \mathcal{L} = \{\text{open \ sets}, \{\text{null \ sets}\}\} \).
  \item \(h : X \to Y\) homeo, \(A\)-borel set \(\Rightarrow f(A)\) is a borel set.
\end{itemize}

Definition 22. A countably additive measure \(\mu : B^{\sigma\text{-alg}} \to [0, +\infty]\) s.t.
\begin{itemize}
  \item \(\mu(\emptyset) = 0\)
  \item \(E_1, E_2, \ldots \in B\), disjoint, \(\Rightarrow \mu(\bigcup E_i) = \sum \mu(E_i)\).
\end{itemize}

properties:
\begin{itemize}
  \item Measure \((X, B, \mu)\) is complete if \(F \subset E \in B, \mu(E) = 0 \Rightarrow F \in B\).
  \item Given \(f(x) : X \to [0, +\infty], B = 2^X, \mu(E) = \# f(E) = \sum_{x \in E} f(x)\) is a measure.
\end{itemize}
Theorem 12. Monotone convergence for seq. of functions

\[ \lim_{n \to \infty} \int X \, d\mu_n = \int X \, d\mu \]

Definition 24. Give a measure

Definition 23. Modes of convergence

Properties of convergence: finite measure case, dominated case, True for subsequence

Theorem 13. Properties of convergence: finite measure case, dominated case, True for subsequence

- For \( f = \mathbb{1}_{\{y\}} \Rightarrow \#(E) = 1 \iff y \in E, 0 \text{ otherwise, we get Dirac measure } = \delta_y. \)
- The measure is finite if \( \mu(X) < \infty \), probability space if \( \mu(X) = 1 \)
- The measure is \( \sigma \)-finite if \( X = \bigcup_{n \in \mathbb{N}} E_n, E_n \in \mathcal{B}, \mu(E_n) < \infty \).
  - \( E_n \) can be taken disjointed OR nested.
  - \( \# f \) is \( \sigma \)-finite \( \iff \forall x: f(x) < \infty \text{ AND } \{f(x) \neq 0\} \text{ is countable.} \)
- Measurability of function is as before: preimage of open set is measurable. Integral is defined as sup of simple integrals bounded by the function.
- \( f, g \) measurable. \( \int (f + g) = \int f + \int g \). If \( f \leq g \Rightarrow \int f \leq \int g \), equal \( \iff g = f \) a.e.

Theorem 12. Monotone convergence for seq. of functions \( 0 \leq f_1 \leq f_2 \leq \ldots \) means on \( X \) then

\[ \lim_{n \to \infty} \int f_n \, d\mu = \int \lim_{n \to \infty} f_n \, d\mu. \]

- **Tonelli’s for integrals** \( f_1, f_2, \ldots: X \to [0, +\infty] \) measurable \( \Rightarrow \sum f_n = \sum f_n. \)

- **Example of measure** Given \( g: X \to [0, +\infty] \) alter the measure function \( \mu_g(E) = \int g \, d\mu. \)

- **Fatu’s for functions** \( f_1, f_2, \ldots: X \to [0, +\infty] \Rightarrow \lim \inf f_n \leq \lim \inf \int f_n. \)

- **Lebesgue dominated conv.** \( f_1, f_2, \ldots: X \to \mathbb{C}, f_n \to f \text{ a.e., } G: X \to [0, +\infty] \text{ integrable, } |f_n| \leq G \text{ a.e. } \Rightarrow \int f \to f \text{ and } \int |f_n - f| \to 0. \)

Definition 23. \( L_1 \)-norm \( \|h\|_{L_1} = \int |h|d\mu \).

Definition 24. Give a measure \( (X, \mu, \mathcal{B}) \) and an integrable function \( g \) define a new measure \( (X, \mu_g, \mathcal{B}), \mu_g(E) = \int g \, d\mu. \) Properties:

\[ \int f \, d\mu_g = \int fg \, d\mu. \]

Definition 25. Modes of convergence

- \( f_n \to f \) pointwise a.e. \( \iff f_n \to f \) pointwise on \( X\setminus\{nulll - set\} \).

- \( f_n \to f \) uniformly a.e. \( \iff f_n \to f \) uniformly on \( X\setminus\{nulll - set\} \).

