Calculus 115 - Notes

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1 Syllabus

1. Exam 1 - Sections 1.1-1.8, 2.1-2.4
2. Exam 2 - 2.5, 2.6, 3.1-3.7, 3.9 (plus quadratic approximation), 3.10 (MVT only), 4.1-4.3
3. Exam 3 - 4.4-4.6, 5.1-5.4, 6.1, 6.2

1.1 Functions and Change

Definition 1.1.1. A **function** is a rule that takes a certain numbers as input and assigns to each input number a single output. The set of the input numbers is called the **domain** of the function. The set of the output numbers is called the **Range**.

There are 4 ways to describe a function — **Rule of Four**:

1. Tables
2. Graphs
3. Formulas
4. Words

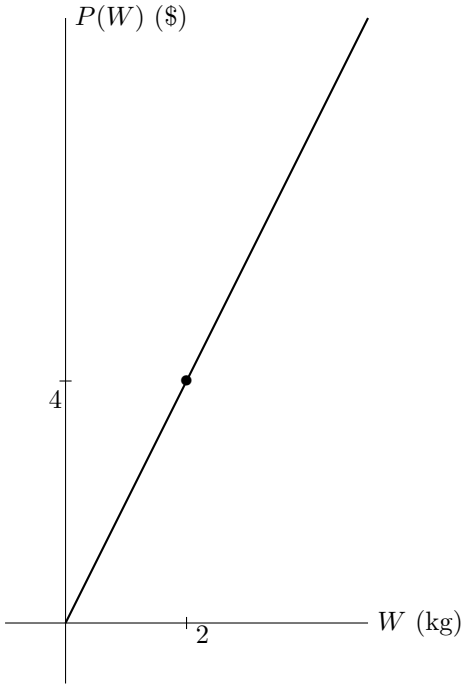
Example 1.1.2.

1. Words: "we assign a dollar price for a bag of tomatoes based on its weight. The price of 1 kg[1] of tomatoes is \$2 and for a different weight the price is calculated proportionally". As one might see, using words is understood by virtually everyone but you need a lot of words to describe a simple concept.
2. Table:

Weight(kg)	Price(\$)
0.5	1
1	2
2	4
0.2	0.4

   Notice that it does not explicitly tells us what is the price of 3kg of tomatoes.
3. Formula: Usually, this is the most concise and accurate way of describing a function. Let P be the price in \$ of W kg of tomatoes.
$$P(W) = 2 \cdot W \quad (\$)$$
4. Graph:

[1] Units are important in this class. Please get comfortable with the metric system, especially the future engineers among you



Many of us are "visual thinkers" and sketching a function as a graph can be very enlightening.

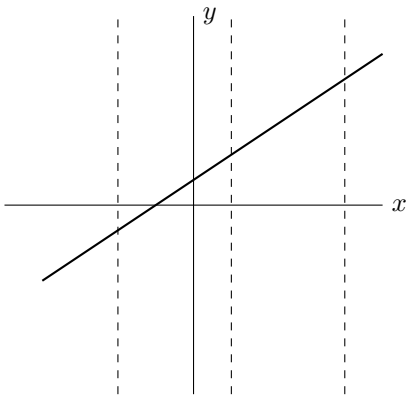
In the example the domain is $0 \leq W$ (we also use this notation $[0, \infty)$ which reads "from 0 to infinity, including 0") The range is $0 \leq P$ or $[0, \infty)$.

Useful notation for domain and ranges, demonstrated using examples:

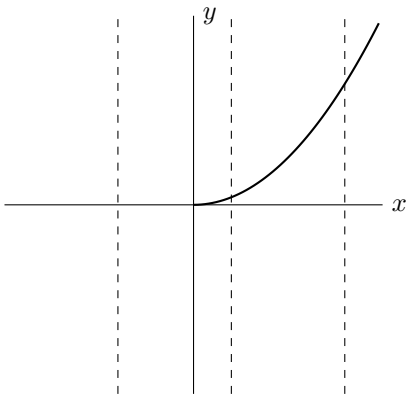
- $1 \leq x \leq 23$ denoted as $[1, 23]$
- $1 \leq x < 23$ denoted as $[1, 23)$
- $1 < x \leq 23$ denoted as $(1, 23]$
- $1 < x < 23$ denoted as $(1, 23)$

Question 1.1.3. When a rule of input and output is not a function?
Based on **1.1.1**, the only thing that can prevent a rule from being function is assigning an input element more than one output element. It is really convenient to test for this property using graphical interpretation of the rule. Given a graph, we use the **vertical line test**: if any vertical line that can be drawn on the same coordinate system intersects the graph zero or one times, but never more, then the graph represent a function. Otherwise the graph represent a rule that is not a function.
The following are a few examples of graphs and some vertical (dashed) lines that are used to test wether the graph represents a function or not.

This is a function

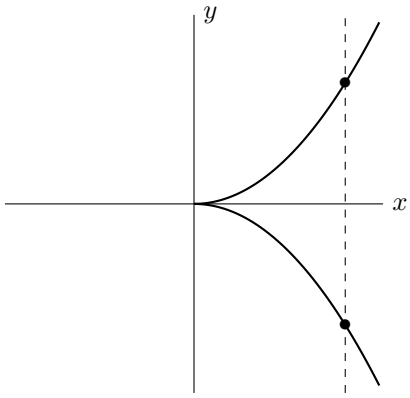


This is a function



This is NOT a function

one vertical line is enough to fail a graph from being a function



Exercise 1.1.4 (section 1.1, problem 38)**.** Which graph in Figure 1.13 best matches each of the following stories? Write a story for the remaining graph.

(a) I had just left home when I realized I had forgotten my books, and so I went back to pick them up.

(b) Thing went fine until I had a flat tire.

(c) I started out calmly but sped up when I realized I was going to be late.

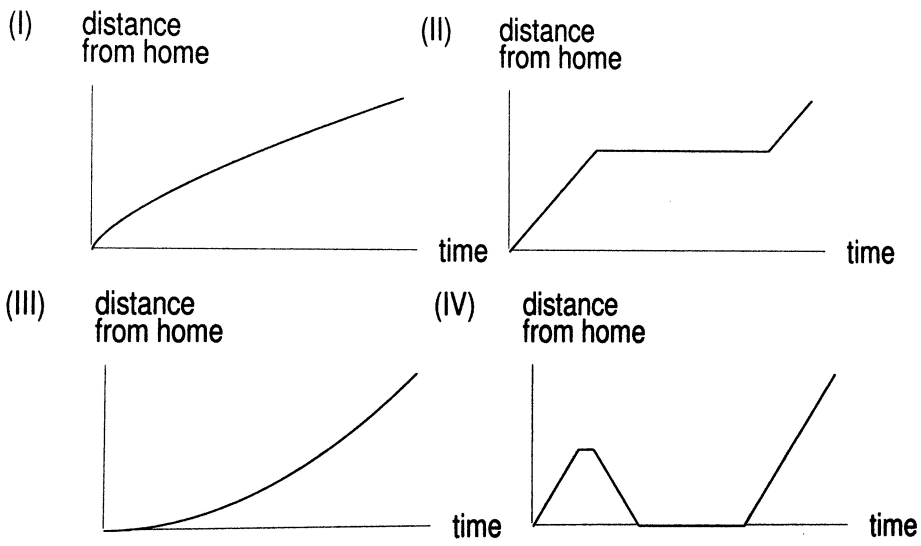


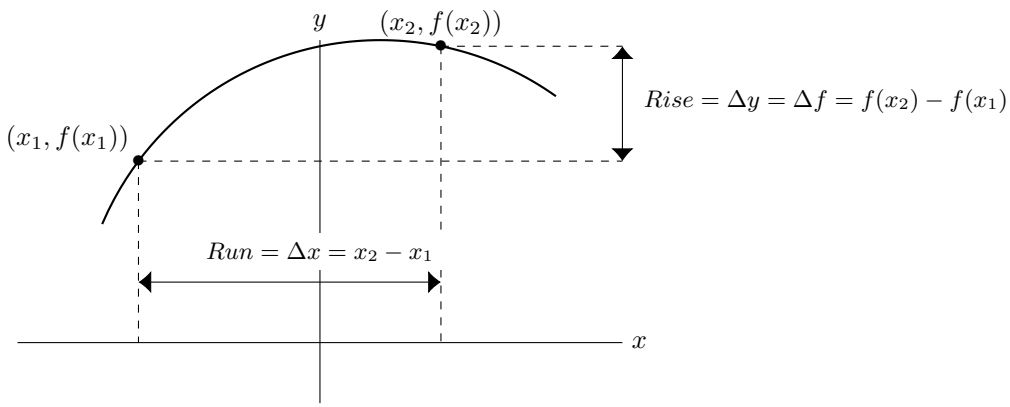
Figure 1.13

Solution. In the story (a), my distance from home increases then I stop and starting to going back home, thus my distance from home is decreasing. So we expect for a part of the graph to represent a decreasing behavior. Only (IV) fits. Notice that between the time when I stop walking and the time when I was starting to go back, there is a small horizontal portion of the graph. Meaning - for a small period of time, the distance from home stayed the same. I was probably scratching my head, trying to remember where are my books.
In (b), the flat tire must be represented by a horizontal portion of the graph. So (II) fits.
For (c), my speed becomes greater as time goes by. Graphically it means that the slope of the graph becomes steeper. We will give an exact mathematic interpretation for the slop in the next discussion. Only (III) fits.
(I) pictures the scenario of me starting out fast and then realizing I don't want to be out of breath for the fast day of school and slowing down to a slower constant speed □

Definition 1.1.5. Given a function $f(x)$ and two points of the graph $(x_1, f(x_1))$ and $(x_2, f(x_2))$, the **slope**, or **Rate of Change**, of the $f(x)$ between these two points is[2]:

$$\boxed{m = \frac{Rise}{Run} = \frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$

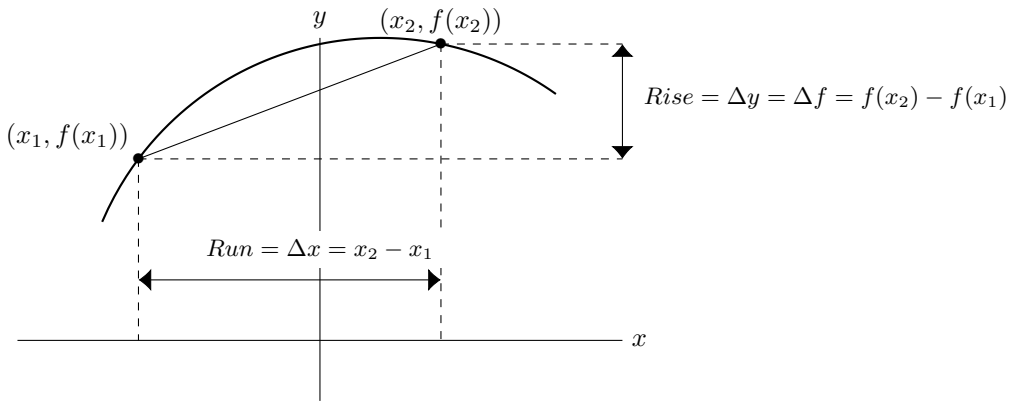
Graphically:



Remark 1.1.6. Don't be afraid of Δx, it is just a notation. Same for Δf. Sometimes we denote a function as $y = f(x)$ or just $y(x)$ and thus Δy is the same as Δf.

Remark 1.1.7. A slope is a number. We can compute this number given a function and two points on its graph. A specific function may have a different slope for every pair of points.

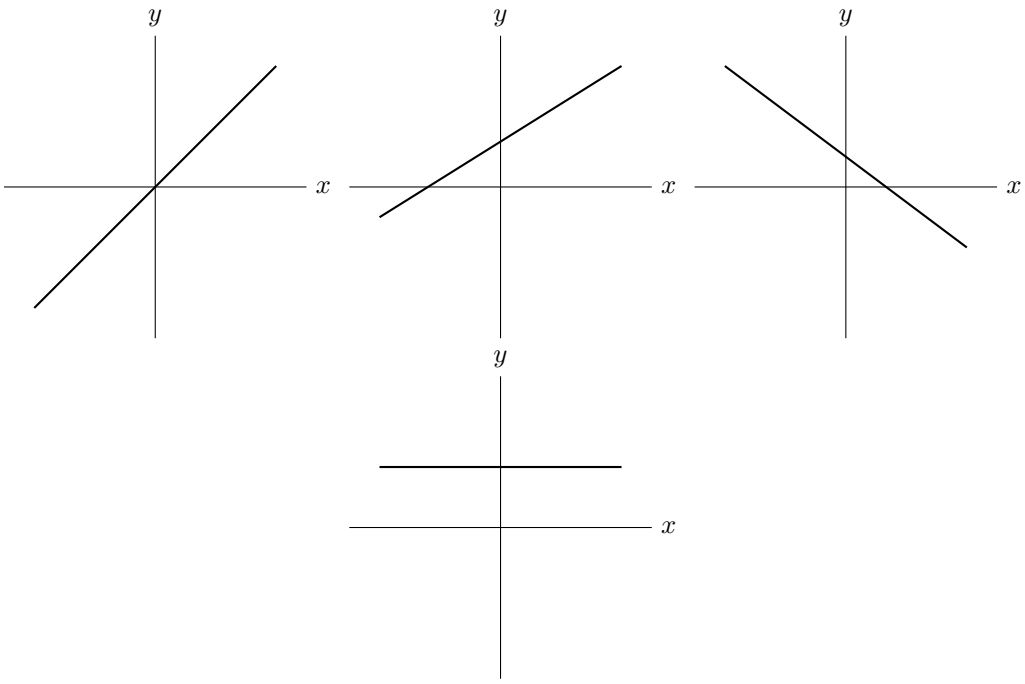
Remark 1.1.8. A graphical way to represent the slope is the connect the given two points and observe the slope of the secant line.



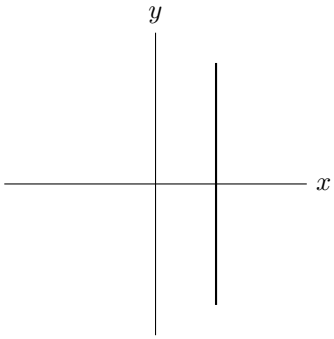
Definition 1.1.9. A function is called a **Linear Function** if it has the same slope at any pair of points.

Graphically, a linear function must look like a line on the coordinate system. Here are a few examples:

[2]This is the first time we use a "box" around a formula. This box implies that this information is important and thus it is recommended to put it in your allowed two sides of a 3" x 5" note card



Question 1.1.10. Is the following a linear function?



The answer is NO! it is not a function since it fails the vertical line test

So graphically it is easy to spot a linear function. What about when the function is given in a different method?

Example 1.1.11. The following is a table of the temperature T as a function of time t of a glass of liquid in a microwave.

t(min)	$T(^oC)$
0	20
1	40
1.5	50
2	60
4	100

Is it a linear function?

Solution. We need to compute the slope for different pairs of points. Let us do it for any adjacent two points:

t(min)	$T(^oC)$	$\Delta y/\Delta x$
0	20	
		(40-20)/(1-0)=20
1	40	
		(50-40)/(1.5-1)=20
1.5	50	
		(60-50)/(2-1.5)=20
2	60	
		(100-60)/(4-2)=20
4	100	

It seems that the slope is the same for every pair that we have picked. Can we deduce that the function is linear? Well, no. The accurate answer is "we don't have enough information". We can't really tell from the table what happens to the temperature between, for example, 2 and 4 minutes. Questions like this usually comes with a disclaimer: assume that the table represents the behavior of the function, or something of that sort. Given this assumption, we can say that since the slope is the same at every pair of points the function is linear.

Trick 1.1.12. If the table is evenly spaced for the input the *Run* is the same for any pair. So suffices to check the *Run* for any pair, meaning to check that the output is evenly spaced. For example:

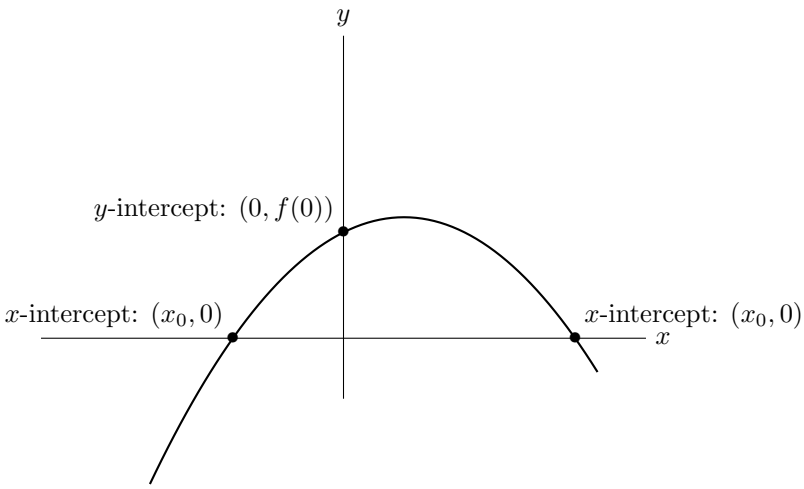
t(min)	$T(^oC)$
0	20
1	40
2	60
3	80
4	100

□

Our next goal is to describe a linear function using a formula. For that we need the following:

Definition 1.1.13. Let $f(x)$ be a function. **The y-intercept** is the value $f(0)$. It is the point where the graph of $f(x)$ meets the y-axis. There is only one such intercept or none if 0 is not in the domain.
An x-intercept is the value x_0 such that $f(x_0) = 0$. It is the values of x where the graph of $f(x)$ meets the x-axis. can be none, one or many,

$y-$ intercept:	$f(0)$
$x-$ intercept,the solutions of:	$0 = f(x)$



Remark 1.1.14. We know that if $y = f(x)$ is a linear function it has one slope for every pair of points. We denote it as m. The formula for a liner function is set by two numbers, the slope m, and the **the y-intercept** b, which is the y value of the point $(0, b)$ where the function meets the y-axis. This is the formula:

$$\boxed{y = f(x) = mx + b, \;\; b = f(0)}$$

Let us demonstrate that such a function has the same slope at any pair of points. Let's take $f(x) = 3x + 1$. The first point is $(x_1, 3x_1 + 1)$. The second point is $(x_2, 3x_2 + 1)$. So

$$\frac{\Delta y}{\Delta x} = \frac{3x_2 + 1 - 3x_1 - 1}{x_2 - x_1} = \frac{3(x_2 - x_1)}{x_2 - x_1} = 3$$

Exercise 1.1.15. The following is a table of the temperature T as a function of time t of a glass of liquid in a microwave. Assume that the function is linear and find a formula for $T(t)$

t(min)	$T(\,^oC)$
0	20
1	40
1.5	50
2	60
4	100

Solution. This is the same table as in **1.1.11**. We already calculated the slope $m = 20$. We just need to find the y-intercept. So far we have $T(t) = 20t + b$. From the table we see that $b = 20$. We can also plug in one point to get b. For example, plug in $(1, 40)$:

$$40 = T(1) = 20 \cdot 1 + b \Rightarrow b = 20$$

Remark 1.1.16. Remark about the units. T is in ^oC. t is in minutes. So the slope being $\Delta T/\Delta t$ is in the units of Celsius degrees per minute: $\frac{^o\text{C}}{\text{min}}$. In words we say that the rate of change is 20 degrees of Celsius per minutes. Equivalently, we can say that the rate is 20 degrees per 60 seconds or 2 degrees per 6 seconds or 1 degree per 3 seconds or... You get the point. The units matter.

Trick 1.1.17. Given two points $(x_1, y_1), (x_2, y_2)$, we can find the linear function passing through them (the line passing through them) using a single formula:

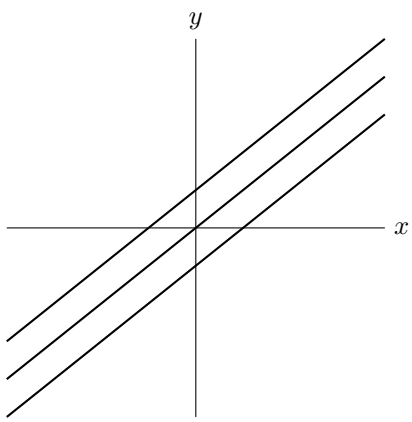
$$\boxed{f(x) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1}$$

After plugging in, we can simplify if needed.

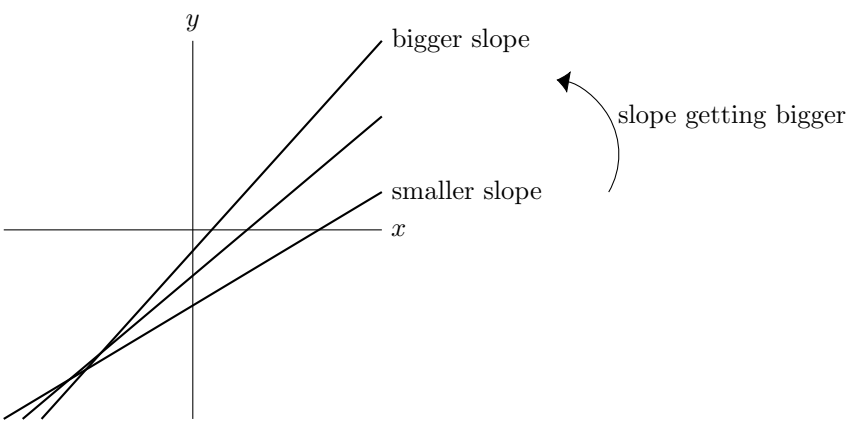
□

Question 1.1.18. Consider the function $f(x) = 4$. Is it a linear function?
YES. It fits the formula in **1.1.14**, with $m = 0$ and $b = 0$. It is called a **constant function**.