- \( f_n \to f \) almost uniformly \( \iff \forall \epsilon, \exists B, \mu(B) < \epsilon, f_n \to f \) uniformly on \( X\setminus B \).

- \( f_n \to f \) in \( L_1 \) \( \iff \|f_n - f\|_{L_1} \to 0 \) \( \iff \int |f_n - f| \, d\mu \to 0 \).

- \( f_n \to f \) in measure \( \iff \exists \epsilon, \mu(\{|x| |f_n - f| > \epsilon\}) \to 0 \).

- Fast \( L_1 \): \( \sum_{n} |f_n - f| < \infty \).

- Fast in measure: \( \forall \epsilon, \sum \mu(\{|x| |f - f_n| > \epsilon\}) < \infty \).

Theorem 13. Properties of convergence: finite measure case, dominated case, True for subsequence

- \( f_n \to f \) is any mode, \( f_n \to f \) is other any mode \( \Rightarrow f = \bar{f} \text{ a.e.} \)

- In the dominated case, \( |f_n| \leq G \text{ a.e, } f_n \to f \text{ in any mode, } \Rightarrow |f| \leq G \text{ a.e.} \)

- **General Egorov:** \( f_n \to f \) p.w. a.e on a finite measure \( \Rightarrow f_n \to f \) almost uniformly.

- Absolute continuity of the integral - Vitaly \( f \) integrable, \( \epsilon > 0 \Rightarrow \exists \delta \) s.t. if \( \mu(E) < \delta \) then \( \int_E |f| < \epsilon \) \( \mu(E) \to 0 \Rightarrow \int_E |f| \to 0 \).

\[\text{Unif } \rightarrow \text{ almost-unif } \rightarrow \text{ p.w. a.e}\]

\[\text{Fast } L_1 \rightarrow \text{ Fast in measure}\]

Fast - L1 \( \rightarrow \) Fast - in - measure

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Definition 26. p-norms: \( 0 < p < \infty, \|f\|_p = \left( \frac{1}{\mu(X)} \int_X |f|^p \, d\mu \right)^{1/p} \). \( L^p(X, \mathcal{B}, \mu) \) = all measures \( f \) s.t. \( |f|_p < \infty \).

- \( \mu \)-finite, \( 0 < p_1 < p_2 < \infty, f \in L^{p_2} \Rightarrow f \in L^{p_1} \).
- \( 0 < p_1 < p_2 < \infty, f \in L^{p_1}(X, 2^X, \#) \Rightarrow f \in L^{p_2}(X, 2^X, \#) \).
- For Lebesgue \( L^{p_1} \subseteq L^{p_2}, L^{p_1} \supset L^{p_2} \).

Chebyshev: \( \mu(\{|f| \geq \lambda\}) \leq \frac{|f|_p^p}{\lambda^p} \).

\( \|f_n - f\|_p \to 0 \Leftrightarrow f_n \to f \) in measure \( \Rightarrow \) for subsequence we have fast conv. in measure and p.a.e.

Definition 27. Convex functions: I interval in \( \mathbb{R} \), \( \phi : I \to \mathbb{R} \). \( \phi \) is convex \( \iff \) \( a, b \in I, t \in [0,1] \) implies \( \phi((1-t)a + tb) \leq (1-t)\phi(a) + t\phi(b) \) \( \iff \) \( a < x_0 < b \) implies \( \frac{\phi(x_0) - \phi(a)}{x_0 - a} \leq \frac{\phi(b) - \phi(x_0)}{b - x_0} \).

- \( \phi \) CTS on \( I \), \( \phi' \) is (strictly) increasing on \( I^p \) \( \Rightarrow \) \( \phi \) is (strictly) convex.
- \( \phi \) (strictly) convex on \( I \), then \( R(x_0, x_1) = \frac{\phi(x_1) - \phi(x_0)}{x_1 - x_0} \) is a (strictly) increasing function of \( x_0 \) on \( I \setminus \{x_0\} \) when fixing \( x_1 \).
- \( 0 < p < 1, x, y \geq 0 \Rightarrow (x + y)^p \leq x^p + y^p \).

Theorem 14. p-norm:

- \( 0 < p < 1 \) : \( \|f + g\|_p^p \leq \|f\|_p^p + \|g\|_p^p \)
- \( 1 \leq p < \infty : \|f + g\|_p \leq \|f\|_p + \|g\|_p \) (minkowski)

- For \( 0 < p < \infty \) : \( L^p := \{f^{\text{meas}} \mid \|f\|_p < \infty \} \) is a vector space.
- \( d_p(f, g) := \|f - g\|_p^p \) for \( 0 < p < 1 \) and \( \|f - g\|_p \) for \( 1 \leq p \). Is a pseudo norm. On \( L^p = L^p/(\{f^{\text{meas}} \mid f = 0 \text{ a.e.}\}), d_p \) is a norm.

- Riesz–Fisher \((L^p, d_p)\) is complete.
- \{Simple functions\} \( \cap L^p \) is dense in \( L^p, 0 < p < \infty \).
- Support is \( \{x \mid f(x) \neq 0\} \) (in Tau, sometimes the the closure of that called support). \( C_c(\mathbb{R}^d) \) - is the functions with bounded support.
- \( f \) CTS with bounded support \( \Rightarrow \) uniformly CTS.
- \( C_c(\mathbb{R}^d) \) are dense in \( L^p(\mathbb{R}^d, m) \), also \( C_c^\infty(\mathbb{R}^d) \)

Theorem 15. \( \infty \)-norm:

- \( \|f\|_\infty = \inf\{\lambda \mid \mu(\{|f| > \lambda\}) = 0\} = \inf\{\lambda \mid \mu(\{|f| \geq \lambda\}) = 0\} \)
- \( L^\infty = \{f^{\text{meas}} \mid \|f\|_\infty < \infty\} \iff \exists g^{\text{bdd}} \text{ s.t. } g = f \text{ a.e} \)
- \( L^\infty = L^\infty/\mathbb{N} \).

- \( \|f_n - f\|_\infty \to 0 \iff f_n \to f \) unif.a.e.
- \( L^\infty \) is complete.

Cor/Lemma 36. translation and reflection:

- \( T_h(f) = f(x - h) \)
- \( R(f) = f(-x) \)
- \( T_aT_b = T_{a+b} \)
- \( T_h(f \cdot g) = T_h(f) \cdot T_h(g) \)
- \( R(f \cdot g) = R(f) \cdot R(g) \)
- \( RT_h = T_{-h}R \)
- \( \|T_h(F)\|_p = \|R\|_p = \|f\|_p \)
- \( f \in L^p(\mathbb{R}^d, m) \Rightarrow T_h f \to f \) in \( L^p \) as \( h \to 0 \).
6.1 From HW 8

Theorem 16. Holder inequality: \( \sum_{x} |f(y)| \leq \|f\|_p \cdot \|g\|_q \) for \( p > 1 \) or \( p = \infty \) and \( 1/p + 1/q = 1 \).

\((1 < p < \infty)\) Equality \( \Longleftrightarrow |f|^p = C |g|^q \) a.e. or \( g = 0 \) a.e.

\( p = q = 2 \) Cauchy-Schwartz

6.2 Back to lectures - p 32

Definition 28. Convolution

- \((f * g)(x) = \int f(y) \cdot RT_x(g(y)) \, dm(y) = \int f(y) \cdot g(x - y) \, dm(y)\)
- \( z + y = x \Rightarrow (f * g)(x) = \text{a function of } f(y), g(z) \).
- \( 1 \leq p, q \leq \infty, 1/p + 1/q = 1, \, f \in L^p, g \in L^q \Rightarrow (f * g)(x) \text{ is well defined for all } x, \text{ uniformly CTS and bounded, and } \|f * g\|_{\infty} \leq \|f\|_p \cdot \|g\|_q \)
- \( \text{(using later results - Schur’s test) For } k(x, y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}, \text{ define } T_k(f)(y) = \int k(x, y) f(x) \, dm(x), T_h(f) = f(x - h). \text{ So:} \)
- \(- T_h \circ T_k = T_k \circ T_h, \forall h \in \mathbb{R}^d \Longleftrightarrow k(x, y) = g(x - y). \text{ Then, } T_k(f) = f * g \)
- \(- f \in L^p, g \in L^q \Rightarrow f * g \text{ exists a.e. and } \|f * g\|_p \leq \|f\|_p \cdot \|g\|_q \)
- \(- \text{ Youngs inequality: } 1 \leq p, q \leq \infty, 1 + 1/r = 1/p + 1/q, \, f \in L^p, g \in L^q \Rightarrow \|f * g\|_r \leq \|f\|_p \cdot \|g\|_q \)