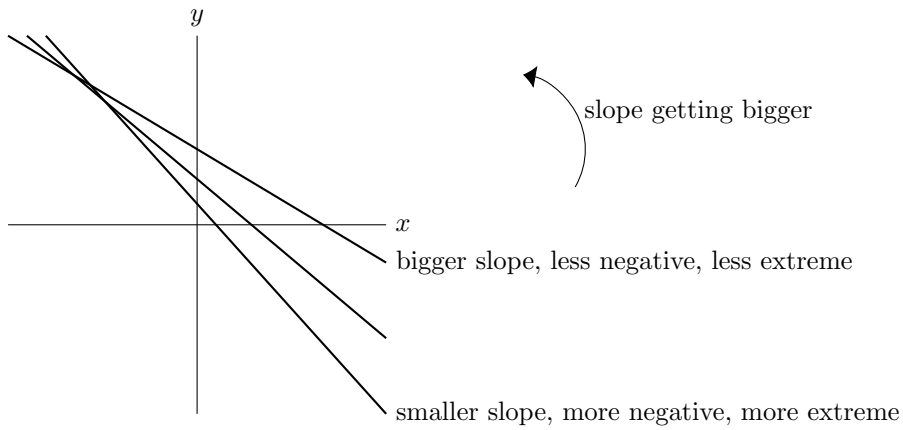
Discussion 1.1.19. Graphical Interpretation of the Slope:
Linear function with the same slope represented by parallel lines.



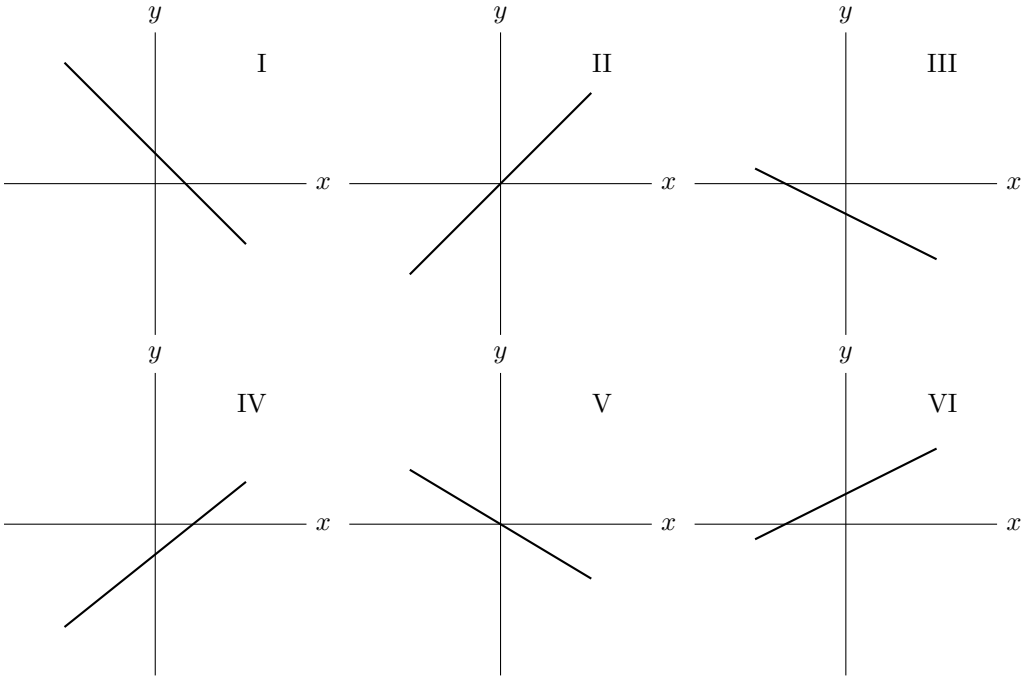
When the slope is positive, the function is increasing (we should imagine ourselves walking on the graph from left to right). The bigger the slope is, the steeper the graph looks.



When the slope is negative, the function is decreasing (again, we should imagine ourselves walking on the graph from left to right). The smaller the slope is (but bigger in absolute), the steeper the graph looks.


y
x
slope getting bigger
bigger slope, less negative, less extreme
smaller slope, more negative, more extreme

Example 1.1.20. Match a graph to each formula. Give a formula for the remaining graph:



1. $y = -2x$
2. $y = 0.1 + 0.0001x$
3. $y = 30 - 2x$
4. $y = 2x - 30$
5. $y = -6 - 10x$

Solution. Notice that we are not given that the coordinate system are all of the same scale. Thus we cannot get any information from the actual values of the slope m and the y-intercept b, but we can get information from the sign - negative or positive:

1. $b = 0, m < 0$ - the graph should be increasing and passing through (0,0). Must be V.
2. $b > 0, m > 0$ - the graph should be increasing and meeting the y-axis at the upper half of it. Must be VI.
3. $b > 0, m < 0$ - the graph should be decreasing and meeting the y-axis at the upper half of it. Must be I.
4. $b < 0, m > 0$ - the graph should be increasing and meeting the y-axis at the lower half of it. Must be IV.
5. $b < 0, m < 0$ - the graph should be decreasing and meeting the y-axis at the lower half of it. Must be III.

For graph II, we need $b = 0$ and a positive slope. $y = x$ would be the classic choice. □

Remark 1.1.21. The lines $y = m_1x + b_1$ and $y = m_2x + b_2$ are:

Parallel if $m_1 = m_2$
Perpendicular if $m_1 = \frac{-1}{m_2}$.

1.2 Exponential Functions

Definition 1.2.1. $P(t)$ is an **Exponential Function** if it can be represented using the following formula:

$$\boxed{P = P_0a^t, \qquad a > 0}$$

a is called the **base** of the function, and P_0 is called the initial value.

Discussion 1.2.2. Exponential functions usually appear in the context of a quantity that changes over time with a fixed ratio. The classical example is population that doubles every year or triples every month, etc... P_0 is called the initial value because this the value of $P(t)$ at $t = 0$ (beginning of time). It is also the y-intercept. a is the base of the function. It is also the fixed ratio between $P(1)$ and $P(0)$, $P(2)$ and $P(1)$, etc.

$$a = \frac{P(1)}{P(0)} = \frac{P(2)}{P(1)} = \frac{P(3)}{P(2)} = ...$$

These type of functions are better understood once we introduce logarithms (section 1.4). In the meantime, given two point on the a graph, $(t_1, P_1), (t_2, P_2)$ of an exponential function, we can find the formula as follows $P_0 a^{t_1} = P_1, P_0 a^{t_2} = P_2$ divide one by the other to get $a^{t_2 - t_1} = P2/P1$. Then $P_0 = P_1/a^{t_1}$. We deduce:

$$\boxed{P(t) = P_0 a^t,\ a = \left(\frac{P_2}{P_1}\right)^{\frac{1}{t_2 - t_1}},\ P_0 = \left(\frac{P_1}{a^{t_1}}\right)} \tag{1}$$

If P_0 is known to us, we just need an addition point (t_1, P_1), and then the formula becomes:

$$\boxed{P(t) = P_0 a^t = P_0 \left(\frac{P_1}{P_0}\right)^{\frac{t}{t_1}}} \tag{2}$$

As a reminder, here are a few algebra rules for powers:

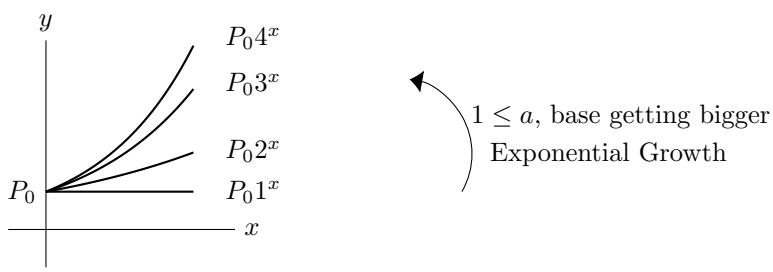
- $\boxed{a^n \cdot a^m = a^{n+m}}$ (common mistake: $a^m + a^n \neq a^{mn}$ or a^{n+m} in general)
- $\boxed{(a^n)^m = a^{nm}}$
- $\boxed{a^n b^n = (ab)^n}$
- $\boxed{a^{-n} = \frac{1}{a^n}}$
- $\boxed{a^1 = a,\ a^0 = 1}$

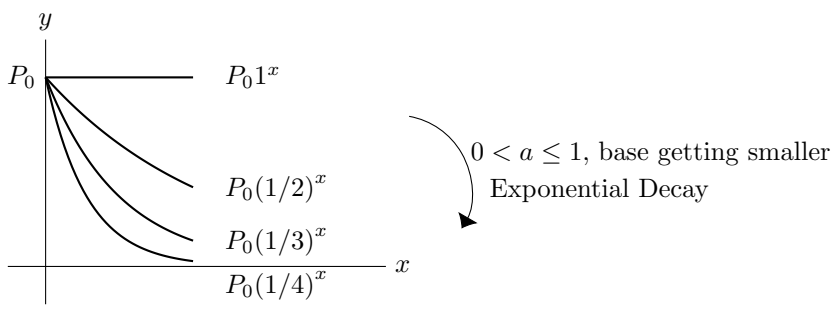
Example 1.2.3. The population of Ann Arbor $P(t)$ double itself every 2.3 years. It was 50,000 on at the beginning of the year 2000. When t defined as "years after 2000", we use (**2**) with $P_0 = 50000$ and the point $(2.3, 2 \cdot P_0)$ and get:

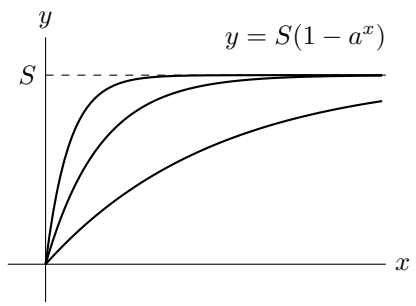
$$P(t) = 50000 \left(\frac{2 \cdot 50000}{50000}\right)^{t/2.3} = 50000(2)^{t/2.3}$$

The base here is $a = 2^{(1/2.3)}$.

Discussion 1.2.4. Graphical Representation of Exponential Functions:







With $0 < a \leq 1$. Function approaches to S from below.
Approaches faster as a gets smaller
S is the **saturation** value

Exercise 1.2.5.

3. [10 points] Jim's new car came with an information sheet about the typical fuel efficiency of the car at different speeds. The fuel efficiency, E, is measured in miles per gallon (mpg) and the speed, v, is measured in miles per hour (mph). A portion of the spreadsheet is given here:

E	15	20	22.925	25	26.61	27.925
v	10	20	30	40	50	60

a. [4 points] Jim notices that, for the range of values in this table, v grows exponentially with E. Find an exponential function f so that $v = f(E)$.

Solution. We are required to find a relation: $v = v_0 a^E$. Let's use two (convenient) points from the table: (15,10) and (25,40)

$$\begin{aligned}
&10 = v_0 a^{15} \\
&40 = v_0 a^{25} \Rightarrow \text{divide the second by the first} \Rightarrow \quad 4 = a^{25-15} = a^{10} \Rightarrow \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad a = 4^{1/10} = (2^2)^{1/10} = 2^{1/5} \\
&\text{now, plug in one of the points to get} \qquad\qquad 10 = v_0 2^{15/5} = v_0 \cdot 8 \Rightarrow v_0 = 5/4
\end{aligned}$$

So we get:

$$v(E) = \frac{5}{4} 2^{E/5} \ \ (mph)$$

Note that the units of v_0 are mph, giving v the units mph. Usually, the base and the power has no units. We expect to have some fraction of the form $(E/something)$ as the power in order to make it a "unit-less" number. The technical term is **Dimensionless**.

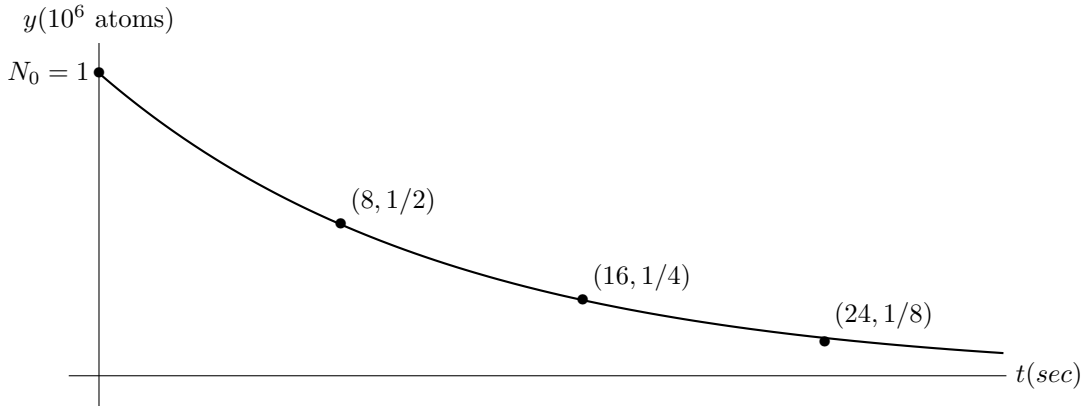
<u>Another solution</u>: We know that v doubles every 5 $E's$. So we expect the formula to be $v = v_0 \cdot 2^{(E/5)}$. □

Exercise 1.2.6. Consider a container with 1 million radioactive atoms at the beginning of the experiment. The **half life period** of the atoms is 8 second, meaning, in a every period of 8 seconds half of the number of the atoms undergo a radioactive decay and being annihilated while exerting energy. The When the number of atoms is 125,000? Sketch a graph of the number of atoms vs. time. Make sure the label the axis, units and mark significant points of the graph.

Solution. The is a classic example. Whenever we are dealing with half-life period, the equation is:

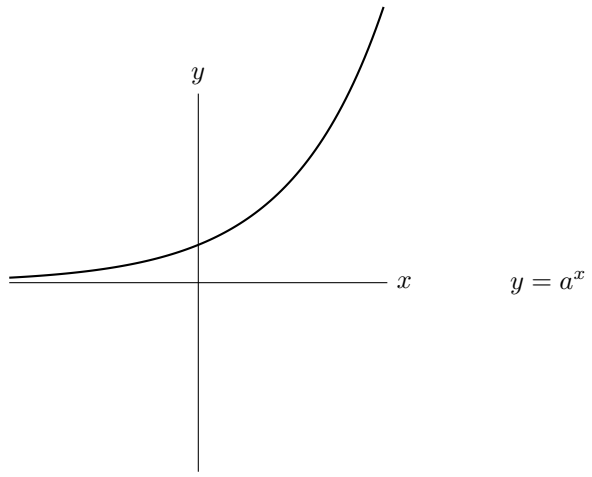
$$\boxed{N_0\left(\frac{1}{2}\right)^{t/t_0}}$$

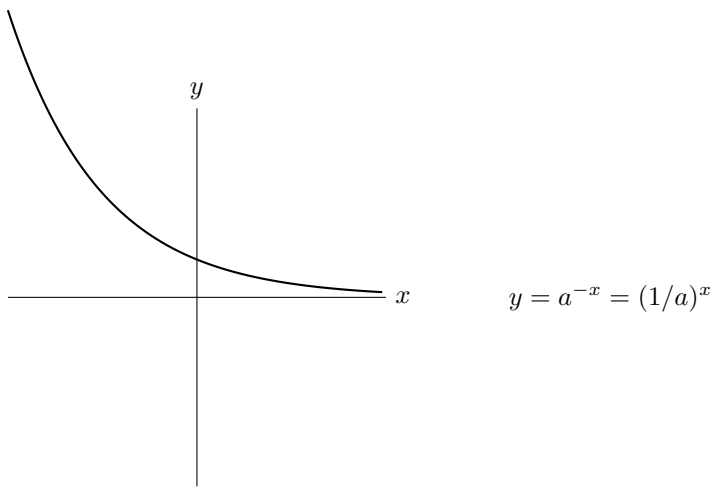
where t_0 is the half-life period. In our case, $N = 1000000(0.5)^{t/8}$ where t is in seconds and N is in number of atoms. We can also write $N = (0.5)^{t/8}$ where N is measured in millions of atoms. N will be 125000, when 3 half life periods will pass, that it, 24 seconds after the beginning of the experiment. Here is the graph:

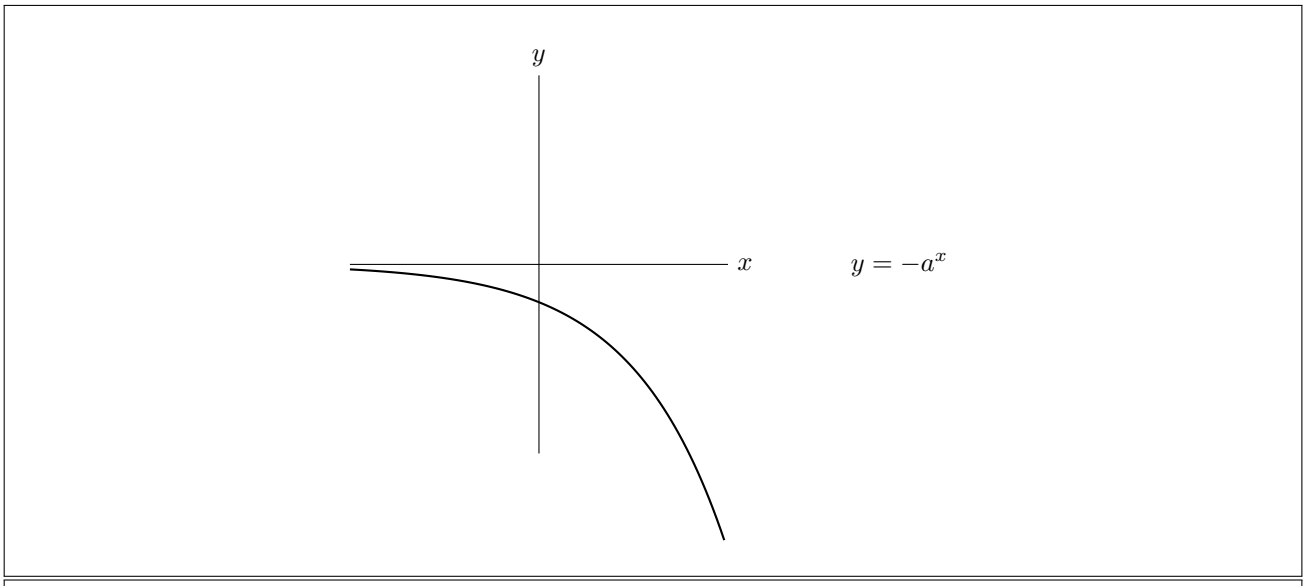


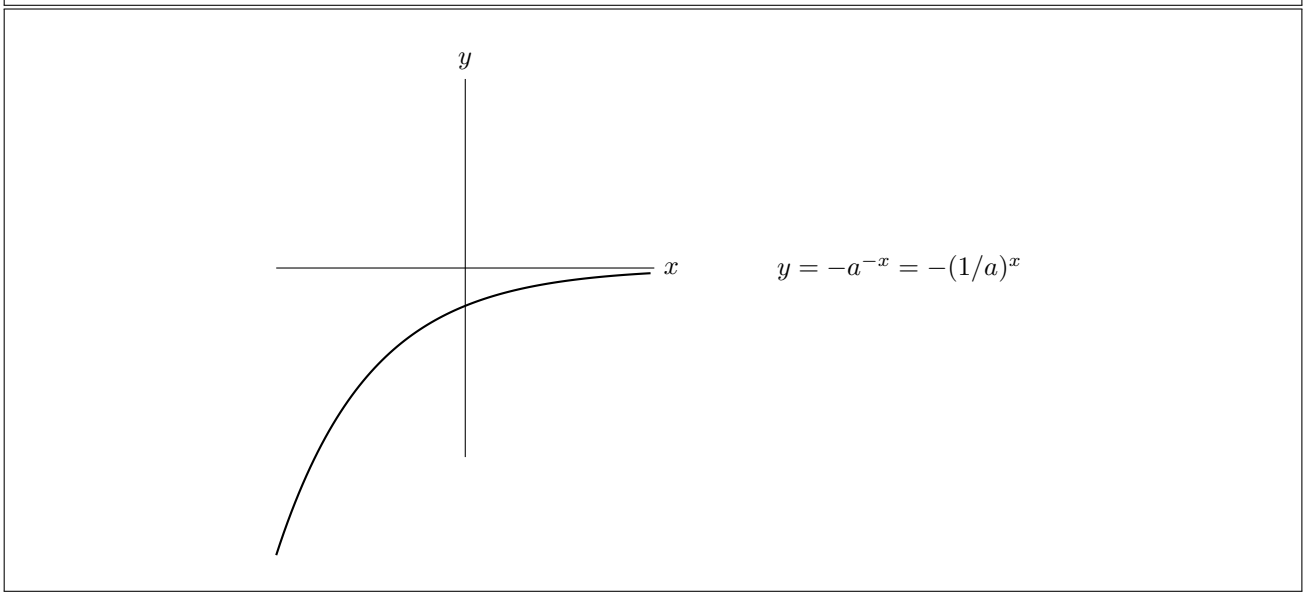
□

Example 1.2.7. Here are some important examples. Consider a base $a > 1$. Note that $a^{-1} = 1/a$ is less than 1. Notice when each function is positive, negative, goes to 0 or goes to infinity:









Remark 1.2.8. We need logarithms (section 1.4) to understand how to move from one basis to another. There is a special number $e = 2.71828...$ which is, by convention, often chosen to serve as a base.

Example 1.2.9. $f(x) = 3(a)^x$ and suppose that we know that $e^k = a$ for some k. Then $f(x) = 3(e^k)^x = 3e^{kx}$. For exponential growth, $k > 0$, and for exponential decay, $k < 0$. That k is called the **continuous rate**.

Math 115, Course Policy Quiz #0 **Name:** _____

Recall that filling out the *online student data form* is worth 5 points on this quiz, so there are a total of 95 possible points on this quiz. (10 points per question, unless mentioned otherwise).

1. Write your name at the top of this page. (5pt)

Decide whether each of the statements below is *True* or *False* as stated. Circle your answer.
You do not need to explain.

2. If you do not complete a WeBWorK or Team Homework assignment by the deadline, you can finish it within the following day for partial credit.

   TRUE FALSE

3. Every idea on which you will be tested on exams will be covered by in-class examples.

   TRUE FALSE

4. If you miss a quiz, you can make arrangements to make it up later.

   TRUE FALSE

5. Each member of the team is responsible for writing up and turning in the solution for at least one problem on the team homework every week.

   TRUE FALSE

6. Grades in Math 115 are based on a traditional scale (90–100 A, 80–89 B, etc.).

   TRUE FALSE

7. In order to be successful in this course, a student should expect to spend a total of about 4 hours outside of class per week working with the course material (doing homework, reading the text, etc.).

   TRUE FALSE

8. At the end of the semester, every section of Math 115 will have the same number of A's, B's, etc, so what is important is how you do in comparison to the other students in your section.

   TRUE FALSE

9. If you are performing poorly on quizzes and team homework due to lack of effort, your course grade can be lowered by a full letter grade.

   TRUE FALSE

Note: The following question is not true/false.

10. What is the penalty for failing to pass the gateway exam before the deadline in Math 115?

Solution.

1. I hope you wrote your name correctly.
2. FALSE. You must complete it on time. Deadlines are firm.
3. FLASE. You must read the textbook and do the HW in order to get all the required material.
4. FALSE. No makeup quizzes.
5. FALSE. One person is the scribe, who is responsible for writing up all the solutions.
6. FALSE. Scale is based on difficulty.
7. FALSE. At least 8 hours per week.
8. FALSE. The scale is calculated course-wide.
9. TRUE.
10. The penalty is a full letter grade.

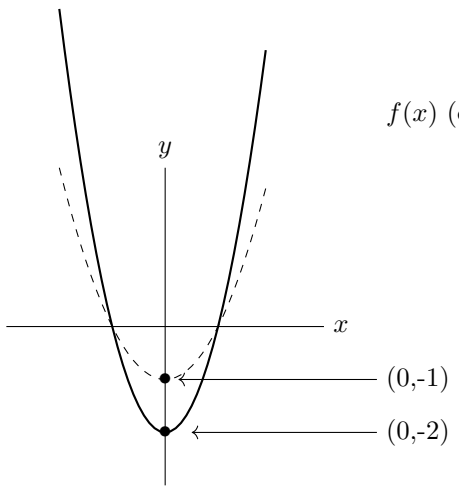
□

1.3 New Functions From Old

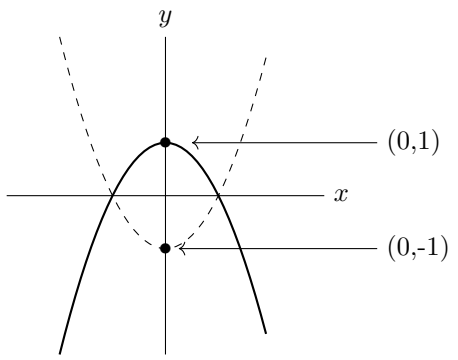
Given a function $f(x)$ we would like to manipulate it and get new functions.

$$g(x) = C \cdot f(x)$$

- $|C| > 1$ - vertical stretch.
- $|C| < 1$ - vertical shrink.
- $C < 0$ vertical flip, with respect to the x-axis, in addition to the above.
- **the x-intercepts stay put**



$f(x)$ (dashed), and $g(x) = 2f(x)$

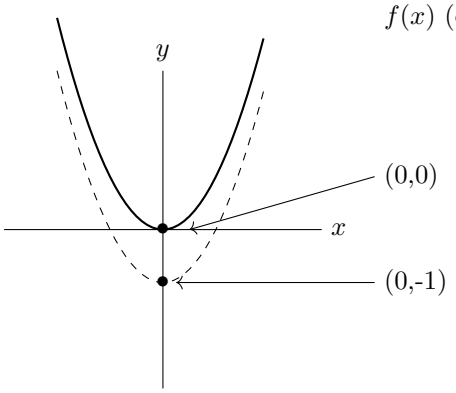


$f(x)$ (dashed), and $g(x) = -f(x)$

$$g(x) = f(x) + C$$

- $C > 0$ - move up.
- $C < 0$ - move down.

$f(x)$ (dashed), and $g(x) = f(x) + 1$



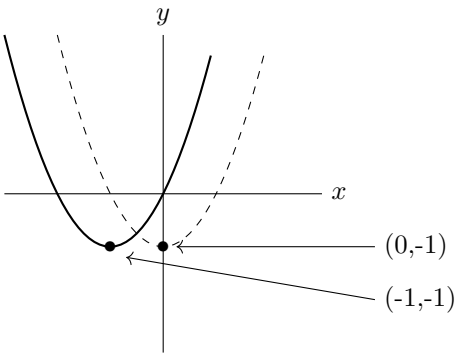
$$g(x) = f(x + h)$$

- $h > 0$ - shift left.
- $h < 0$ - shift right.
- **Mind the signs!**

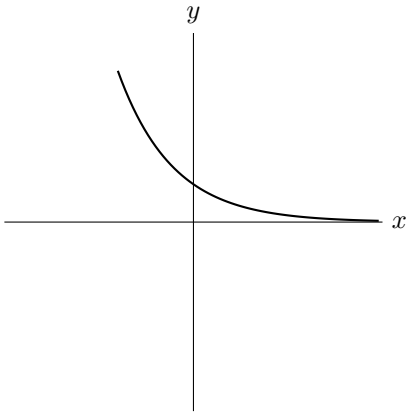
The letters may confuse. Suppose $f(x) = x^2 - 1$. What is $g(x) = f(x + 1)$? We have too many x's. The trick is to write f using a different variable, or even using a shape: $f(\square) = \square^2 - 1$. Then replace $\square$ with $x + 1$ to get:

$$g(x) = f(x + 1) = (x + 1)^2 - 1 = x^2 + 2x + 1$$

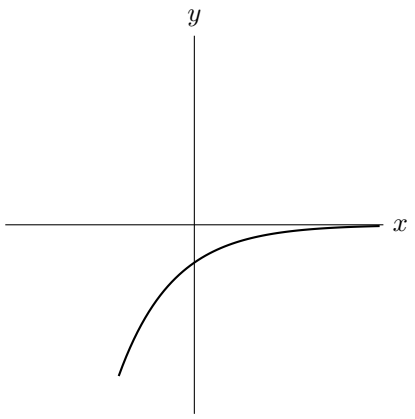
$f(x)$ (dashed), and $g(x) = f(x + 1)$



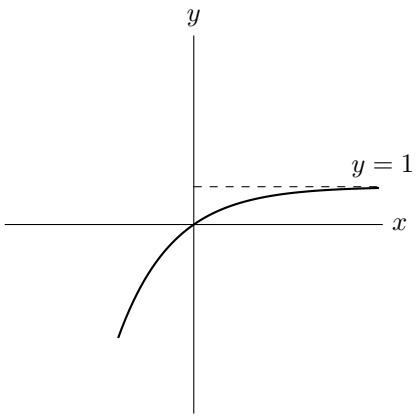
Example 1.3.1. Let's demonstrate how to sketch $g(x) = 3(1 - 0.5^x)$ using the known graph of $f(x) = 0.5^t$



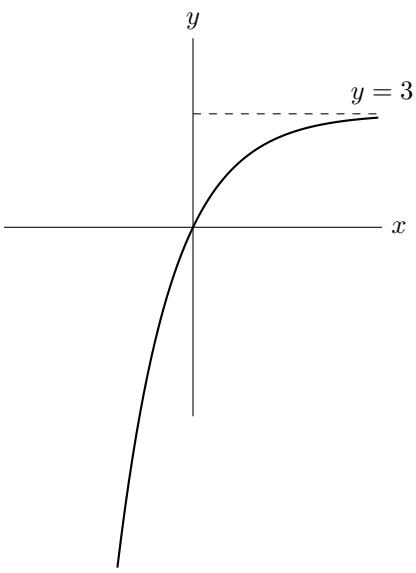
Start with $y = (0.5)^x$



Apply vertical flip $y = -(0.5)^x$



Move up one unit $y = 1 - (0.5)^x$

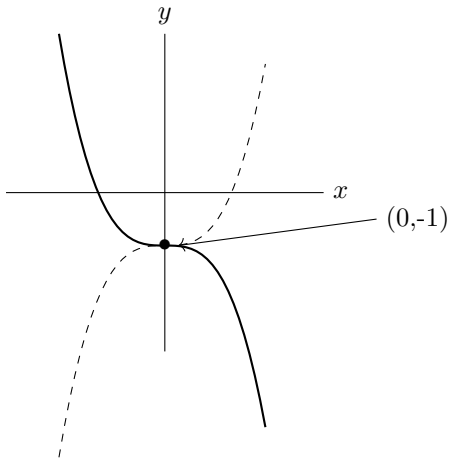


Finally, stretch by 3 units: $y = 3(1 - (0.5)^x)$

$$g(x) = f(-x)$$

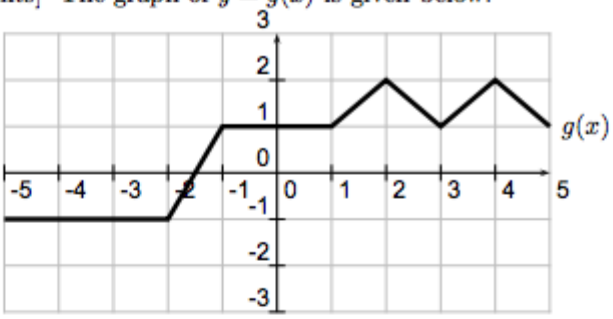
- Horizontal Flip with respect to the y-axis.

$f(x) = x^3 - 1$ (dashed), and $g(x) = f(-x) = -x^3 - 1$

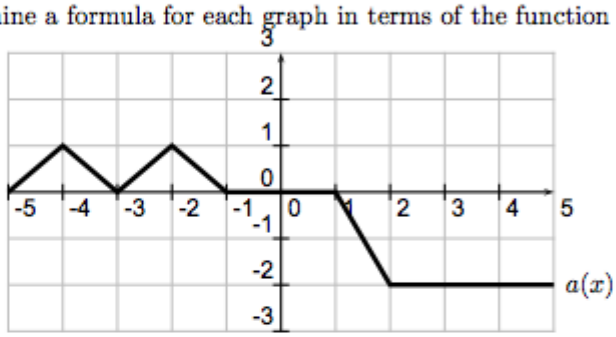


Exercise 1.3.2.

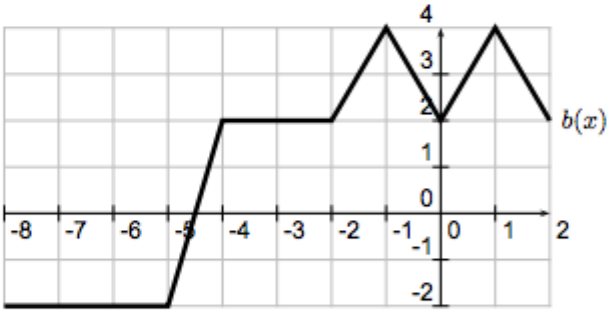
5. [10 points] The graph of $y = g(x)$ is given below.



a. [6 points] The graphs of the following two functions are related to the graph of g. Determine a formula for each graph in terms of the function g.



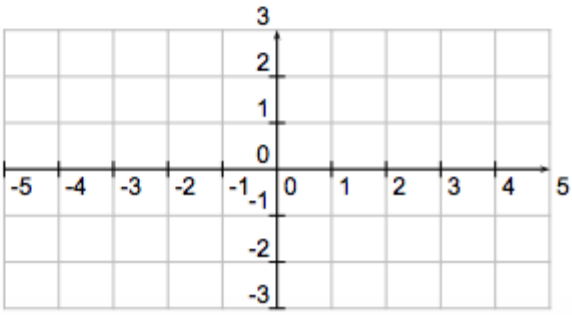
$a(x) =$ _____



$b(x) =$ _____

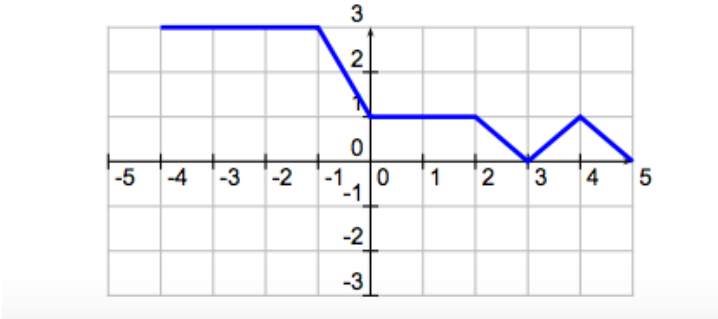
b. [4 points] Carefully sketch as much of the function $c(x)$ as will fit on the axes below, where

$$c(x) = -g(x-1) + 2.$$



Solution. $a(x)$ represents a flip with respect to the y-axis and then moving one unit down. So
$a(x) = g(-x) - 1$
$b(x)$ represents a stretch of 2 units and a move of 3 units left. $b(x) = 2g(x+3)$
For $c(x)$ we have a shift 1 to the right, a flip with respect to the x-axis, and then move up by 2. We get:

$$c(x) = -g(x-1) + 2.$$



□

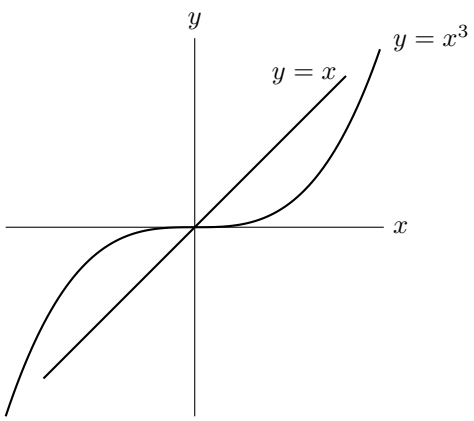
Composite Function: Once we have two function, $f(x)$ and $g(x)$, we can create a composite function $f(g(x))$ and $g(f(x))$.

Example 1.3.3. $f(x) = \sqrt{x+4}$, $g(x) = x^2$. Then:

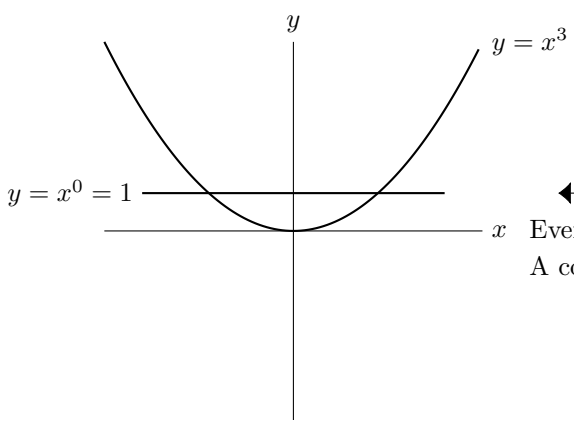
$$f(\square) = \sqrt{\square + 4},\ f(g(x)) = \sqrt{g(x)+4} = \sqrt{x^2+4}$$

$$g(\square) = \square^2,\ g(f(x)) = (f(x))^2 = (\sqrt{x+4})^2 = x+4$$

$f(x)$ is an **odd function** if $f(-x) = -f(x)$. It is invariant to 180° rotation. E.g. $x, x^3, x^5, \ldots.$ and $\sin(x)^3$.
$f(x)$ is an **Even function** if $f(-x) = f(x)$. It is invariant horizontal flip. E.g. $x^2, x^4, x^6, \ldots.$ and $\cos(x)^4$



Odd functions - invariant under 180° rotation



Even functions - invariant under horizontal flip
A constant function is an even function. $f(x) = 3$ is an even function

Remark 1.3.4. An odd function $y = f(x)$ with 0 at its domain must satisfy $f(0) = 0$. Why?

$$-f(x) = f(-x) \Rightarrow f(0) = f(-0) = -f(0) \Rightarrow 2f(0) = 0 \Rightarrow f(0) = 0$$

Question 1.3.5. Is there a function $y = f(x)$ that is invariant under a vertical flip? YES. If we apply a vertical flip to $f(x)$ we get $-f(x)$ so

$$f(x) = -f(x) \Rightarrow 2f(x) = 0 \Rightarrow f(x) = 0,$$

so $f(x)$ must be the constant 0 function. Try to see is graphically.

Inverse function:
Suppose we have a function given in a form of a table. For example, recall **1.1.15**:

t(min)	$T(^oC)$
0	20
1	40
1.5	50
2	60
4	100

We can think of a function $T(t)$ that take time as an input and gives temperature as an output. However, we have another way to produce a different function, $t(T)$, which takes the temperature as an input and gives time as an output. This would be the **inverse function** of $T(t)$. Recall that we had $T = 20t + 20$. Finding the formula for the inverse function is a matter of algebra:

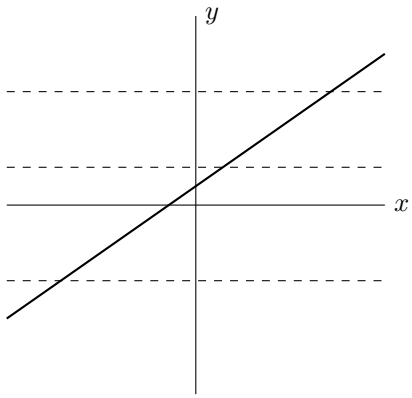
$$T = 20t + 20 \Rightarrow T - 20 = 20t \Rightarrow t = T/20 - 1,$$

and we conclude that the inverse function is $t(T) = T/20 - 1$.

Remark 1.3.6. The inverse function does NOT always exist. If $f(x)$ gives a certain input more than one output, the inverse rule would not be a function because it suppose to assign a certain input more than one output. We have a graphical test for that. Given the graph of $f(x)$, it has an inverse function if and only if it passes the **horizontal line test**: if any horizontal line that can be drawn on the same coordinate system intersects the graph zero or one times, but never more, then one can find an inverse function.

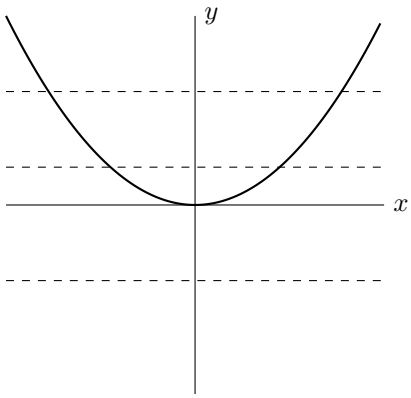
Example 1.3.7. Any linear function has an inverse, except the constant function:

Inverse function exists



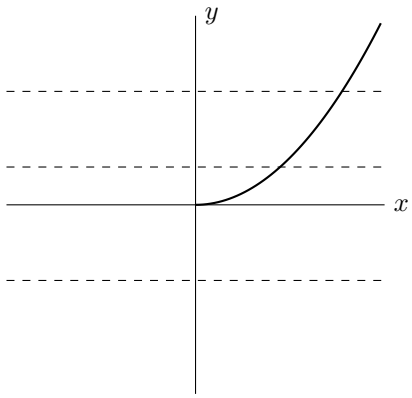
Example 1.3.8. What about $y = f(x) = x^2$?

Inverse function does NOT exists



However, we can limit the domain and make x to be in $[0, \infty)$. Then we get:

Inverse function does exists

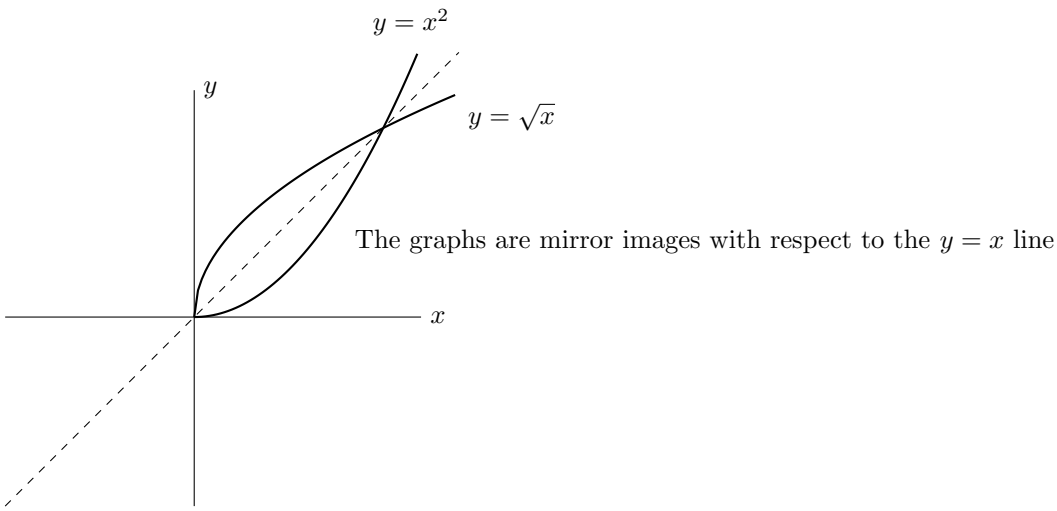


Now, let's find the inverse function's formula:

$$y = x^2 \Rightarrow \pm\sqrt{y} = x^5$$

Using the algebra, we get two inverse functions: $x(y) = \sqrt{y}$ and $x(y) = -\sqrt{y}$. Since x is non negative, only the former applies. So the inverse function is $x(y) = \sqrt{y}$.

Notation: The inverse function of $y = f(x)$ is $x = f^{-1}(y)$. Choosing the same letter to describe the input for f and f^{-1} is confusing, but possible. The inverse function of $f(x) = x^2$ is $g(x) = \sqrt{x}$. This allows us to put them on the same coordinate system, and observe the graphical relations between the graphs:



When composed together $f(f^{-1}(x)) = x$, $f^{-1}(f(x)) = x$

Example 1.3.9. Let $f(x) = x^2, f^{-1}(x) = \sqrt{x}$. Then $f(f^{-1}(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$. In addition, $f^{-1}(f(x)) = f^{-1}(x^2) = \sqrt{x^2} = x = x$

Remark 1.3.10. There is a difference between $f^{-1}(x)$ and $(f(x))^{-1} = 1/f(x)$. They love this common mistake in the homework. Beware.

Exercise 1.3.11.

[5] recall that taking a square root results in two solutions, and we need to check both

9. [15 points] Suppose that $W(h)$ is an invertible function which tells us how many gallons of water an oak tree of height h feet uses on a hot summer day.

a. [9 points] Give practical interpretations for each of the following quantities or statements.

- $W(50)$

- $W^{-1}(40)$

Solution.

- $W(50)$

Solution: The expression $W(50)$ represents how many gallons of water a 50 foot tall oak tree uses on a hot summer day.