Theorem 17. Lebesgue Differentiation Thm - LDT:

- Notation: \( cB(x, r) = B(x, cr) \)
- Vitaly covering lemma Given \( B_1, \ldots, B_n \) balls in \( \mathbb{R}^d \). Then we can pick a subset of disjoint balls \( s.t. \cup^0_0 B_j \subset \cup^m_{m=1} 3B_{1,k} \).
- For \( f \in L^1(\mathbb{R}^d, m) \), define \( Mf(x) = \sup_{r > 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} \|f\| \, dm \).
- \( \text{(HW9Q1) } f \in L^1 \) does not imply \( Mf \in L^1 \).
- Inner regularity \( E \) measurable, than \( m(E) = \sup_{K \subset E, \text{compact}} m(K) \).
- Hardy-Littlewood Inequality \( m(\{x : Mf(x) > \lambda\}) \leq \frac{c\|f\|_1}{\lambda} \)
- \( \mu \)-finite borel measure, \( \mu(x) = \sup_{r > 0} \frac{\mu(B(x, r))}{m(B(x, r))} \), then \( m(\{x : M\mu(x) > \lambda\}) \leq \frac{c\|\mu\|_{\infty}}{\lambda} \)
- \( \text{(LDT) } f \in L^1(\mathbb{R}^d) \Rightarrow \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(x) - f(y)| \, dm(y) \to 0 \text{ as } r \to 0^+. \text{ for a.e. } x \)
- \( \text{(LDT) } f \in L^1(\mathbb{R}^d) \Rightarrow \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) \, dm(y) \to f(x) \text{ as } r \to 0^+. \text{ for a.e. } x \).
- \( \text{This defines a natural value for } [f](x) \text{ in } L^1 = L^1/N. \)
- Points where LDT hold - Lebesgue points.
- \( \{E_r\} \text{ Shrinks Nicely} \text{ to } x \text{ if } E_r \subset B(x, r) \text{ and } m(E_r) > \alpha \cdot m(B(x, r)) \).
- \( \text{(LDT) } f \in L^1(\mathbb{R}^d), x\text{-lebesgue point} \Rightarrow \frac{1}{m(E_r)} \int_{E_r} |f(x) - f(y)| \, dm(y) \to 0 \text{ as } r \to 0^+. \)
- \( \text{(LDT) } f \in L^1(\mathbb{R}^d), x\text{-lebesgue point} \Rightarrow \frac{1}{m(E_r)} \int_{E_r} f(y) \, dm(y) \to f(x) \text{ as } r \to 0^+. \)
- \( \text{(FTC1) } \text{Given } f \in L^1(\mathbb{R}^d), \text{ define } F(x) = \int_{[-\alpha, \alpha]} f \, dm. \text{ Then } F \text{ is CTS and } \frac{F(x + r) - F(x)}{r} \to f(x) \text{ at each lebesgue point.} \)
6.3 Monotone Functions p36

For $F: \mathbb{R} \to \mathbb{R}$ increasing:

- $F_-(x) = \lim_{y \to x^-} F(y)$
- $F_+(x) = \lim_{y \to x^+} F(y)$
- $F_\pm$ are increasing, $F_-(x) \leq F(x) \leq F_+(x)$.
- $F(x)$ left(right) CTS at $x$ if $F(x) = F_-(x)(F(x) = F_+(x))$

**lem:** The number of points in which monotone $F$ is discontinuous is countable.

**Jumps**

- $\gamma: \mathbb{R} \to [0, \infty)$ s.t. $\sum_{\mathbb{R}} \gamma(x) < \infty (\iff \#(\mathbb{R}) < \infty)$
- Def $A_\gamma = \{x : \gamma(x) > 0\}$ (Countable by HW1Q1).
- For $\theta: A_\gamma \to [0, 1]$ Def
  \[
  J_{\gamma, \theta}(x) = \begin{cases} 
  \#_\gamma(-\infty, x) = \#_\gamma(-\infty, x) & x \notin A_\gamma \\
  \#_\gamma(-\infty, x) + \theta(x)\gamma(x) & x \in A_\gamma
  \end{cases}
  \]
- $(J_{\gamma, \theta})_0 = J_{\gamma, 0}, (J_{\gamma, \theta})_+ = J_{\gamma, 1}$
- Discontinuity of $J_{\gamma, \theta} \iff x \in A_\gamma$
- $(J_{\gamma, \theta})'(x) = 0$ for all $x \notin A_\gamma$.
- For $F: \mathbb{R} \to \mathbb{R}$ increasing and bounded, let $\gamma = F_+ - F_-, \theta = (F - F_-)/\gamma$ then:
  - $\#(\mathbb{R}) < \infty$
  - $F = J_{\gamma, \theta} + F_c$, $F_c$-CTS, increasing.

**Theorem 18. Differentiability of increasing**

- $F$ CTS increasing, then differentiable a.e.
- $F$ bounded and increasing(or decreasing), then differentiable a.e.

**Dini Derivatives:**

- $D^+ F(x) = \limsup_{h \to 0^+} \frac{F(x+h) - F(x)}{h}$
- $D^+ F(x) = \liminf_{h \to 0^+} \frac{F(x+h) - F(x)}{h}$
- $D^- F(x) = \limsup_{h \to 0^-} \frac{F(x+h) - F(x)}{h}$
- $D^- F(x) = \liminf_{h \to 0^-} \frac{F(x+h) - F(x)}{h}$
- When $F(x)$ increasing - all are non negative
- When $F(x)$ increasing - all are finite almost everywhere
- $F(x)$ diff at $x \iff$ all the Dini’s exists and positive.

**Theorem 19. Total variation**

- $H: \mathbb{R} \rightarrow \mathbb{R}$, Def $|H|_{TV} = \sup_{0 \leq x_1 \leq \ldots \leq x_n} \sum_1^n |H(x_j) - H(x_{j-1})|$.
- $BV = \{H : \|H\|_{TV} < \infty\}$, Norm on $BV\\setminus\{\text{constants}\}$.
- $H \in BV \iff H = F^{\text{bdd+increasing}} + G^{\text{bdd+decreasing}}$. In such case $|H|_{TV} = \|F\|_{TV} + \|G\|_{TV} \iff F(x) = \text{const} + \sup_{0 \leq x_1 \leq \ldots \leq x_n} \sum_1^n \max(H(x_j) - H(x_{j-1})), 0$
- $H \in BV \Rightarrow H$ diff’ almost everywhere, $H' \in \mathcal{L}^1(\mathbb{R}, m), |H'|_1 \leq |H|_{TV}$
Theorem 20. FTC2

- \( F : [a, b] \to \mathbb{R} \), increasing, then \( F' \) defined a.e. and \( \int_a^b F' dm \leq F(b) - F(a), \ F' \in L^1([a, b], m) \).
- \( F \) bdd and inc. on \( \mathbb{R} \Rightarrow F' \in L^1(\mathbb{R}, m) \)
- Def: \( H : [a, b] \to \mathbb{R} \) is Lipshitz CTS if \( |H(x) - H(y)| \leq M |x - y| \).
- Lipshitz CTS \( \Rightarrow H \in BV([a, b]), \text{ diff a.e. and } \int_a^b H' dm = H(b) - H(a) \).
- A function \( F : \mathbb{R} \to \mathbb{R} \) is absolutely CTS if for \( \epsilon > 0 \), \( \exists \delta > 0 \) s.t. \( \sum_{i=1}^n |F(b_i) - F(a_i)| \leq \epsilon \) for any disjoint \( (a_1, b_1), \ldots, (a_n, b_n) \) of total length \( \leq \delta \).
- \( f \in L^1 \Rightarrow F(x) = \int_a^x f dm \) is abs.CTS.

Lipschiz \( \Rightarrow \) abs.CTS \( \Rightarrow \) unif.CTS.

- \( F \) abs.CTS at \( [a, b] \Rightarrow F \in BV([a, b]) \Rightarrow \) diff. a.e.
- \( F \) abs.CTS \( \Rightarrow F(x) = F(a) + \int_a^x F' dm \).
- abs.CTS functions form a vector space. This is Additive.
- \( F : [a, b] \to \mathbb{R} \) diff everywhere, \( F' \in L^1 \) then \( F(b) - F(a) = \int_a^b F' \).

Decomposition of functions

- \( F : [a, b] \to \mathbb{R} \) increasing, then \( F = J_{\gamma, \theta} + \int_a^b F' dm + F_{sc} \).