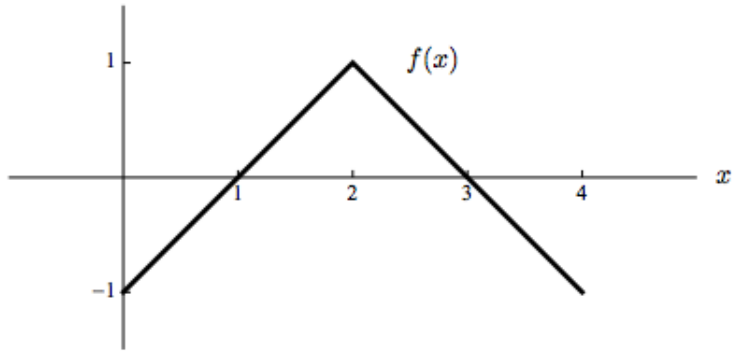
- $W^{-1}(40)$

Solution: The expression $W^{-1}(40)$ represents the height of an oak tree (in feet) which uses 40 gallons of water on a hot summer day.

□

Exercise 1.3.12.

1. [10 points] Suppose $g(x) = x^2$. The graph of a function $f(x)$ is given below. For parts (a)-(c) below, write *all* real numbers z that make the statement true. If no values of z make the statement true, write "NONE". You do not need to show your work.



a. [2 points] $f(g(z)) = 1$.

$z =$ _____

b. [2 points] $g(f(z)) = 0$.

$z =$ _____

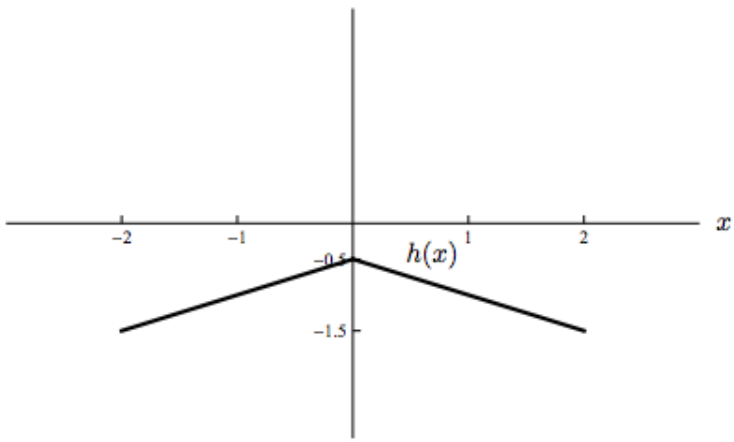
c. [2 points] $f(f(z)) = 0$.

$z =$ _____

d. [4 points] The function $h(x)$ is given by the formula $h(x) = \frac{1}{2}f(x+2) - 1$. On the axes provided below, draw a well-labeled graph of $h(x)$.

Solution.

1. $f(\square) = 1$. Based on the graph, $\square = 2$. Then $g(z) = z^2 = 2 \Rightarrow z = \pm\sqrt{2}$.
2. $g(\square) = 0 \Rightarrow \square^2 = 0 \Rightarrow \square = 0 \Rightarrow f(z) = 0 \Rightarrow z = 1, 3$.
3. $f(\square) = 0 \Rightarrow \square = 1, 3 \Rightarrow f(z) = 1, 3$. For 1, z can be only 2. For 3, we do not have any solutions for z.
4. We shift left by 2 to get $f(x+2)$ then stretch by 0.5 (shrink) and then move down by 1. We expect the graph to stay in its $\wedge$ shape, but with different values. It is enough to compute the the apex point and the two edge point. Originally, they are at $x = 0, 2, 4$. Due to the "shift left", those points are moved to $x = -2, 0, 2$ respectively. The rest of the manipulations does not change the x-coordinate. Now, we just plug in.
$$\begin{aligned} h(-2) &= 0.5f(-2+2) - 1 = 0.5f(0) - 1 = -0.5 - 1 = -1.5 \\ h(0) &= 0.5f(0+2) - 1 = 0.5f(2) - 1 = 0.5 - 1 = -0.5 \\ h(2) &= 0.5f(2+2) - 1 = 0.5f(4) - 1 = -0.5 - 1 = -1.5 \end{aligned}$$



$\square$

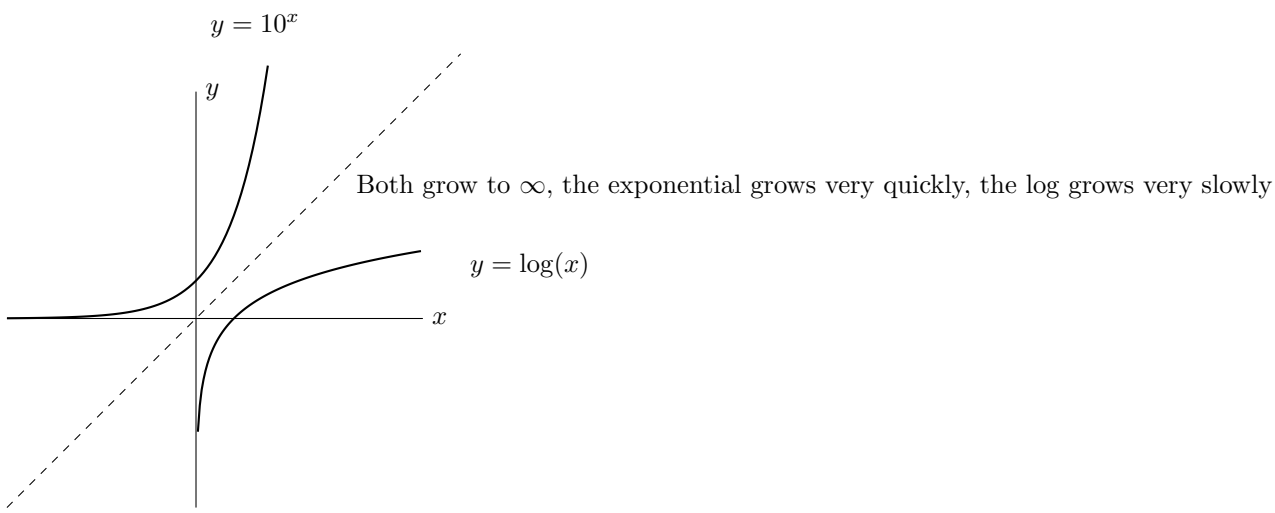
1.4 Logarithmic Functions

Definition 1.4.1. $\log_{10}(x)$ is the inverse function of $f(x) = 10^x$. It may be denoted just as $\log(x)$.

A better way to describe it is

$$\log(x) = c \iff 10^c = x. \qquad \text{The domain is } (0, \infty)$$

Graphically:

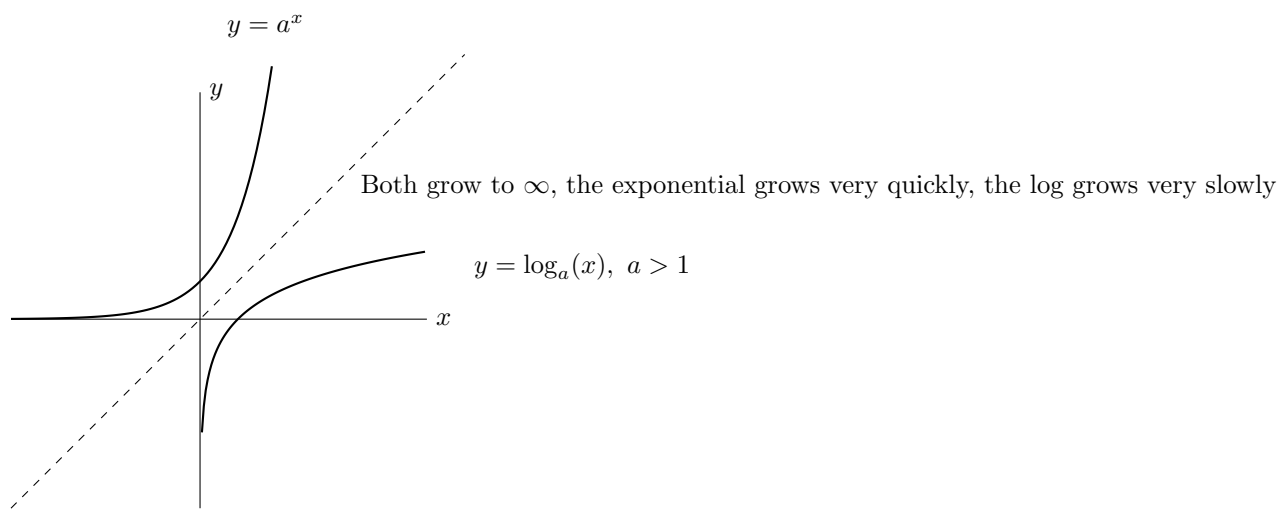


We can repeat the same construction for any base $a > 1$:

Definition 1.4.2.

$$\boxed{\log_a(x) = c \iff a^c = x. \qquad \text{The domain is } (0, \infty)}$$

And the graphical representation is the same.



There is a special number, $e = 2.71828...$, that we often use as base[6]. It is so special that we have a special notation $\log_e(x) = \ln(x)$:

$$\boxed{\ln(x) = c \iff e^c = x. \qquad \text{The domain is } (0, \infty)}$$

In order to move between bases we have:

$$\boxed{\log_a(x) = \frac{\log(x)}{\log(a)} = \frac{\ln(x)}{\ln(a)}} \tag{3}$$

Example 1.4.3. $3^x = 4$, $x =?$ By definition, $x = \log_3(4)$. To compute, using a calculator, we use **3** and punch in $\ln(4)\ln(3)$

Algebraic relations:

1. $\log_a(1) = 0$
2. $\log_a(AB) = \log_a(A) + \log_a(B)$
3. $\log_a(A/B) = \log_a(A) - \log_a(B)$
4. $\log_a(A^p) = p \log_A(A)$
5. $\log_a(a^x) = x$
6. $a^{\log_a(x)} = x$

Example 1.4.4. Let $f(x) = 3(5)^x$ be an exponential function. Find the continuous rate (see **1.2.9**).
We need to replace 5 with e^k so

$$5 = e^k \Rightarrow k = \ln(5).$$

So, the continuous rate is $\ln(5)$.

Example 1.4.5. Solve for x: $7^x = e^{x+3}$

$$7^x = e^{x+3} \Rightarrow 7^x/e^x = e^3 \Rightarrow (7/e)^x = e^3 \Rightarrow x = \ln(e^3)/\ln(7/e) = 3/(\ln(7) - 1)$$

[6] the reason will be revealed later

Exercise 1.4.6. Find the half-life (in hours) of a radioactive substance that is reduced by 35 percent in 20 hours.

Solution. We use $P = P_0 a^t$. We are given the point $(20, 0.65P_0)$. Plug in:

$$P_0 a^{20} = 0.65P_0 \Rightarrow a = (0.65)^{1/20} \Rightarrow P = P_0(0.65)^{t/20}$$

Now, we look for t such that $P = 0.5P_0$. So solve for t: $P = P_0(0.65)^{t/20} = 0.5P_0$:

$$t/20 = log_{0.65}(0.5) = \ln(0.5)/\ln(0.65) \Rightarrow t = 20\ln(0.5)/\ln(0.65) = 32.180811(hours)$$

□

Exercise 1.4.7.

3. [13 points] A wedge of cheese in Zack's refrigerator has become home to a colony of bacteria. Let $A(t)$ be the surface area of the colony (in cm^2) t days after the expiration date of the cheese.

a. [4 points] For the first 20 days after the expiration date, the surface area of the colony grows exponentially. During this time, it takes the colony 5 days to double. Write a formula for $A(t)$ on the domain $0 \leq t \leq 20$. (Your formula may involve an unknown constant, but be sure to specify what this constant means in terms of bacteria.)

b. [3 points] How many days does it take for the surface area of the colony to triple? (Your answer does not need to be a whole number.)

c. [3 points] Twenty days after the expiration date, the bacteria mysteriously begin to die off. The surface area of the colony on the cheese decreases linearly at a rate of 0.3 cm^2/day starting at $t = 20$, and by $t = 22$ the surface area has fallen to 9 cm^2. Given that $A(t)$ is a continuous function, what was the surface area of the colony on the expiration date of the cheese?

Solution.

3. [13 points] A wedge of cheese in Zack's refrigerator has become home to a colony of bacteria. Let $A(t)$ be the surface area of the colony (in cm^2) t days after the expiration date of the cheese.

a. [4 points] For the first 20 days after the expiration date, the surface area of the colony grows exponentially. During this time, it takes the colony 5 days to double. Write a formula for $A(t)$ on the domain $0 \leq t \leq 20$. (Your formula may involve an unknown constant, but be sure to specify what this constant means in terms of bacteria.)

Solution:

$$A(t) = A_0 2^{t/5}, \text{ where } A_0 \text{ is the initial surface area of the colony.}$$

Beginning with $A(t) = A_0 b^t$ we have that the surface area of the bacteria doubles in 5 days, so we set $2A_0 = A_0 b^5$. Then $2^{1/5} = b$.

b. [3 points] How many days does it take for the surface area of the colony to triple? (Your answer does not need to be a whole number.)

Solution:

$$5\frac{\log(3)}{\log(2)} \approx 7.9248 \text{ days}$$

Beginning with our equation from (a) we set $3A_0 = A_0 2^{t/5}$. Taking ln of both sides and simplifying we have $\ln 3/\ln 2 = t/5$.

c. [3 points] Twenty days after the expiration date, the bacteria mysteriously begin to die off. The surface area of the colony on the cheese decreases linearly at a rate of 0.3 cm^2/day starting at $t = 20$, and by $t = 22$ the surface area has fallen to 9 cm^2. Given that $A(t)$ is a continuous function, what was the surface area of the colony on the expiration date of the cheese?

Solution: Working backwards from $t = 22$ we have that the surface area was 0.6 cm^2 more at $t = 20$ than at $t = 22$. This means it was 9.6 cm^2 at $t = 20$ where the exponential growth stopped. Setting

$$9.6 = A_0(2)^{20/5}$$

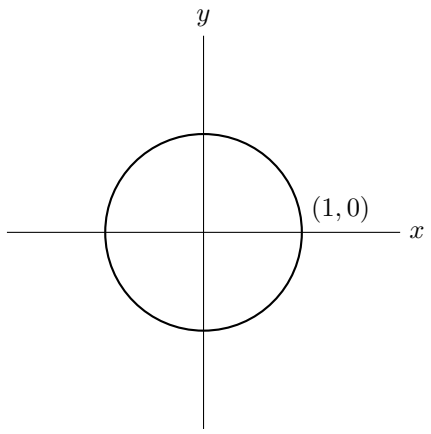
we have

$$A_0 = 0.6\text{cm}^2$$

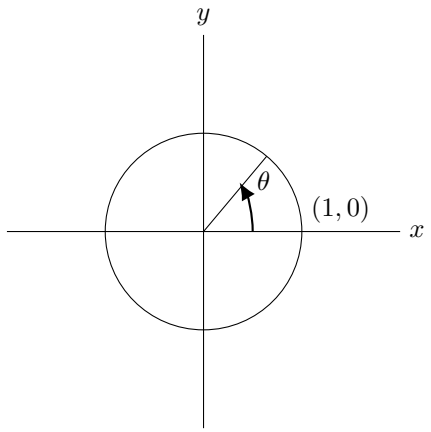
□

1.5 Trig Functions

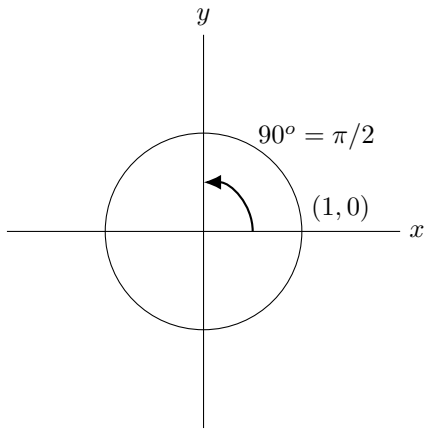
This is the unit circle. It is a circle of radius 1 centered at the origin.



We measure degrees starting from the positive x-axis



We define the angle that cuts out an arc of length 1 to be an angle of 1 radian. It turns out that 180° are π radians, 90° are $\pi/2$.



The conversion is:

$$\boxed{(degrees)\frac{\pi}{180} = (radians), \ \ (degrees) = \frac{180}{\pi}(radians)}$$

Example 1.5.1. Convert $\pi/6$ to degrees. $\frac{pi}{6}\frac{180}{\pi} = 180/6 = 30^o$.

Degreed can be measured in any circle of radius r. The formula of the arc length (cut by an angle θ) is:

$$r\theta \ \text{ (when } \theta \text{ is in radians)}, \ r\theta\frac{\pi}{180} \ \text{ (when } \theta \text{ is in degrees)}$$

Back to the unit circle. We define two special functions. $\cos(t), \sin(t)$ give us the x, y coordinates of a point on the unit circle for a given angle t:

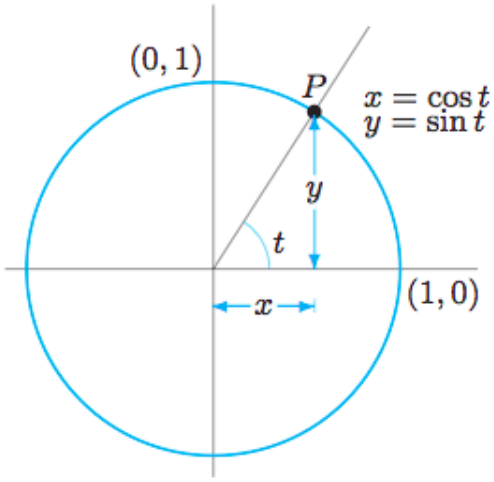
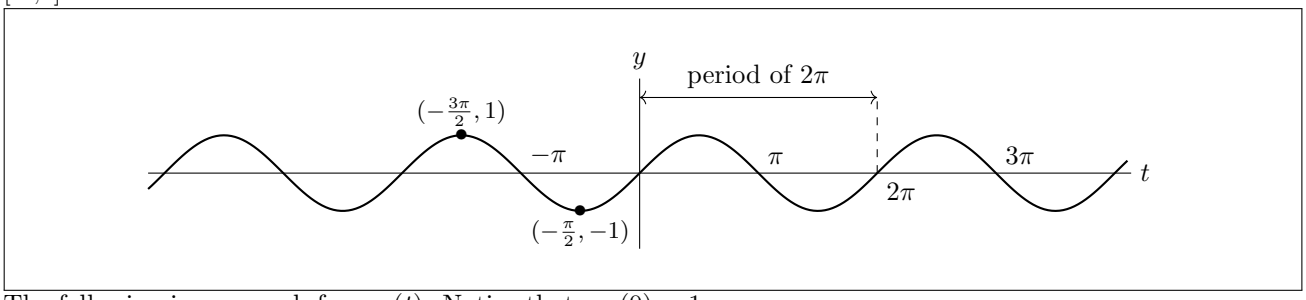
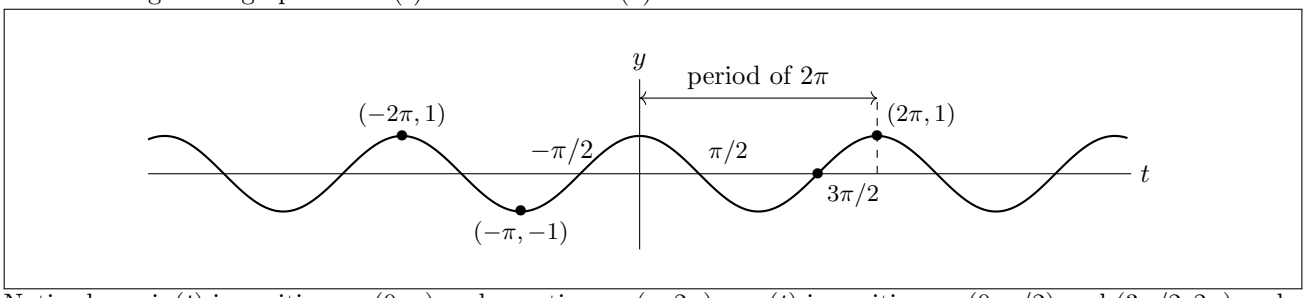


Figure 1.50: The definitions of $\sin t$ and $\cos t$

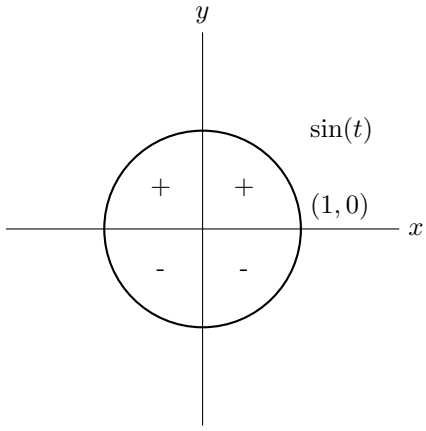
The graph of $\sin(t)$ looks like the following. It repeats itself every 2π, so we say that $\sin(t)$ is a periodic function with period 2π. The range is $-1 \leq \sin(t) \leq 1$. We say that $\sin(t)$ has amplitude 1 since the range is [-1,1].

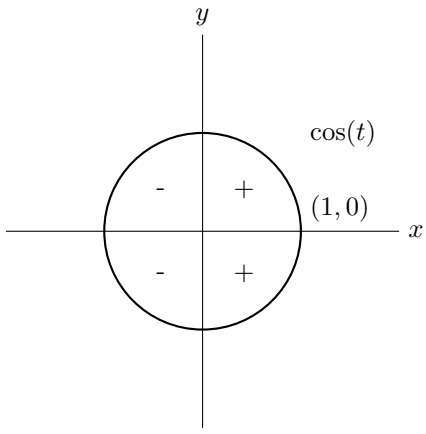


The following is a a graph for $\cos(t)$. Notice that $\cos(0) = 1$.



Notice how $\sin(t)$ is positive on $(0, \pi)$ and negative on $(\pi, 2\pi)$. $\cos(t)$ is positive on $(0, \pi/2)$ and $(3\pi/2, 2\pi)$ and negative on $(\pi/2, 3\pi/2)$. This is how you can remember it:





Formulas:

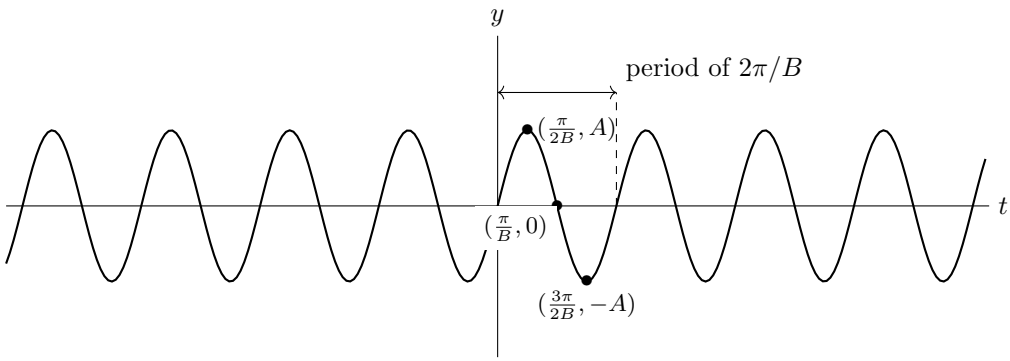
$\sin(-t) = -\sin(t)$	an odd function
$\cos(-t) = \cos(t)$	an even function
$\cos(t) = \sin(t + \pi/2)$	
$\sin(t) = \cos(t - \pi/2)$	
$\cos^2(t) + \sin^2(t) = 1$	

A **Sinusoidal Function** is a function which can be represented like so:

$$f(t) = A\sin(Bt), \text{ or } f(t) = A\cos(Bt)$$

- The amplitude is $A = (\text{max-min})/2$.
- The period is $2\pi/B$.
- One can add vertical or horizontal shifts.
- When the function "starts" at a max or min point, it is better to work with $f(t) = A\cos(Bt)$. If it satrts from the "middle", it is better to work with $f(t) = A\sin(Bt)$

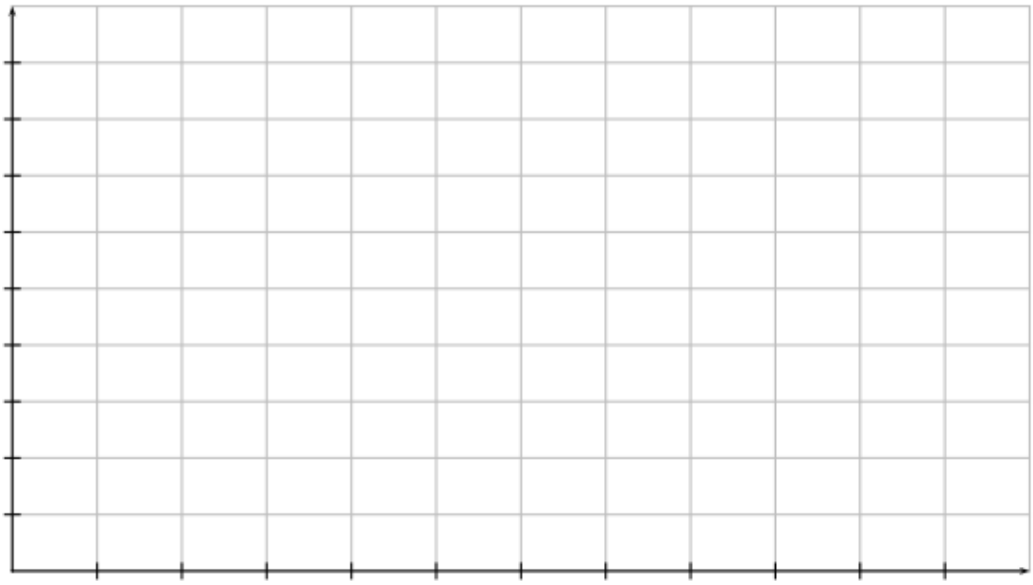
For example, this is $f(t) = A\sin(Bt)$ with $A > 1, B > 1$.



Exercise 1.5.2.

6. [12 points] At the county fair, there is a ferris wheel with radius 40 feet. Riders board at the lowest point of the ferris wheel, from a platform 10 feet off the ground. Once the ride begins, the ferris wheel completes 3 revolutions in 120 seconds. Suppose that you are the last rider to board (so you begin at the lowest point), and the function $H(t)$ measures your height off the ground (in feet), t seconds after the ride starts.

a. [4 points] On the grid below, sketch a graph $H(t)$ for one complete ride (3 revolutions). Be sure to carefully label the axes.



b. [4 points] Find the period and amplitude of $H(t)$.

period = _____

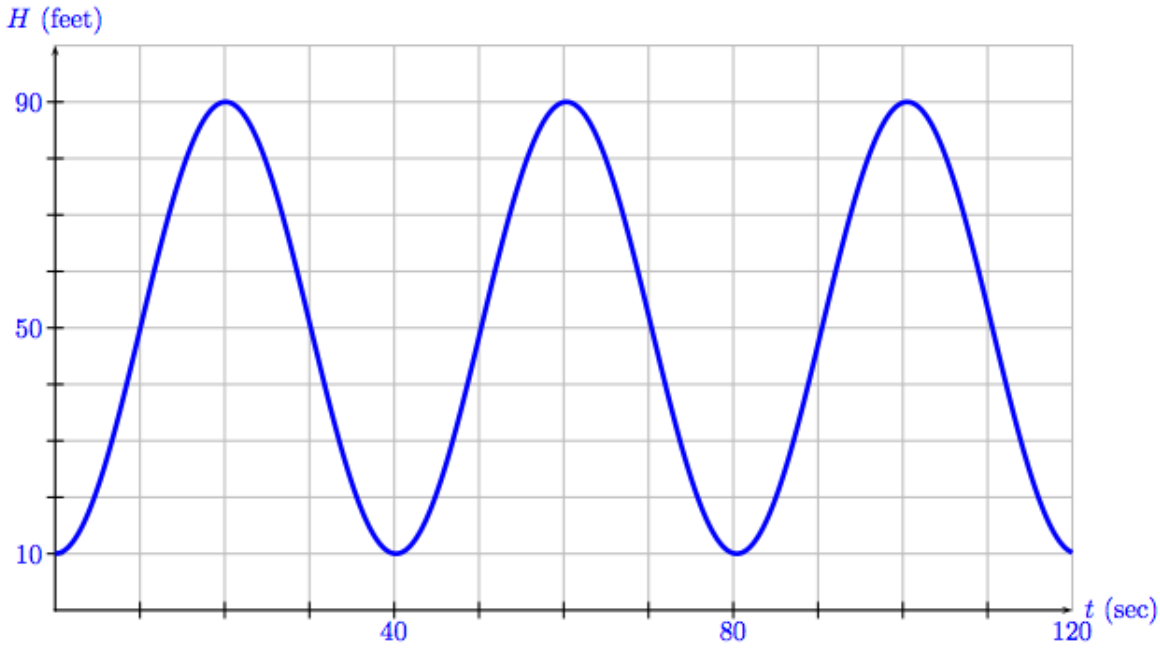
amplitude = _____

c. [4 points] Find a formula for $H(t)$.

Solution.

6. [12 points] At the county fair, there is a ferris wheel with radius 40 feet. Riders board at the lowest point of the ferris wheel, from a platform 10 feet off the ground. Once the ride begins, the ferris wheel completes 3 revolutions in 120 seconds. Suppose that you are the last rider to board (so you begin at the lowest point), and the function $H(t)$ measures your height off the ground (in feet), t seconds after the ride starts.

a. [4 points] On the grid below, sketch a graph $H(t)$ for one complete ride (3 revolutions). Be sure to carefully label the axes.



b. [4 points] Find the period and amplitude of $H(t)$.

Solution: 40 seconds

period = _____

Solution: 40 feet

amplitude = _____

c. [4 points] Find a formula for $H(t)$.

Solution:

$$H(t) = 50 - 40\cos\left(\frac{\pi}{20}t\right)$$

□

For (c): since the period is $2\pi/B = 40$ then $B = \pi/20$. Then we better pick cos rather then sin since we begin at the min/max points instead of the middle point. We need to apply vertical flip, and move to match the graph. The amplitude is (max-min)/2=(90-10)/2=40.

Exercise 1.5.3.

7. [15 points] In each of the following problems, give a *formula* for a function whose domain is all real numbers, with *all* of the indicated properties. If there is no such function, then write "NO SUCH FUNCTION EXISTS". You do not need to show your work.

a. [6 points] A sinusoidal function $P(t)$ with the following three properties:
(i.) The period of the graph of $P(t)$ is 7.
(ii.) The graph of $P(t)$ attains a maximum value at the point $(1, 20)$.
(iii.) The graph of $P(t)$ attains a minimum value at the point $(-2.5, -6)$.

$P(t) =$ _____

b. [3 points] A function $h(x)$ with the following two properties:
(i.) $h(x)$ is concave down for all x
(ii.) $h(x) > 0$ for all x.

$h(x) =$ _____

c. [3 points] A function $j(x)$ with the following two properties:
(i.) $j(x)$ is decreasing for all x.
(ii.) $j(x)$ is concave up for all x.

$j(x) =$ _____

Solution.

7. [15 points] In each of the following problems, give a *formula* for a function whose domain is all real numbers, with *all* of the indicated properties. If there is no such function, then write "NO SUCH FUNCTION EXISTS". You do not need to show your work.

a. [6 points] A sinusoidal function $P(t)$ with the following three properties:
(i.) The period of the graph of $P(t)$ is 7.
(ii.) The graph of $P(t)$ attains a maximum value at the point $(1, 20)$.
(iii.) The graph of $P(t)$ attains a minimum value at the point $(-2.5, -6)$.

Solution: If we move the function to the left by 1, we get a cosine function with period 7, amplitude 13, and vertical shift 7. Call this shifted function $\widetilde{P}(t)$. Since

$$\widetilde{P}(t) = 13\cos\left(\frac{2\pi}{7}t\right) + 7$$

and $P(t)$ is $\widetilde{P}(t)$ shifted right by one, we get

$$P(t) = 13\cos\left(\frac{2\pi}{7}(t-1)\right) + 7$$

b. [3 points] A function $h(x)$ with the following two properties:
(i.) $h(x)$ is concave down for all x
(ii.) $h(x) > 0$ for all x.

Solution: No such function exists. If the function is decreasing at some point, it will decrease faster and faster until it touches the x-axis. If the function is increasing at some point, similar logic applies (reading right to left, rather than left to right). The only other possibility is that the function is not increasing or decreasing anywhere, but then it would just be a horizontal line with no concavity.

c. [3 points] A function $j(x)$ with the following two properties:
(i.) $j(x)$ is decreasing for all x.
(ii.) $j(x)$ is concave up for all x.

Solution: One example of such a function that we have encountered is an exponential decay function, for instance $j(x) = e^{-x}$.

□

The next trig function:

$$\boxed{\tan(t) = \frac{\sin(t)}{\cos(t)}, \quad \text{period}=\pi}$$

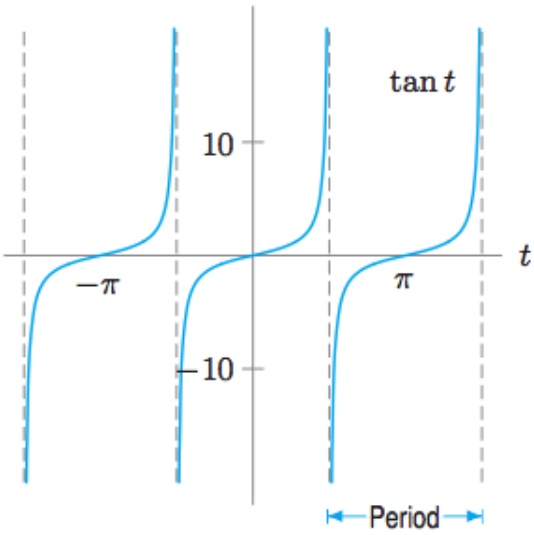
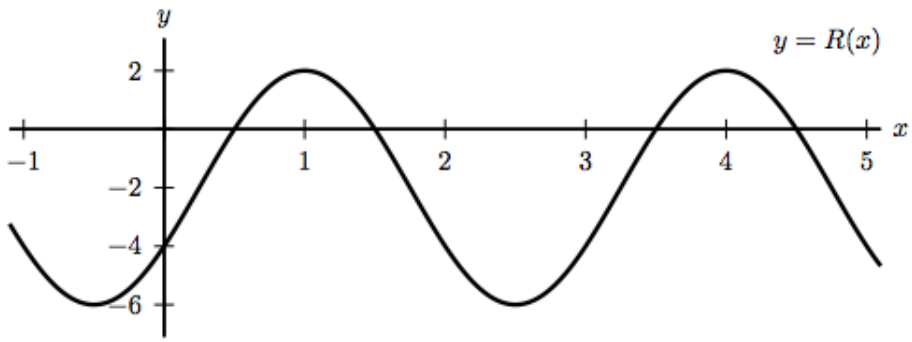


Figure 1.56: The tangent function

Exercise 1.5.4.

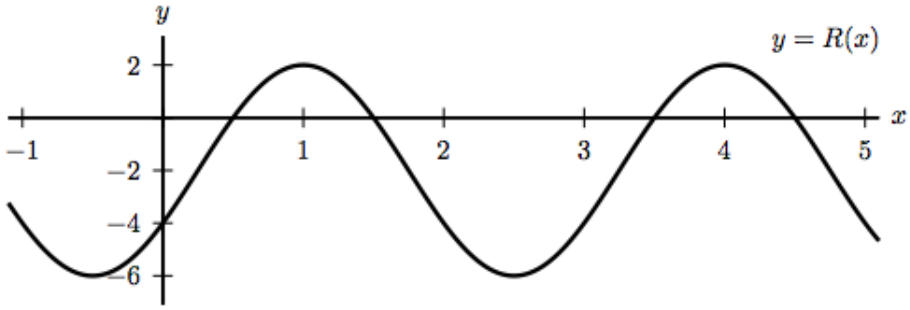
8. [6 points] Given below is the graph of a sinusoidal function $R(x)$.



Find a possible formula for $R(x)$.

Solution.

8. [6 points] Given below is the graph of a sinusoidal function $R(x)$.



Find a possible formula for $R(x)$.

Solution: The graph shown above is of a sinusoidal function with amplitude 4, period 3, and midline $y = -2$. We first consider the graph of

$$y = 4\cos\left(\frac{2\pi}{3}x\right) - 2\,.$$

This graph has the proper amplitude, period, and midline. We shift this graph over to the right 1 unit to obtain the graph of $y = R(x)$. Thus, one possible formula for $R(x)$ is given by

$$R(x) = 4\cos\left(\frac{2\pi}{3}(x-1)\right) - 2.$$

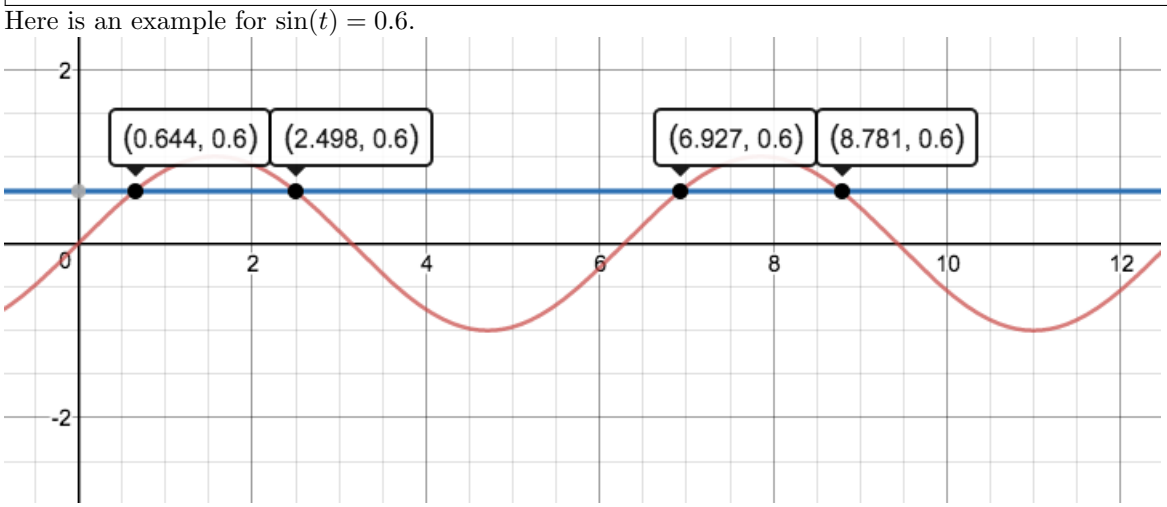
Answer: $R(x) = 4\cos\left(\frac{2\pi}{3}(x-1)\right) - 2$

□

Important note about the inverse of trig functions:
When we solve for t: $\sin(t) = c$, we expect a lot of solutions. The function $\sin(t)$ does not has an inverse as is, but if we limit the domain to $[-\pi/2, \pi/2]$ then we can define the inverse $\sin^{-1}(c) = t$, and the answer is between $-\pi/2$ to $\pi/2$. But, we need to find many solutions, and we find them using the value of $\sin^{-1}(c)$ and the
symmetry of the function. we have 2 solutions "main" $\sin^{-1}(c), (\pi - \sin^{-1}(c))$ and plus or minus multiples of 2π:

$$\sin(t) = c \Rightarrow \begin{aligned} &t = \sin^{-1}(c) \pm 2\pi, \sin^{-1}(c) \pm 4\pi, \sin^{-1}(c) \pm 6\pi, \ldots \\ &t = (\pi - \sin^{-1}(c)) \pm 2\pi, (\pi - \sin^{-1}(c)) \pm 4\pi, (\pi - \sin^{-1}(c)) \pm 6\pi, \ldots \end{aligned}$$

Here is an example for $\sin(t) = 0.6$.



For $\cos(t) = c$ the 2 "main" solutions are $\pm\cos^{-1}(c)$, and plus or minus multiples of 2π:

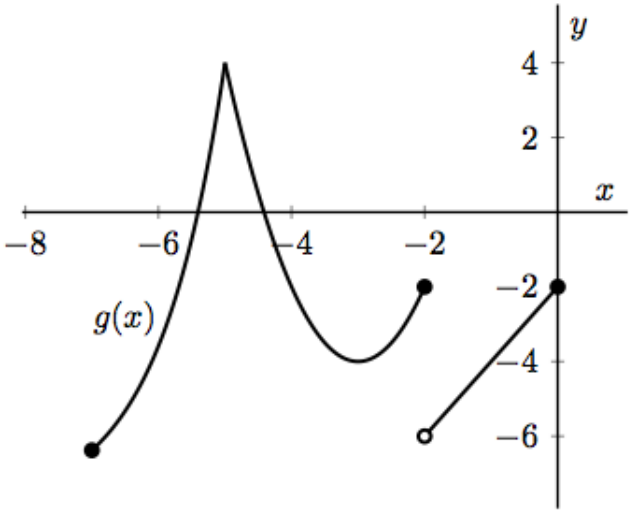
$$\cos(t) = c \Rightarrow$$
$$t = \pm\cos^{-1}(c) \pm 2\pi, \pm\cos^{-1}(c) \pm 4\pi, \pm\cos^{-1}(c) \pm 6\pi, ...$$

1.6 Polynomials and Rational Functions

We will come back to this section once we are done with continuity.

1.7 Continuity

Graphically, a function is continuous (CTS) on the domain $[a, b]$ if we can sketch its graph using one stroke. For example, observe $g(x)$:

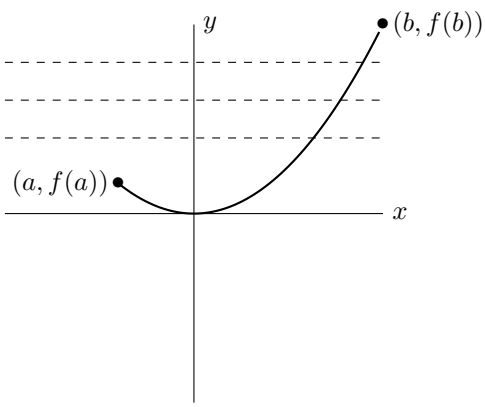


It is not continuous on [-7,0], or on [-3,-1], but it IS continuous on [-7,-2] or on [-1,0].
Notice, the function is not continuous on [-2,0] because the value at -2 is -2, but the value immediately to the right of it are close to -6. By convension, if $f(x)$ is not defined at $x = a$, then it is not continuous there.
There is one important, albeit intuitive, theorem about continuity. From the textbook:

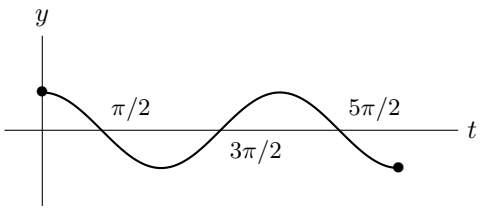
Theorem 1.1: Intermediate Value Theorem

Suppose f is continuous on a closed interval $[a, b]$. If k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

Graphically, the theorem is almost obvious since the graph must intersect any horizontal line between $f(a)$ and $f(b)$:



Note the the theorem says "at least one". It can be more. $\cos(t)$ is continuous on $[0, 3\pi]$, $\cos(0) = 1$ and $\cos(3\pi) = -1$, so the theorem guarantees at least one c such that $\cos(c) = 0$, but we can find 3:



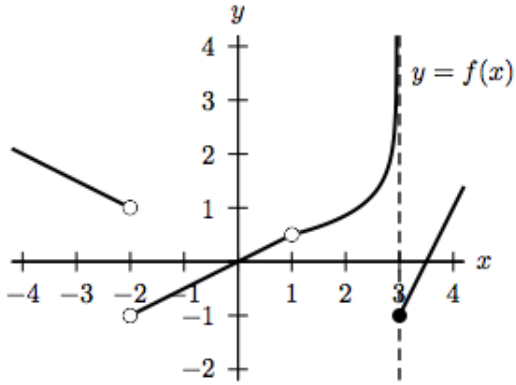
Most functions that you know are continuous when defined:
$y = x, y = x^2, y = x^3, ..., y = a^x, y = \ln(x), \sin(x), \cos(x)$. In order for a function to be discontinuous one should either:

- "cook it up" by using a piecewise function, build out of incompatible graphs like the the first example.
- Use a fraction of functions, and then (usually) when the denominator is 0 we get a point of discontinuity. For example, refer to the graph of $\tan(x) = \sin(x)/\cos(x)$. Whenever $\cos(x)$ is 0, the graph is discontinuous.