Outer/Weak/pre measure

- Outer Measure is \( \mu^* : 2^X \to [0, +\infty] \) s.t. \( \mu^*(\emptyset) = 0, \mu^*(E) \leq \mu^*(F) \) for \( E \subset F, \mu^*(\cup E_n) \leq \sum \mu^*(E_n) \).
- Weak premeasure consists \( B_0 \subset 2^X, \mu_0 : B_0 \to [0, +\infty] \) s.t. \( \emptyset \in B_0, \exists E_1, E_2, \ldots \in B_0 \) s.t. \( \cup E_n = X \) (Sequential Covering).
- Define \( \mu^*(E) = \inf_{E \subset \cup E_n, E_n \in B_0} \sum \mu_0(E_n) \Rightarrow \mu^* \) is an outer measure.
- Given outer measure, Define \( B_{car} = \{ E \subset X : \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E), \forall A \subset X \} = \{ E \subset X : \mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \setminus E), \forall A \subset X \} \).
- Caratheodory's thm: \( B_{car} \) is a \( \sigma-\)alg, and \( \mu = \mu^*|_{B_{car}} \) is a complete measure.
- For \( \mu^* = m^*, B_{car} = L \).
- Standard pre-measure is a weak pre-measure with \( B_0 \) boolean-alg, \( \mu(\emptyset) = 0, E_1, E_2, \ldots \) disjoint \( \in B_0 \cap \cup E_n \in B_0 \Rightarrow \mu(\cup E_n) = \sum \mu(E_n) \).
- Hahn-Kolmogorov extension thm \( \mu_0 \text{ std.pre-meas } \Rightarrow B_0 \subset B_{car} \) induced by the outer measure and \( \mu|_{B_0} = \mu_0 \).

Lebesgue-Stieltjes:

- Given and increasing \( F : \mathbb{R} \to \mathbb{R} \) define \( [a, b]^F = F(b) - F(a), [a, b]^F = F_-(b) - F_-(a), [a, b]^F = F_+(b) - F_+(a), [a, b]^F = F_+(b) - F_-(a), [a, b]^F = F_-(a) - F_-(a) \). Denote \( B_{\#} \) all disjoint unions of intervals and \( \mu_0(\cup \# I_i) = \sum |I_i|^F \). Then \( \mu_0 \) is a std.pre-measure inducing a (complete) Lebesgue-Stieltjes measure \( \mu_F \) on \( B_{car} \) containing Borel-alg.
- \( \mu_F \) is a unique measure on Borel-alg agreeing with \( | \cdot |_F \).
- \( \mu \text{ finite measure on Borel algebra on } \mathbb{R}, \mu = \mu_F \text{ for } F(x) = \mu([-\infty, x]) \).
- \( F \) increasing \( \Rightarrow (F_+)_+ = (F_-)_- = F_+(F_-) = F_-(F_+) = \mu_F = \mu_{F_+} = \mu_{F_-} \).
- Bijection \( \{ F \text{ right CTS, increasing on } \mathbb{R}, F(\infty) = 1, F(-\infty) = 0 \} \leftrightarrow \{ \text{Borel probability meas} \} \).
• $F = F_1 + \ldots + F_n \Rightarrow \mu_{F_1} + \ldots + \mu_{F_n}$ also for countable.

• $\mu_{\gamma} = \# \gamma$

• $F$ abs.CTS $\Rightarrow \mu_F(E) = \int_E F' dm \Rightarrow \mu_F = m_F \Rightarrow \int h d\mu_F = \int h F' dm$.

• **canton measure** $F$ extended canton function, $K$ - closed cantor set, $\mu_F(\mathbb{R}\setminus K) = 0, \mu(K) = 1$.

• $F$ non const, increasing, $F' = 0$ a.e. $\Rightarrow \exists$ borel set $E$ with $m(E) = 0, \mu_F(\mathbb{R}\setminus E) = 0$.

**Product Measure**

• For $(X, B_X, \mu_X), (Y, B_Y, \mu_Y) : R$ - measurable rectangles, $B_0 = \{0, \infty\}, \mu_0(E_1 \times F_1 \cup \ldots \cup E_n \times F_n) = \sum \mu_X(E_i) \mu_Y(F_i)$ is a std.premeasure. Get complete measure $\mu_X \times \mu_Y$ on $B_X \times B_Y$. Denote $\mu_X \times \mu_Y = \mu_X \times \mu_Y$ on $B_X \times B_Y$.