Note that adding, multiplying, and composing continuous functions yields a continuos function.

Exercise 1.7.1.

6. [11 points] Below is the graph of a portion of a function $f(x)$.



a. [2 points] Give all values of a in the interval $-4 < a < 4$ that are not in the domain of $f(x)$. If there are none, write NONE.

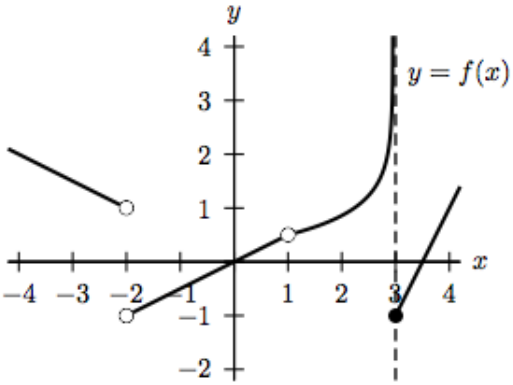
Answer: _____

b. [2 points] Give all values of a in the interval $-4 < a < 4$ where $f(x)$ is not continuous at $x = a$. If there are none, write NONE.

Answer: _____

Solution.

6. [11 points] Below is the graph of a portion of a function $f(x)$.



a. [2 points] Give all values of a in the interval $-4 < a < 4$ that are not in the domain of $f(x)$. If there are none, write NONE.

Answer: $-2,\ 1$

b. [2 points] Give all values of a in the interval $-4 < a < 4$ where $f(x)$ is not continuous at $x = a$. If there are none, write NONE.

Answer: $-2,\ 1,\ 3$

□

Back to Section 1.6 - Polynomials and Rational Functions

This is a power function:

$$f(x) = kx^p$$

where k, p are constant.

Usually, we look at positive integer powers: $x^1, x^2, x^3, \ldots$. There is a difference of behavior between even and odd functions:

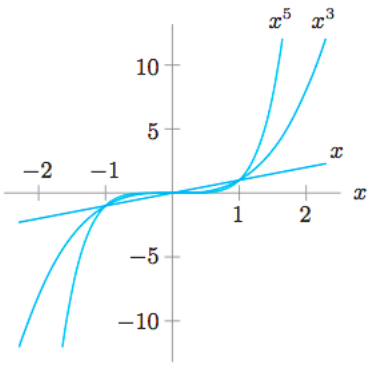


Figure 1.66: Odd powers of x: "Seat" shaped for $n > 1$

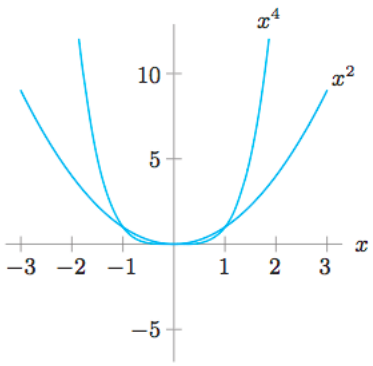


Figure 1.67: Even powers of x: ⋃-shaped

The function become more extreme as the power becomes bigger.

Property: exponential growth a^x with $a > 1$ will eventually be bigger than (dominate) x^p for any $p > 0$.

Here is a demonstration:

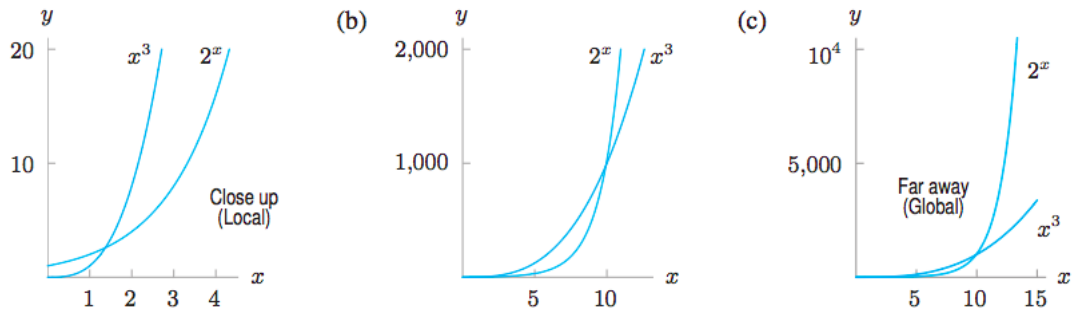
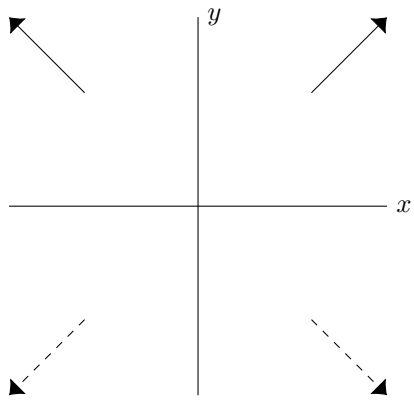


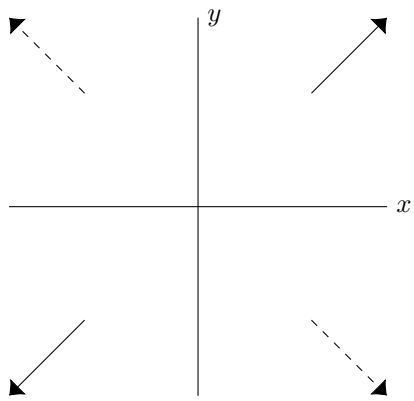
Figure 1.68: Comparison of $y = 2^x$ and $y = x^3$: Notice that $y = 2^x$ eventually dominates $y = x^3$

End behavior is how the functions behave when $x \to \pm\infty$. Notice the long term behavior $f(x) = kx^n$ depends on the sign of k and whether n is even or odd:

$f(x) = kx^n$, $n > 0$ even, k positive, and k negative(dashed)



$f(x) = kx^n$, n odd, k positive, and k negative(dashed)



A **polynomial** is a sum of power functions with integer powers. Example:

- $p(x) = 1 + x^2$
- $p(x) = 2x^3 + 3x^{100}$
- $p(x) = -4x^3 - \pi x^{100} - 1$

When $x \to \pm infty$, $p(x)$ behave like the power function with the highest power in it. The highest power is called the **Degree** of $p(x)$

Example 1.7.2. $p(x) = 100000000000x^3 - 0.00000001x^4$ will go to $-\infty$ when $x \to \infty$ since it behaves like $-0.00000001x^4$ there.

Theorem: For a polynomial $p(x)$, the number of solutions of $p(x) = c$ at most the degree of $p(x)$. If $p(x)$ is of odd degree, then the number of solutions is at least 1.

The graphical interpretation is: every horizontal line intersects at most "degree" times.

exm Multiple choice: How may x-intercepts does $x^3 - x^2$ has:

(a) 0

(b) 2

(c) 4

(d) 8

Solution: we need to solve $p(x) = 0$. So the number of solutions is at least 1 and must be at most 3, so the answer must be (b). We can also see it using a direct computation:

$$0 = x^3 - x^2 = x^2(x-1) = (x-0)(x-0)(x-1),$$

so the roots are 0,0,1.

Example 1.7.3. Find a polynomial of degree 4 that has only 2 solutions for $p(x) = 1$.
We do it using the following trick. We define $q(x) = p(x) - 1$, which is also a polynomial of the same degree. Now, we need to find a polynomial with 2 solutions for $q(x) = 0$ (roots). So take:
$q(x) = (x-0)(x-0)(x-1)(x-1) = x^2(x-1)^2 = x^2(x^2-2x+1) = x^4 - 2x^3 + x^2$ so $p(x) = x^4 - 2x^3 + x^2 + 1$.

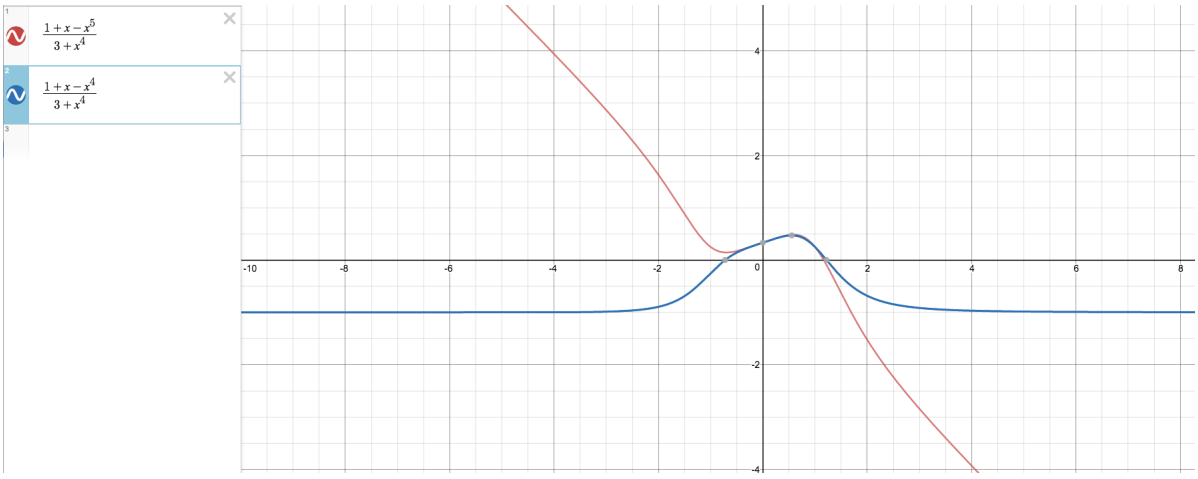
A rational function is a fraction of a polynomial over polynomial $f(x) = \frac{p(x)}{q(x)}$. For example:

- $y = \frac{x+1}{x^5}$
- $y = \frac{4+x+x^3}{1-x-x^2}$
- $y = \frac{ex^3}{\pi x^4}$

End Behavior of $f(x) = \frac{p(x)}{q(x)}$ is set by the highest power function in the numerator and the denominator $\frac{k_1 x^n}{k_2 x^m}$:

- if $n > m$, then when $x \to \pm\infty$, $f(x) \to \pm\infty$.
- If $m > n$, then when $x \to \pm\infty$, $f(x) \to 0$. This is a **Horizontal Asymptote** $y = 0$
- if $n = m$, then when $x \to \pm\infty$, $f(x) \to \frac{k_1}{k_2}$. This is a **Horizontal Asymptote** $y = \frac{k_1}{k_2}$

Example 1.7.4. $f(x) = \frac{1+x-x^5}{3+x^4}$. So when $x \to \pm\infty$, $f(x)$ behaves like $\frac{-x^5}{x^4} = -x$. Thus when $x \to \infty, f(x) \to -\infty$ and when $x \to -\infty, f(x) \to +\infty$.
$f(x) = \frac{1+x-x^4}{3+x^4}$. So when $x \to \pm\infty$, $f(x)$ behaves like $\frac{-x^x}{x^4} = -1$. Thus when $x \to \infty, f(x) \to -1$. It has a horizontal asymptote $y = -1$.



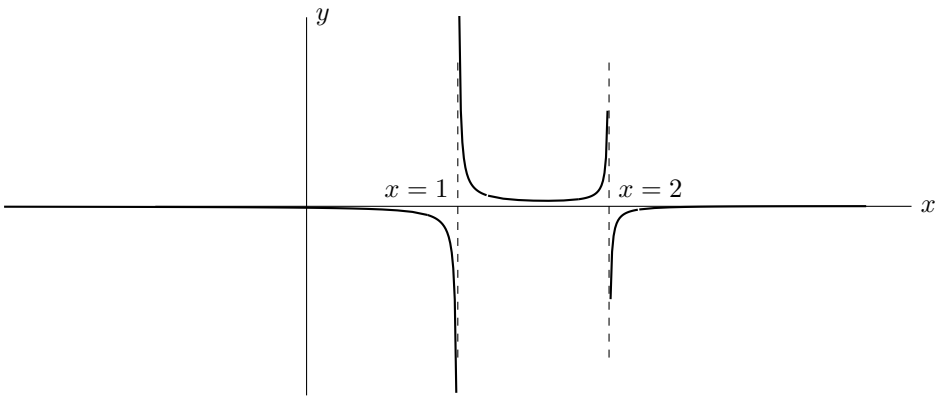
A Vertical Asymptote for $f(x) = \frac{p(x)}{q(x)}$ can occur only at x's which are roots of $q(x)$.

So we find all the candidates by solving $q(x) = 0$ and then checking the neighborhood of those solutions. It is also useful to find the roots of $p(x)$ and try to cancel out candidates.

Example 1.7.5. For $f(x) = \frac{x^2-6x+9}{(x-1)(x-2)(x-3)}$. We have 3 candidates for a vertical asymptote: $x = 1, 2, 3$. For $x = 1$, let us plug in 0.9 and then 0.99. We get approximately $-20, -200$ and then 1.1. For $1.01, 1.1$ we get 200 and 20.
We do the same around 2 and 3. We get:

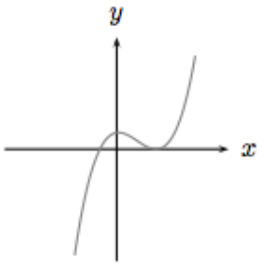
x	f(x)
0.9	-19.09
0.99	-199.01
1.01	201.01
1.1	21.11
1.9	12.22
1.99	102.02
2.01	-98.02
2.2	-3.33
2.9	-0.06
2.99	-0.01
3.01	0.00
3.1	0.04

So we get that actually $f(3) = 0$ and we do not have an asymptote there. Here is the graph:



Exercise 1.7.6.

7. [8 points] For each of the graphs below, select the formula beneath the graph which best fits the behavior of the graph. In each case, assume that A, B, C, D, E, F, and G are positive constants. (Circle your choice. No work or explanation is necessary.)

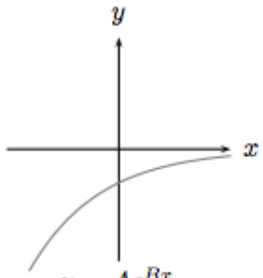


$y = A(x-B)(x+C)$

$y = A(x-B)^2(x+C)$

$y = -A(x+B)^2(x-C)$

$y = A(x+B)^2(x-C)$

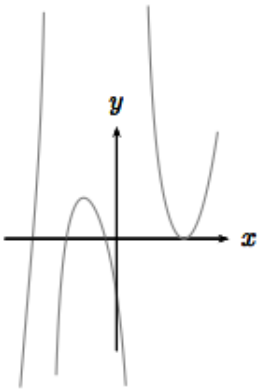


$y = Ae^{Bx}$

$y = Ae^{-Bx}$

$y = -Ae^{Bx}$

$y = -Ae^{-Bx}$

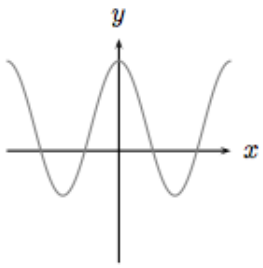


$$y = \frac{A(x-B)(x-C)(x-D)(x+E)^2}{(x+F)(x-G)}$$

$$y = \frac{A(x+B)(x+C)(x+D)(x-E)}{(x+F)(x-G)}$$

$$y = \frac{A(x+B)(x+C)(x+D)(x-E)^2}{(x+F)(x-G)}$$

$$y = \frac{-A(x+B)(x+C)(x+D)(x-E)^2}{(x+F)(x-G)^2}$$



$y = -B\cos(Cx) - A$

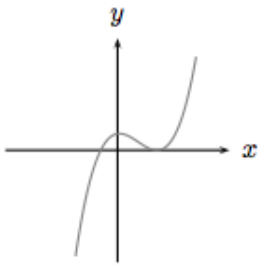
$y = A + B\cos(Cx)$

$y = -A + B\sin(Cx + D)$

$y = A - B\sin(Cx)$

Solution.

7. [8 points] For each of the graphs below, select the formula beneath the graph which best fits the behavior of the graph. In each case, assume that A, B, C, D, E, F, and G are positive constants. (Circle your choice. No work or explanation is necessary.)

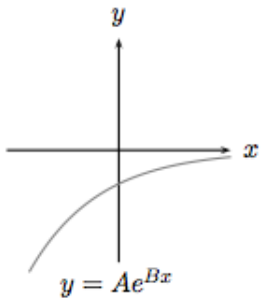


$y = A(x-B)(x+C)$

$\boxed{y = A(x-B)^2(x+C)}$

$y = -A(x+B)^2(x-C)$

$y = A(x+B)^2(x-C)$

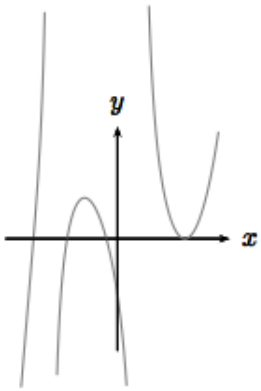


$y = Ae^{Bx}$

$y = Ae^{-Bx}$

$y = -Ae^{Bx}$

$\boxed{y = -Ae^{-Bx}}$

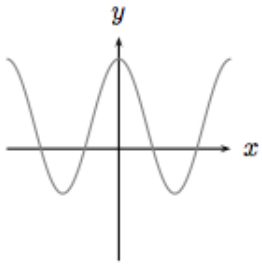


$y = \dfrac{A(x-B)(x-C)(x-D)(x+E)^2}{(x+F)(x-G)}$

$y = \dfrac{A(x+B)(x+C)(x+D)(x-E)}{(x+F)(x-G)}$

$\boxed{y = \dfrac{A(x+B)(x+C)(x+D)(x-E)^2}{(x+F)(x-G)}}$

$y = \dfrac{-A(x+B)(x+C)(x+D)(x-E)^2}{(x+F)(x-G)^2}$



$y = -B\cos(Cx) - A$

$\boxed{y = A + B\cos(Cx)}$

$y = -A + B\sin(Cx + D)$

$y = A - B\sin(Cx)$

Upper left: the degree must be at least 3, the end behavior match to Ax^3, and it must have 2 intercepts. Further we wee that the x axis is tanget to the positive root so it must be at least a double root. In general:

$\boxed{\text{If } f(a) = 0 \text{ and the } x\text{-axis is tanget to the graph of } f(x) \text{ at } x = a \text{ then } x = a \text{ is at least a double root of } f(x).}$

Upper right: This is a exponential decay, flipped.
Lower left: end behavior should be Ax^n with n an odd integer. This disqualifies options 2 and 4. Notice it has 3 negative $x - intercepts$ so only option 3 is valid.
Lower right: Must be cosine without a flip.

□

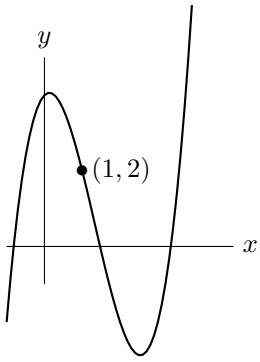
1.8 Limits

From the book:

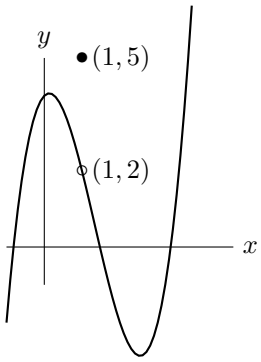
> We write $\lim_{x \to c} f(x) = L$ if the values of $f(x)$ approach L as x approaches c.

The process of finding the limit is a dynamic one. The value L is a result of an estimation based on the values of $f(x)$ around $x = c$. Graphically, we need to "walk" on the graph from left to right until we hit $x = c$, record the values and then estimate what $f(c)$ should have been based on the values we have seen so far. We do the same for right to left. These are called the left side limit $\lim_{x \to c^-} f(x)$, and the right side limit $\lim_{x \to c^+} f(x)$, and we need them to be equal in order to conclude that the limit exists.
Note: the actual value of $f(x)$ at $x = c$ does NOT affect the limit at $x \to c$. $f(x)$ may not be even defined at $x = c$.

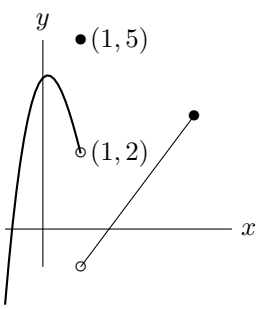
Example 1.8.1.



$\lim_{x \to 1} f(x) = 2$
Can be seen by "walking" on the graph around $x = 1$



$\lim_{x \to 1} f(x) = 2$
$f(1) = 5$ but is does not change the limit computation



The limit is 2 from the left and -1 from the right,
so the limit does not exist

When a function is given as a formula, we can estimate limits using a calculator.[7]

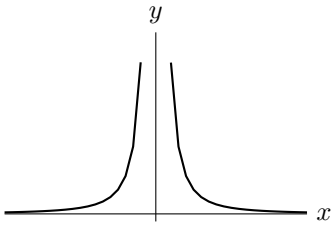
Example 1.8.2. Compute: $\lim_{t \to 0} \frac{\sin(t)}{t}$. Note, $f(t) = \frac{\sin(t)}{t}$ is not defined at $t = 0$ but that does not matter for the limit computation. We are using a table of values to produce an estimation:

t	sin(t)/t
-0.100	0.99833417
-0.010	0.99998333
-0.001	0.99999983
0.001	0.99999983
0.010	0.99998333
0.100	0.99833417

[7] the trick is to use the option to store values as instructed in class

So we estimate $\lim_{x\to 1} f(t) = 1$.

A limit can be also infinite. For example $f(x) = 1/x^2$ has a vertical asymptote at $x = 0$ so $\lim_{x\to 0} f(x) = \infty$



Arithmetics of Limits:

Theorem 1.2: Properties of Limits

Assuming all the limits on the right-hand side exist:

1. If b is a constant, then $\lim_{x\to c}(bf(x)) = b\left(\lim_{x\to c} f(x)\right)$.
2. $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$.
3. $\lim_{x\to c}(f(x)g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$.
4. $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided $\lim_{x\to c} g(x) \neq 0$.
5. For any constant k, $\lim_{x\to c} k = k$.
6. $\lim_{x\to c} x = c$.

Limits and Continuity: A function is continuous around $x = c$ if and only if $\lim_{x\to c} f(x) = f(c)$.
So, for continuous functions we do not need to go through a dynamic process since we have this shortcut — just take the value at $x = c$.
Limits at Infinity: For $x \to \pm\infty$ we need to provide an estimate for the "End behavior" based on either "walking" on the graph to $\pm\infty$, or providing an estimate by plugging in larger and larger x's (for $x \to +\infty$) or smaller and smaller x's (for $x \to -\infty$). If one of those exists, we have a horizontal asymptote.

Exercise 1.8.3.

3. [9 points] Consider the function h defined by

$$h(x) = \begin{cases} \dfrac{60(x^2 - x)}{(x^2+1)(3-x)} & \text{for } x < 2 \\ c & \text{for } x = 2 \\ 5e^{ax} - 1 & \text{for } x > 2 \end{cases}$$

where a and c are constants.

a. [5 points] Find values of a and c so that both of the following conditions hold.

- $\lim_{x \to 2} h(x)$ exists.
- $h(x)$ is not continuous at $x = 2$.

Note that this problem may have more than one correct answer. You only need to find one value of a and one value of c so that both conditions above hold. Remember to show your work clearly.

Answer: $a =$ _____ and $c =$ _____

b. [2 points] Determine $\lim_{x \to -\infty} h(x)$. If the limit does not exist, write DNE.

Answer: $\lim_{x \to -\infty} h(x) =$ _____

c. [2 points] Find all vertical asymptotes of the graph of $h(x)$. If there are none, write NONE.

Answer: Vertical asymptote(s): _____

Solution.

3. [9 points] Consider the function h defined by

$$h(x) = \begin{cases} \dfrac{60(x^2-x)}{(x^2+1)(3-x)} & \text{for } x < 2 \\ c & \text{for } x = 2 \\ 5e^{ax} - 1 & \text{for } x > 2 \end{cases}$$

where a and c are constants.

a. [5 points] Find values of a and c so that both of the following conditions hold.

- $\lim_{x\to 2} h(x)$ exists.
- $h(x)$ is not continuous at $x = 2$.

Note that this problem may have more than one correct answer. You only need to find one value of a and one value of c so that both conditions above hold. Remember to show your work clearly.

Solution: In order for $\lim_{x\to 2} h(x)$ to exist, it must be true that $\lim_{x\to 2^-} h(x) = \lim_{x\to 2^+} h(x)$. Now $\lim_{x\to 2^-} h(x) = \dfrac{60(2^2-2)}{(2^2+1)(3-2)} = 24$ and $\lim_{x\to 2^+} h(x) = 5e^{2a} - 1$. So it follows that $5e^{2a} - 1 = 24$. Solving for a, we have

$$\begin{aligned} 5e^{2a} - 1 &= 24 \\ e^{2a} &= 5 \\ 2a &= \ln(5) \\ a &= \ln(5)/2 \approx 0.804. \end{aligned}$$

When $a = \ln(5)/2$, $\lim_{x\to 2} h(x) = 5e^{(\ln(5)/2)*2} = 5e^{\ln(5)} - 1 = 24$. So, h is not continuous at $x = 2$ as long as $\lim_{x\to 2} h(x) \neq h(2)$. Since $h(2) = c$, we can choose c to be any number other than 24.

Answer: $a =$ $\ln(5)/2$ and $c =$ **7 (or any value other than 24)**

b. [2 points] Determine $\lim_{x\to -\infty} h(x)$. If the limit does not exist, write DNE.

Solution: By looking at the rational function

$$\frac{60(x^2-x)}{(x^2+1)(3-x)} = \frac{60(x^2-x)}{-x^3+3x^2-x+3},$$

(the relevant piece of the function here) we see that as $x \to -\infty$, $h(x)$ approaches 0.

Answer: $\lim_{x\to -\infty} h(x) =$ 0

c. [2 points] Find all vertical asymptotes of the graph of $h(x)$. If there are none, write NONE.

Answer: Vertical asymptote(s): NONE

□

Exercise 1.8.4.

9. [4 points] The table below gives several values of a function $w(x)$.

x	4.5	4.9	4.99	5	5.01	5.1	5.5
$w(x)$	-0.879	-0.154	-0.015	0	0.060	0.630	3.750

Use the information in the table above to estimate the following limit.

$$\lim_{h \to 0^-} \frac{w(5+h)}{h}$$

Clearly show any computations that you use to make this estimate.

Answer: $\lim_{h \to 0^-} \frac{w(5+h)}{h} \approx$ _____

Solution.

9. [4 points] The table below gives several values of a function $w(x)$.

x	4.5	4.9	4.99	5	5.01	5.1	5.5
$w(x)$	-0.879	-0.154	-0.015	0	0.060	0.630	3.750

Use the information in the table above to estimate the following limit.

$$\lim_{h \to 0^-} \frac{w(5+h)}{h}$$

Clearly show any computations that you use to make this estimate.

Solution: The left limit can be approximated by $w(5+h)/h$ for small negative values of h. The table of values provided for w allows us to compute this when $h = -0.1$ and when $h = -0.01$. The results are shown in the table below.

h	$\frac{w(5+h)}{h}$
-0.1	$\frac{w(4.9)}{-0.1} = 1.54$
-0.01	$\frac{w(4.99)}{-.01} = 1.5$

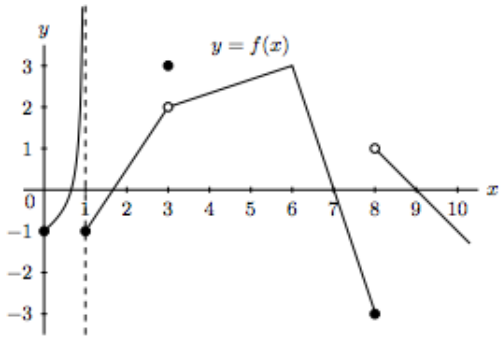
Using these values, we estimate that the desired left-hand limit is approximately 1.5.

Answer: $\lim_{h \to 0^-} \frac{w(5+h)}{h} \approx$ 1.5

□

Exercise 1.8.5.

8. [12 points] A portion of the graph of a function f is shown below.



a. [2 points] Give all values c in the interval $0 < c < 10$ for which $\lim_{x \to c} f(x)$ does not exist. If there are none, write NONE.

Answer: $c =$ _____

b. [2 points] Give all values c in the interval $0 < c < 10$ for which $\lim_{x \to c^+} f(x)$ does not exist. If there are none, write NONE.

Answer: $c =$ _____

c. [2 points] Give all values c in the interval $0 < c < 10$ for which $f(x)$ is not continuous at c. If there are none, write NONE.

Answer: $c =$ _____

d. [6 points] With f as shown in the graph above, define a function g by the formula

$$g(x) = \begin{cases} \dfrac{B + 2x^2 + 3x^3 + Ax^5}{12 + 6x^3 + 4x^5} & \text{if } x \leq 0 \\ f(x) & \text{if } 0 < x < 10 \end{cases}$$

where A and B are nonzero constants.
Find values of A and B so that both of the following conditions hold.

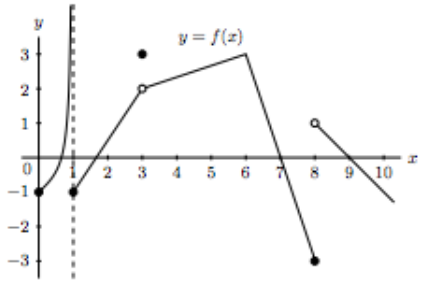
- $g(x)$ is continuous at $x = 0$.
- $\lim_{x \to -\infty} g(x) = \frac{1}{2}$.

If no such values exist, write NONE in the answer blanks.
Be sure to show your work or explain your reasoning.

Answer: $A =$ _____ and $B =$ _____

Solution.

8. [12 points] A portion of the graph of a function f is shown below.



a. [2 points] Give all values c in the interval $0 < c < 10$ for which $\lim_{x \to c} f(x)$ does not exist. If there are none, write NONE.

Answer: $c =$ 1, 8

b. [2 points] Give all values c in the interval $0 < c < 10$ for which $\lim_{x \to c^+} f(x)$ does not exist. If there are none, write NONE.

Answer: $c =$ **NONE**

c. [2 points] Give all values c in the interval $0 < c < 10$ for which $f(x)$ is not continuous at c. If there are none, write NONE.

Answer: $c =$ 1, 3, 8

d. [6 points] With f as shown in the graph above, define a function g by the formula

$$g(x) = \begin{cases} \dfrac{B + 2x^2 + 3x^3 + Ax^5}{12 + 6x^3 + 4x^5} & \text{if } x \leq 0 \\ f(x) & \text{if } 0 < x < 10 \end{cases}$$

where A and B are nonzero constants.
Find values of A and B so that both of the following conditions hold.

- $g(x)$ is continuous at $x = 0$.
- $\lim_{x \to -\infty} g(x) = \frac{1}{2}$.

If no such values exist, write NONE in the answer blanks.
Be sure to show your work or explain your reasoning.

Solution: To satisfy the first condition, we first compute $g(0)$ by plugging in $x = 0$ to the rational function $\dfrac{B + 2x^2 + 3x^3 + Ax^5}{12 + 6x^3 + 4x^5}$ to find $\lim_{x \to 0^-} g(x) = g(0) = \dfrac{B}{12}$. In order for $g(x)$ to be continuous at $x = 0$, we must also have $\lim_{x \to 0^-} g(x) = \dfrac{B}{12}$.
Now $\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} f(x) = -1$ (from the graph), so $\frac{B}{12} = -1$, and $B = -12$.
To satisfy the second condition, we compute that

$$\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} \frac{B + 2x^2 + 3x^3 + Ax^5}{12 + 6x^3 + 4x^5} = \frac{A}{4}.$$

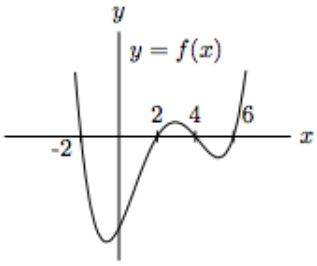
In order for this limit to equal $\frac{1}{2}$, we must have $A/4 = 1/2$, so $A = 2$.

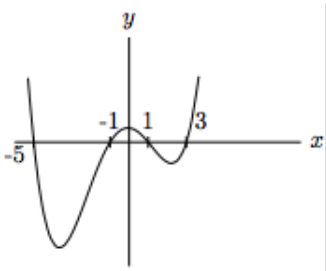
Answer: $A =$ 2 and $B =$ -12

□

Quiz #2. Please write you name: _____ and email: _____
HINT: plugging in help with sanity checks.

3. [9 points] A portion of the graph of a function f is shown below, along with three graphs obtained from f by one or more transformations. Below each of the three graphs is a list of possible formulas for that graph. Find the <u>one</u> correct formula for each graph, and write the corresponding letter in the answer blank provided.
Note that the zeros of each graph are labeled and that the scales on both the vertical and horizontal axes are the same for all the graphs shown.





Answer: _____

A. $f(x-1)$

B. $f(x-2)$

C. $f(x-3)$

D. $f(-(x-2))$

E. $f(-(x-3))$

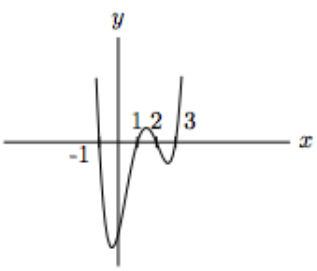
F. $f(x+1)$

G. $f(x+2)$

H. $f(x+3)$

I. $-f(x+1)$

J. $-f(x+3)$



Answer: _____

A. $f(x-1)$

B. $f(x+1)$

C. $f(\frac{1}{3}x)$

D. $f(\frac{1}{2}x)$

E. $f(2x)$

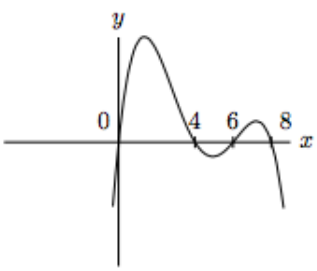
F. $f(3x)$

G. $f(2(x+1))$

H. $f(\frac{1}{2}(x-1))$

I. $f(2(x-1))$

J. $f(\frac{1}{2}(x+1))$



Answer: _____

A. $f(x+2)$

B. $f(x-2)$

C. $f(-(x-2))$

D. $-f(x+1)$

E. $-f(x-2)$

F. $-f(x+2)$

G. $-f(x)$

H. $f(-(x+2))$

I. $-f(-x)$

J. $f(-(x-4))$

Solution. The first graph is obtained by shifting left 3 units, so $f(x+3)$ is correct.
The middle graph is by shrinking the graph horizontally. From the discussion about sinusoidal functions, we can tell that:

$$g(x) = f(Bx)$$

- $|B| > 1$ - horizontal shrink.
- $|B| < 1$ - horizontal stretch.
- $B < 0$ horizontal flip, with respect to the y-axis, in addition to the above.
- **the y-intercept stays put**

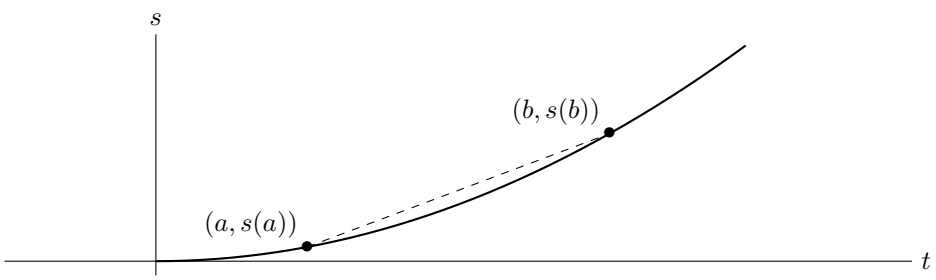
Thus, the answer is $f(2x)$.
The last graph is a result of vertical flip and shifting 2 units to the right, so $-f(x-2)$.

□

2 Derivatives

2.1 Speed

Consider position as function of time $s(t)$.



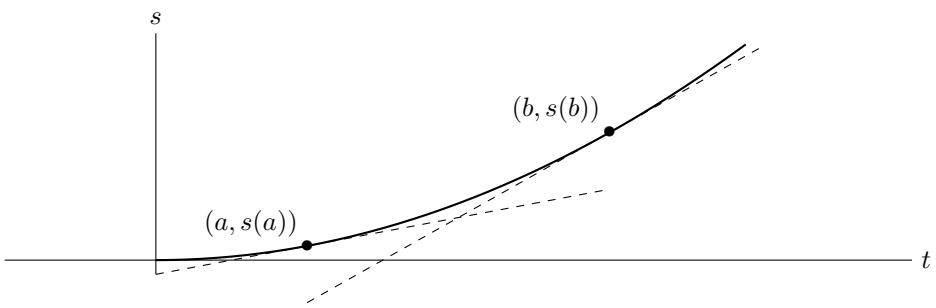
Intuitively we sense that the movement starts slow and gets faster when the time goes by. Mathematically we can quantify this intuition.
The **Average velocity** between $t = a$ and $t = b$ is:

$$\boxed{\frac{s(b) - s(a)}{b - a}}$$

And we know that this is the slop of the secant line, or "rise/run". We sense that the velocity at a is smaller than the velocity at b. We define **Instantaneous velocity** at a be letting the point b get closer and closer to a while we record the slope of the secant line. Mathematically:

$$\boxed{\lim_{h \to 0} \frac{s(a+h) - s(a)}{h}}$$

Note that we must do this dynamic procedure from both lines of a. Graphically, we end us with the slop of the **tanget line** at a. This explains we we "sense" that the velocity at $t = a$ is smaller than at $t = b$.



Remark 2.1.1. While velocity can be positive or negative, **Speed** is defined to be the magnitude of the velocity. Suppose the I drive at a constant speed of 50 mph from point A to point B, and then drive from

point B to point A at 30 mph. The trip is finished at $t = c$. The average velocity between $t = 0$ and $t = c$ is 0 since $s(0) = 0 = s(c)$. However, the average speed is not. Assume that the distance is 150 mile. Then for 3 hours I drove 50 mph, and for 5 hours I drove 30 mph. $c = 8$ hours. The average is

$$\frac{3 \cdot 50 + 5 \cdot 30}{8} = 300/8 = 37.5mph.$$

In general, the speed between $t = a$ and $t = b$ is:

$$\text{speed} = \frac{\text{total distance covered between time } a \text{ and time } b}{b - a}$$

Exercise 2.1.2.

2. [12 points] Angelica Neiring and Simona Koloji decide to enjoy the fall weather by racing each other from the brass block "M" in the center of the Diag along a 2.5 kilometer (2500 meter) route to the Huron River inside the Arb. Let $A(t)$ (respectively $S(t)$) be Angelica's (respectively Simona's) distance along the route (in meters) t seconds after they start racing. Angelica and Simona are both wearing GPS watches that record data about their race. The table of values for the functions A and S below shows some of the resulting data, rounded to the nearest meter. Note that the data is not always recorded at regular intervals.

t	0	30	60	66	72	105	114	120	135	168	180	198	300
$A(t)$	0	55	119	137	156	226	249	265	302	384	415	463	737
$S(t)$	0	57	120	137	156	225	248	264	303	389	422	473	768

Use the data above to answer the questions below. Remember to show your work.

1. Estimate Angelica's instantaneous velocity 3 minutes into the race.
2. Estimate Simona's instantaneous velocity 120 minutes into the race.
3. Who was ahead after 5 minutes — Angelica / Simona / Cannot be determined?
4. Who was running faster exactly 1 minute into the race — Angelica / Simona / Cannot be determined?
5. In describing the race later, Simona says that her average velocity during the entire race was 2.8 meters per second while Angelica says that after the first 5 minutes, her average velocity for the rest of the race was 3.1 meters per second. Assuming their statements and the table of values above are accurate, who won the race? Or is there not enough information to decide? Explain your reasoning.

Solution.

- 3 min = 180 sec. The best estimate is to compute the secant line from the left, from the right and then average. We use a **trick** of calculating the slope using one point to the left and one to the right. It is just as good as an estimation, just quicker. So we use (168,384),(198,463). We get: (463-387)/(198-168)=76/30≈2.53 m/s
- (303-248)/(135-114)=55/21≈2.61 m/s
- At $t = 300$ simona was ahead by 29 meters.
- We can compute the slopes, but it is apparent that the approximations would be too close to give a clear cut answer.
- Based on the average velocity, Simona has finished after (2500)/2.8 = 892.9 seconds. Angelica ran 737 meters at the first 5 minutes. For the rest of the (2500-737) meters , she ran at average velocity of $3.1m/s$ so it took here: (2500-737)/3.1≈568.7 seconds. Add the 5 minutes to it to get a time of 868.7. Thus Angelica has finished the race in about 25 seconds less than Simona.

□

2.2 Derivatives

This is a complete analogy of the previous section, only this time we are talking about how functions $f(x)$ change, not necessarily position as a function of time.
For any pair of points on the graph, we have a formula for the slope of the secant line, using rise over run:

$$\frac{f(b)-f(a)}{b-a}$$

We can talk about instantaneous slope, or the slope of a tangent line, or the derivative at $x=a$, using the same limit procedure of letting the point b be closer and closer to a.

The **derivative of f at a**, written $f'(a)$, is defined as

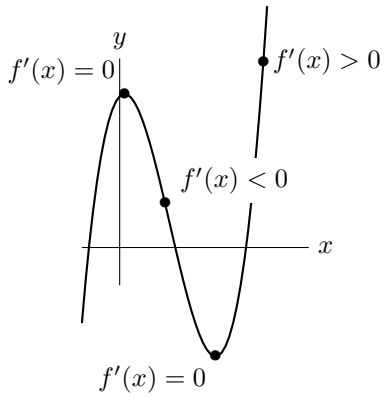
$$\begin{array}{c}\text{Rate of change}\\ \text{of } f \text{ at } a\end{array} = f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

If the limit exists, then f is said to be **differentiable at a**.

Graphically:

- $f'(a) > 0$, the function is increasing around $x = a$.
- $f'(a) < 0$, the function is decreasing around $x = a$.
- $f'(a) = 0$, then the function is neither around $x = a$. This is called a **Critical Point**. Possibly a max/min point.

A critical point of $f(x)$ is a point $x = a$ where $f(a) = 0$ or $f(a)$ does not exists.



Example 2.2.1. Compute algebraically $f'(2)$ for $f(x) = x^2 + x + 1$.

$$f'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{4+4h+h^2+2+h+1-4-2-1}{h} = \lim_{h\to 0}\frac{5h+h^2}{h} = \lim_{h\to 0} 5+h = 5$$

The last inequality is due to the fact that $g(h) = 5 + h$ is a continuous function, thus we can plug in to get the limit.

Exercise 2.2.2.

4. [9 points] Let $P(v) = \begin{cases} v^2 \sin\left(\frac{1}{v}\right) - v\sin(2) & \text{if } v \neq 0 \\ 0 & \text{if } v = 0. \end{cases}$

a. [5 points]
Use the limit definition of the derivative to write down an explicit expression for $P'(0)$.
Your answer should not include the letter P.
Do not attempt to evaluate or simplify the limit.

$P'(0) =$ _____

b. [4 points] Use your answer to (a) to estimate $P'(0)$ to the nearest hundredth.
Be sure to include enough clear graphical or numerical evidence to justify your answer.

Solution.

4. [9 points] Let $P(v) = \begin{cases} v^2 \sin\left(\frac{1}{v}\right) - v\sin(2) & \text{if } v \neq 0 \\ 0 & \text{if } v = 0. \end{cases}$

a. [5 points]
Use the limit definition of the derivative to write down an explicit expression for $P'(0)$.
Your answer should not include the letter P.
Do not attempt to evaluate or simplify the limit.

$$P'(0) = \lim_{h \to 0} \frac{\left((0+h)^2 \sin\left(\frac{1}{0+h}\right) - (0+h)\sin(2)\right) - 0}{h}$$

b. [4 points] Use your answer to (a) to estimate $P'(0)$ to the nearest hundredth.
Be sure to include enough clear graphical or numerical evidence to justify your answer.

Solution: We plug in small values of h approaching 0. Since the difference quotient is an even function of h, we need only check positive values of h (as evenness implies that negative h give precisely the same results).