• **Slices** for $E \subset X \times Y, E_x = \{y : (x, y) \in E\}$. Same for $E^y$.

• **Tonelli’s Thm:**
  1. $e \in B_X \times B_Y \Rightarrow E_x \in B_Y \forall x, E^y \in B_X \forall y$.
  2. $\mu_X, \mu_Y \sigma$-finite, $E \in B_X \times B_Y \Rightarrow x \Rightarrow \mu_Y(E_x) \text{ measurable and } (\mu_X \times \mu_Y)(E) = \int \mu_Y(E_x) d\mu_X(x)$
  3. Further $\mu_Y$ - complete, $E \in \mathcal{B}_X \times B_Y \Rightarrow (\mu_X \times \mu_Y)(E) = \int \mu_Y(E_x) d\mu_X(x)$

• **Tonelli for functions** $\mu_X, \mu_Y \sigma$-finite + complete, $f : X \times Y \rightarrow [0, \infty]$ $\mu_X \times \mu_Y$ measurable, then:

  1. $y \Rightarrow f(x, y)$ is measurable a.e. $x$. (Same for $y$).
  2. $x \Rightarrow \int f(x, y) d\mu_Y(y)$ is measurable (Same for $y$).
  3. $\int_{X \times Y} f(x, y) d\mu_X \times \mu_Y = \int_X \left( \int f(x, y) d\mu_Y \right) d\mu_X$ (Same for the other way).

  4. **Fubiny**: Denote $L'$ as C-valued abs-int functions. So, also, if $f \in L'$ then $1. \in L'(Y)$ a.e $x$ and $2. \in L'(X)$ a.e. $y$ (and reverses). and the above holds.

• Define $T_k(f)(y) = \int k(x, y) f(x) dm(x), T_h(f) = f(x - h)$, for $k$ meas in the product $(X, \mu) \times (Y, \nu)$, So:

  1. $\int |k| dm < C \text{ a.e. } y, f \in L^p \Rightarrow T(f)$ defined a.e. $y$ and $\|T(f)\|_1 \leq C^{1/p} \|f\|_p$.
  2. **Schur’s test**: $\int |k| dm < C \text{ a.e. } y, \int |k| dw < C \text{ a.e. } x, f \in L^p, 1 \leq p < \infty \Rightarrow T(f)$ defined a.e. $y$ and $\|T(f)\|_p \leq C \|f\|_p$.

**Fourier - complex course**

• For $f : \mathbb{R} \rightarrow \mathbb{C}$ we define $\hat{f}(\zeta) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \zeta} dx, \zeta \in \mathbb{R}$.

• Necessary condition for existance: $\int_{\mathbb{R}} |f| < \infty$.

**Fourier**

**Definition 29. Fourier Transform** $f$ int’ on $\mathbb{R}^d, \hat{f}(t) := \int f(x) e^{-2\pi i t \cdot x} dm(x) : \mathbb{R}^d \rightarrow \mathbb{C}$

properties:

• $\hat{f}$ is CTS.

• $\hat{f}(t) = \int f(x) e^{-2\pi i t \cdot x} dm(x)$

• For $f \in L^1$:

  $- T_h f = e^{-2\pi i h \cdot t} \hat{f}$

  $- e^{-2\pi i h \cdot t} f = T_h \hat{f}$

  $- \hat{f} = R \hat{f}$

• $1 \leq p \leq \infty, 1 = \frac{1}{p} + \frac{1}{q}, f \in L^p, g \in L^q \Rightarrow T_h(f \ast g) = (T_h f) \ast g = f \ast (T_h g)$
Fourier in

Fourier Operator

Hausdorff Measure
• **Def** a weak premeasure \((h_{d,r})_0 : \{\emptyset\} \cup \{B(x, \rho) : x \in \mathbb{R}^n, \rho \in [0,r]\}\) \(\rightarrow [0, \infty]\), \((h_{d,r})_0(B(x, \rho)) = \rho^d\). Derive an outer measure \(h_{d,r}^*\).

• Define **Hausdorff outer measure** \((\mathcal{H}^d)^0 = \lim_{r \rightarrow 0^+} h_{d,r}^*\). Obtain a borel measure \(\mathcal{H}^d\).

• \(\mathcal{H}^d\) is a borel measure.

• For \(d_1 < d_2\), \((\mathcal{H}^{d_1})^*(E) < \infty \Rightarrow (\mathcal{H}^{d_2})^*(E) = 0\). Equivalently \((\mathcal{H}^{d_1})^*(E) > 0 \Rightarrow (\mathcal{H}^{d_2})^*(E) = \infty\).

• Define **Hausdorff dimension** as the number \(\gamma = \dim_\mathcal{H} E\) s.t. \((\mathcal{H}^d)^*(E) = 0\) for \(d > \gamma\) and \((\mathcal{H}^d)^*(E) = \infty\) for \(d < \gamma\).

• \((\mathcal{H}^1)^* = \#\).

• \((\mathcal{H}^n) = \frac{1}{m(B[0,1])} m\) (lebesgue). \(m(E) > 0 \Rightarrow \dim_\mathcal{H} E = n\).

• For \(F : [a,b] \rightarrow \mathbb{R}\) CTS+injective, \((\mathcal{H}^1)(F[a,b]) = \frac{1}{2} \|F\|_{TV}\).

### 7 Examples and Counter Examples

- \(\mathbb{Q}, \mathbb{R}\setminus\mathbb{Q}\)
  - "middle thirds" cantor set, and "fat" Cantor set.
- Cantor function \([0, 1] \rightarrow [0, 1]\) - CTS and increasing with \(F' = 0\) a.e. NOT abs.CTS
- \(\sqrt{x}\) is abs.CTS but not Lipschitz.
- \(r_n\) enumeration of \(\mathbb{Q} \cap [0, 1]\), so \(f(x) = \sum_{n} \frac{1}{n^2 \sqrt{|x - r_n|}}\) finite a.e.

- **TypeWriter function** \(f_n := \mathbb{1}_{[(n-2^k)/(n-2^{k+1}), (n-2^k+1)/n]}\) when we choose \(k\) to be s.t. \(2^k \leq n < 2^{k+1}\). We get a function that is one over the following: \([0,1], [0,1/2], [1/2,1], [0,1/4], [1/4,2/4], [2/4,3/4],\ldots\) Coverages to 0 in \(L_1\) and in measure, but not p.w.a.e.

- \(f_n = \mathbb{1}_{[n,n+1]}\) escape to horizontal infinity
- \(f_n = \mathbb{1}_{[1/n,2/n]}\) escape to vertical infinity
- \(f_n = 1/n\mathbb{1}_{[0,n]}\)

- \(f(x)\) is 1 on \((1,1.2),(2.01,3),(3.001,4),\ldots, 0\) on \((1.1,1),(2.201),(3.3001)\ldots\) This function is constant on \(A = f^{-1}(1)\) and \(B = f^{-1}(0)\), and CTS on \(A \cup B\) but not uniformly CTS, Since there isn’t good choice for \(\delta\).

### 8 QR

**may2012, q 6**

W.T.S that \(f_n\) converges to 0 fast in measure, (and then we are done since fast in measure implies almost unif’ implies p.w.a.e). Using chevichev:

\[
\sum_n \mu(|f_n| \geq \epsilon) \leq \sum_n \frac{\|f\|^2}{\epsilon}
\]

Since the function \(x^2\) is less than \(x\) for small positive numbers, we can leave out finitely many numbers and then loose the upper 2 to get

\[
\sum_n \mu(|f_n| \geq \epsilon) \leq \sum_n \frac{\|f\|^2}{\epsilon} = C + \sum_n \frac{\|f\|^2}{\epsilon} < \infty
\]

**may2014, q 6**

For \(h \in \mathbb{R}\) and \(q \in \mathbb{Q}\) define \(f_{h,q} = |f(x+h) - f(x) - q|\). This is a CTS function as well. Notice, \(f\) is diff at \(x \iff\) for any \(\epsilon\) exist \(\delta\) and \(q\) s.t. \(|h| < \delta\) implies \(f_{h,q} < \epsilon\) (since the rational are dense). Notice that we can take \(\epsilon, \delta\) to be rational as well. So, the set \(\{x \mid f_{h,q} < \epsilon\}\) is open by continuity.