$h = 0.1$:
$$\frac{0.1^2 \sin(1/0.1) - 0.1\sin(2) - 0}{0.1} \approx -0.964$$

$h = 0.01$:
$$\frac{0.01^2 \sin(1/0.01) - 0.01\sin(2) - 0}{0.01} \approx -0.914$$

$h = 0.001$:
$$\frac{0.001^2 \sin(1/0.001) - 0.001\sin(2) - 0}{0.001} \approx -0.908$$

$h = 0.0001$:
$$\frac{0.0001^2 \sin(1/0.0001) - 0.0001\sin(2) - 0}{0.0001} \approx -0.909$$

We see at this point that the numbers seem to have stabilized to the nearest hundredth at -0.91.

□

Exercise 2.2.3.

1. [11 points] The table below gives several values of a continuous, invertible function $f(x)$. Assume that the domain of both $f(x)$ and $f'(x)$ is the interval $(-\infty, \infty)$.

x	0	3	6	9	12	15	18	21	24
$f(x)$	-7	-3.5	-2	3	4.5	6	7	9	19

a. [3 points] Evaluate each of the following.

(i) $f(f(15))$

Answer: $f(f(15)) =$ _____

(ii) $f^{-1}(3)$

Answer: $f^{-1}(3) =$ _____

(iii) $f^{-1}(2f(12))$

Answer: $f^{-1}(2f(12)) =$ _____

b. [2 points] Compute the average rate of change of f on the interval $3 \leq x \leq 18$.

Answer: _____

c. [2 points] Estimate $f'(19)$.

Answer: $f'(19) \approx$ _____

d. [2 points] Let $g(x) = f^{-1}(x)$. Estimate $g'(5)$.

Answer: $g'(5) \approx$ _____

e. [2 points] Suppose $f'(0) = 2$. Find an equation for the tangent line to the graph of $y = f(x)$ at $x = 0$.

Answer: _____

Solution.

1. [11 points] The table below gives several values of a continuous, invertible function $f(x)$. Assume that the domain of both $f(x)$ and $f'(x)$ is the interval $(-\infty, \infty)$.

x	0	3	6	9	12	15	18	21	24
$f(x)$	-7	-3.5	-2	3	4.5	6	7	9	19

a. [3 points] Evaluate each of the following.

(i) $f(f(15))$

Solution: $f(f(15)) = f(6) = -2.$

Answer: $f(f(15)) =$ -2

(ii) $f^{-1}(3)$

Answer: $f^{-1}(3) =$ 9

(iii) $f^{-1}(2f(12))$

Solution: $f^{-1}(2f(12)) = f^{-1}(2(4.5)) = f^{-1}(9) = 21.$

Answer: $f^{-1}(2f(12)) =$ 21

b. [2 points] Compute the average rate of change of f on the interval $3 \le x \le 18$.

Solution: This average rate of change is equal to the difference quotient

$$\frac{f(18) - f(3)}{18 - 3} = \frac{7 - (-3.5)}{15} = \frac{10.5}{15} = \frac{7}{10} = 0.7.$$

Answer: $10.5/15 = 7/10 = 0.7$

c. [2 points] Estimate $f'(19)$.

Solution: We approximate $f'(19)$ by the average rate of change of f on the interval $18 \le x \le 21$.

$$f'(19) \approx \frac{f(21) - f(18)}{21 - 18} = \frac{9 - 7}{3} = \frac{2}{3}.$$

Answer: $f'(19) \approx$ $2/3 \approx 0.67$

d. [2 points] Let $g(x) = f^{-1}(x)$. Estimate $g'(5)$.

Solution: We approximate $g'(5)$ by the average rate of change of $g(x)$ on the interval $4.5 \le x \le 6$.

$$g'(5) \approx \frac{g(6) - g(4.5)}{6 - 4.5} = \frac{f^{-1}(6) - f^{-1}(4.5)}{1.5} = \frac{15 - 12}{1.5} = \frac{3}{1.5} = 2.$$

Answer: $g'(5) \approx$ $3/1.5 = 2$

e. [2 points] Suppose $f'(0) = 2$. Find an equation for the tangent line to the graph of $y = f(x)$ at $x = 0$.

Solution: This is the line with slope $f'(0) = 2$ that passes through the point $(0, f(0)) = (0, -7)$. An equation for this line is $y = 2x - 7$.

Answer: $y = 2x - 7$

We can give a general formula for the tangent line of $f(x)$ at a point $x = a$ when $f(x)$ is differentiable there:

$$\boxed{\ell(x) = f'(a)(x - a) + f(a)}$$

□

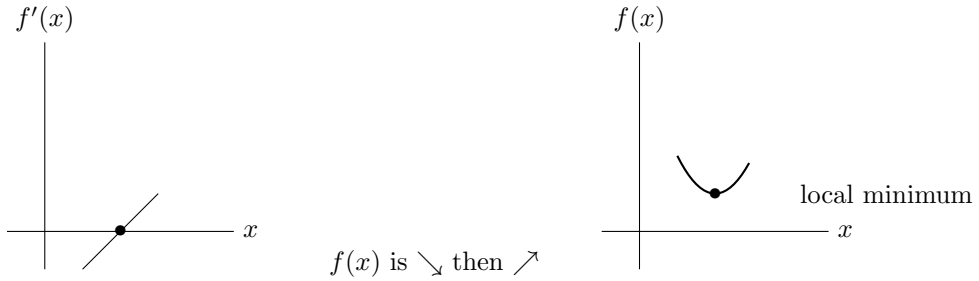
2.3 The Derivative Function

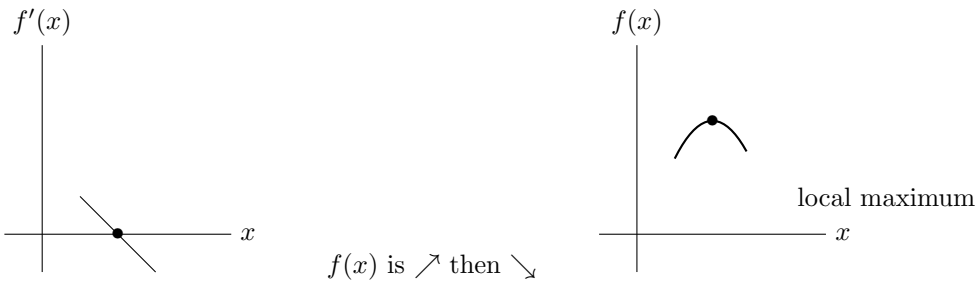
If $f(x)$ is differentiable on an interval, we can define a function assigning to each point its derivative. We get the derivative function:

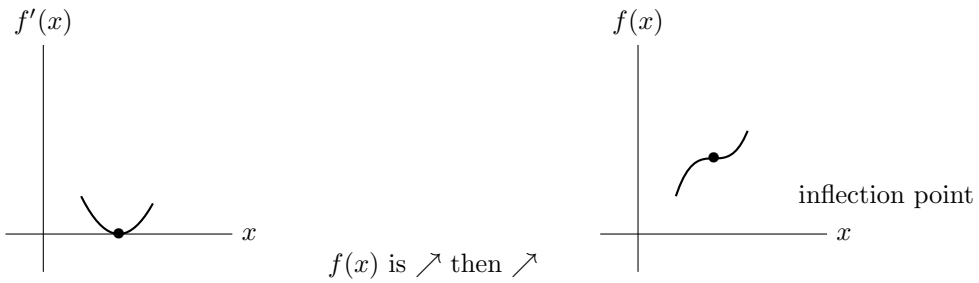
For any function f, we define the **derivative function**, f', by

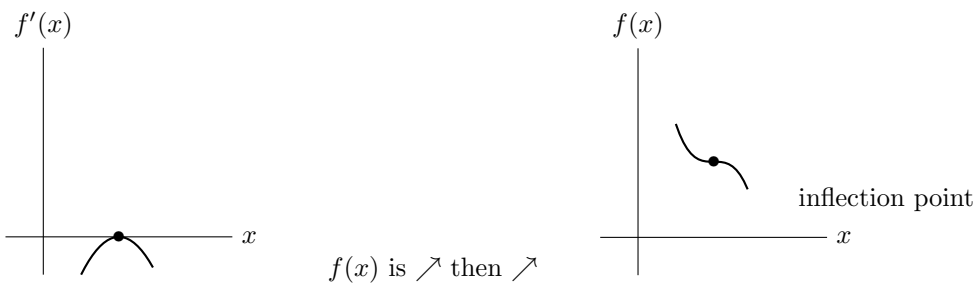
$$f'(x) = \text{ Rate of change of } f \text{ at } x = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Now we can relate the graph of $f'(x)$ to the graph of $f(x)$:









We can always move $f(x)$ up or down without changing the slope, thus without changing the derivative. A few formulas:

$$f(x) = k \Rightarrow f'(x) = 0$$
$$f(x) = mx + b \Rightarrow f'(x) = m$$
$$g(x) = f(x) + k \Rightarrow g'(x) = f'(x)$$
$$\boxed{f(x) = x^n \Rightarrow f'(x) = nx^{n-1}}$$

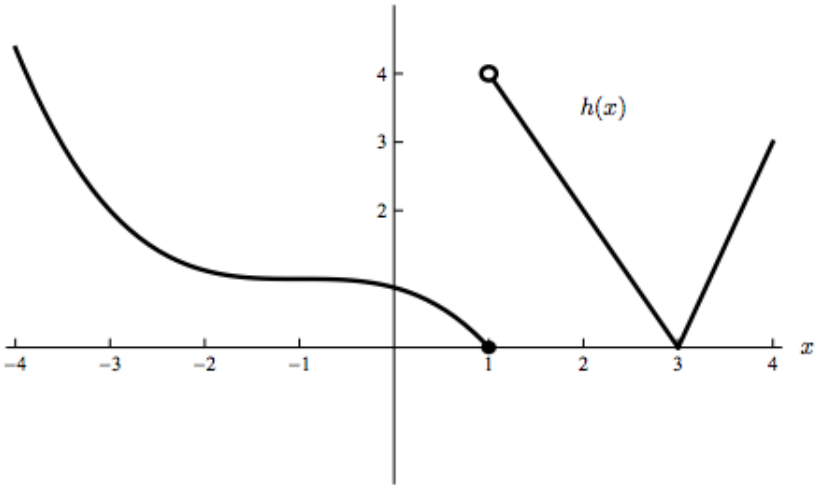
Remark 2.3.1. If $f'(a)$ exists then $f(x)$ is continuous at $x = a$. So $\boxed{\text{differentiability implies continuity}}$ but NOT the other way around.

When a function is not differentiable?

- Not defined at the point.
- Not continuous at the point.
- Not "smooth" at the point, i.e. has a sharp edge or V-like shape.

Exercise 2.3.2.

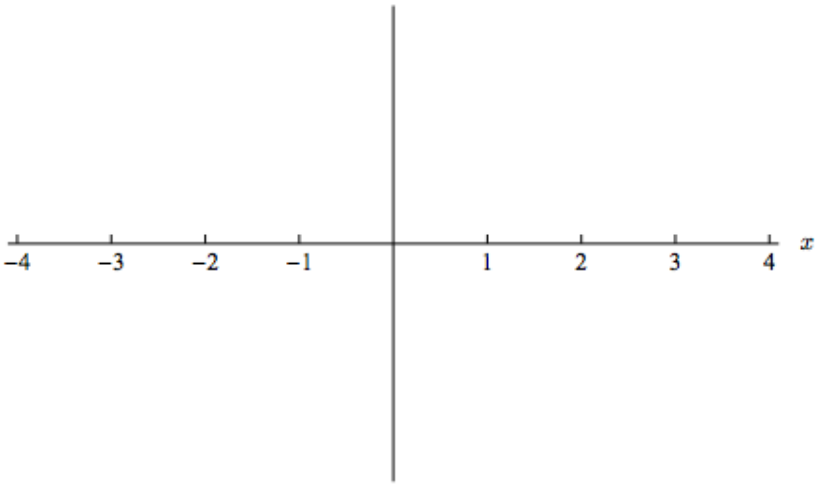
8. [10 points] The graph of a function $h(x)$ is given below.



a. [1 point] List all x-values with $-4 < x < 4$ where $h(x)$ is not continuous. If there are none, write NONE.

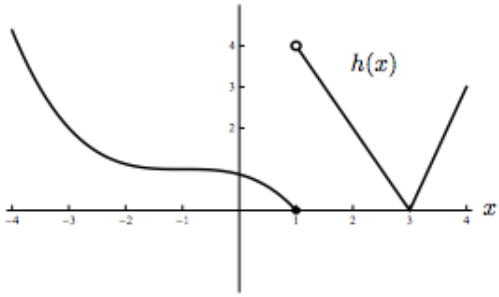
b. [1 point] List all x-values with $-4 < x < 4$ where $h(x)$ is not differentiable. If there are none, write NONE.

c. [8 points] On the axes provided, carefully draw a graph of $h'(x)$. Be sure to label important points or features on your graph.



Solution.

8. [10 points] The graph of a function $h(x)$ is given below.



a. [1 point] List all x-values with $-4 < x < 4$ where $h(x)$ is not continuous. If there are none, write NONE.

Solution:

$$x = 1$$

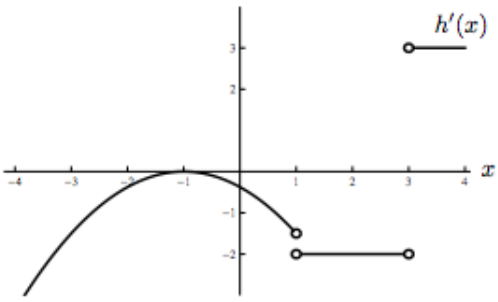
b. [1 point] List all x-values with $-4 < x < 4$ where $h(x)$ is not differentiable. If there are none, write NONE.

Solution:

$$x = 1, 3$$

c. [8 points] On the axes provided, carefully draw a graph of $h'(x)$. Be sure to label important points or features on your graph.

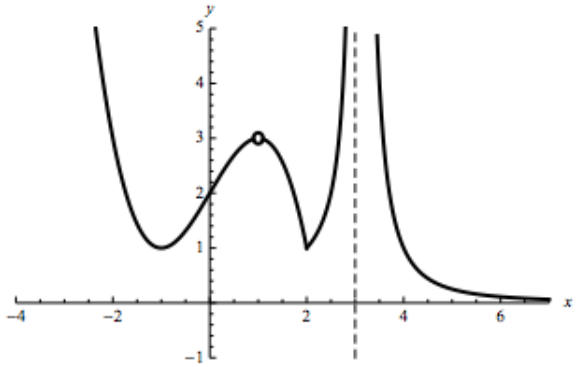
Solution:



Note that the derivative does not exists for $x = 1, x - 3$, and the slope to the left of $x = 1$ is a little less extreme than the one to the right of $x = 1$. □

Exercise 2.3.3.

1. [15 points] The following figure shows the graph of $y = f(x)$ for some function f. The dotted line signifies a vertical asymptote.



a. [12 points] Using the graph, give the values of each of the following quantities if they exist. Choose your answer in each part from the numbers $0, 1, 2, 3$ or the words "Does not exist." Answers may be used more than once–or not at all.

i) $f(1) =$ _____

ii) $f(2) =$ _____

iii) $f(3) =$ _____

iv) $f'(-1) =$ _____

v) $f'(1) =$ _____

vi) $f'(2) =$ _____

vii) $\lim_{x\to+\infty} f(x) =$ _____

viii) $\lim_{x\to 3} f(x) =$ _____

ix) $\lim_{x\to 2} f(x) =$ _____

x) $\lim_{x\to 1} f(x) =$ _____

xi) $\lim_{x\to -1} f(x) =$ _____

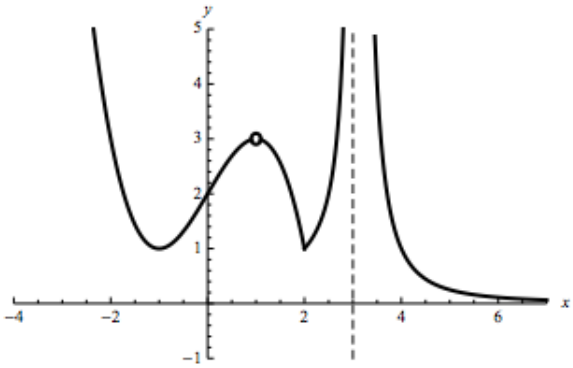
xii) $\lim_{x\to -\infty} f(x) =$ _____

b. [3 points] Still looking at the graph, is f continuous at the following x values? (Yes or No)

i) $x = 1$ _____ ii) $x = 2$ _____ iii) $x = 3$ _____

Solution.

1. [15 points] The following figure shows the graph of $y = f(x)$ for some function f. The dotted line signifies a vertical asymptote.



a. [12 points] Using the graph, give the values of each of the following quantities if they exist. Choose your answer in each part from the numbers $0, 1, 2, 3$ or the words "Does not exist." Answers may be used more than once–or not at all.

i) $f(1) =$Does not exist.

ii) $f(2) = 1$

iii) $f(3) =$Does not exist.

iv) $f'(-1) = 0$.

v) $f'(1) =$ Does not exist.

vi) $f'(2) =$Does not exist.

vii) $\lim_{x\to+\infty} f(x) = 0$.

viii) $\lim_{x\to 3} f(x) =$Does not exist.

ix) $\lim_{x\to 2} f(x) = 1$.

x) $\lim_{x\to 1} f(x) = 3$.

xi) $\lim_{x\to -1} f(x) = 1$.

xii) $\lim_{x\to -\infty} f(x) =$ Does not exist.

b. [3 points] Still looking at the graph, is f continuous at the following x values? (Yes or No)

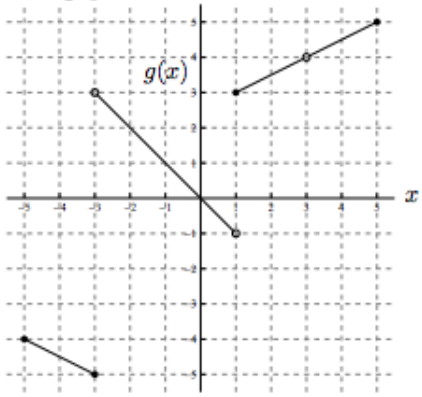
i) $x = 1$, No. ii) $x = 2$, Yes. iii) $x = 3$, No.

□

Quiz #3. Please write you name: _____ and email: _____

2. [13 points] Below is a table of values for an invertible, differentiable function $f(x)$ and the graph of a function $g(x)$. Use these to answer the following questions:

x	0	1	2	3	4	5
$f(x)$	8	7	3	2	1.5	1



a. [1 point] Give one number in the interval $[-5, 5]$ that is *not* in the domain of g.

b. [1 point] Give one number in the interval $[-5, 5]$ that is *not* in the domain of g^{-1}.

c. [8 points] Evaluate the following:

(i) $f(f(5))$

(ii) $g^{-1}(f^{-1}(2))$

(iii) $\lim_{x \to 3} g(x)$

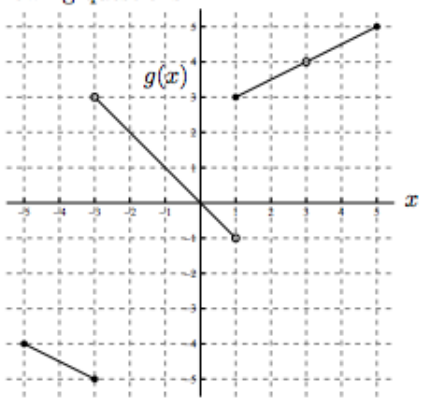
(iv) $g'(1 + f(2))$

d. [3 points] Approximate $f'(3)$. (Be sure to show your work.)

Solution.

2. [13 points] Below is a table of values for an invertible, differentiable function $f(x)$ and the graph of a function $g(x)$. Use these to answer the following questions:

x	0	1	2	3	4	5
$f(x)$	8	7	3	2	1.5	1



a. [1 point] Give one number in the interval $[-5, 5]$ that is *not* in the domain of g.

Solution: 3

b. [1 point] Give one number in the interval $[-5, 5]$ that is *not* in the domain of g^{-1}.

Solution: Anything in $(-4, -1] \cup \{4\}$. For example, 4.

c. [8 points] Evaluate the following:

(i) $f(f(5))$

Solution: $f(f(5)) = f(1) = 7$.

(ii) $g^{-1}(f^{-1}(2))$

Solution: $g^{-1}(f^{-1}(2)) = g^{-1}(3) = 1$.

(iii) $\lim_{x \to 3} g(x)$

Solution: 4, found from graph of g.

(iv) $g'(1 + f(2))$

Solution: $g'(1 + f(2)) = g'(4) = \frac{1}{2}$ looking at the slope on the graph of g at $x = 4$.

d. [3 points] Approximate $f'(3)$. (Be sure to show your work.)

Solution: Acceptable answers: -1, $-\frac{1}{2}$, $-\frac{3}{4}$. Found approximating the derivative via a difference quotient.

□

2.4 Interpretation of The Derivative

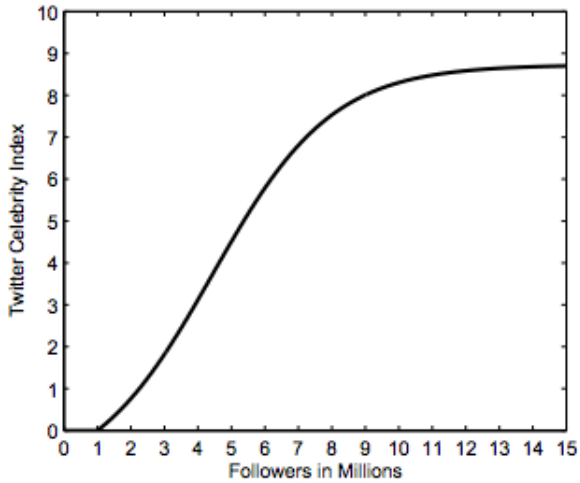
$f'(x)$ represent the instantaneous change of the function. Another notation is

$$f'(x) = \frac{df}{dx} = \frac{dy}{dx}$$

In term of units, it has units of $\boxed{\text{(units of y)/(units of x)}}$. (Note to instructor - the following example should be presented on the board. The next two exercises are group work).

Exercise 2.4.1.

4. [12 points] The Twitter Celebrity Index (TCI) measures the celebrity of Twitter users; the function $T(x)$ takes the number of followers (in millions) of a given user and returns a TCI value from 0 to 10. Below is a graph of this function.



Use the graph above to help you answer the following questions.

a. [3 points] Explain in practical terms what $T(13.72) = 8.67$ means.

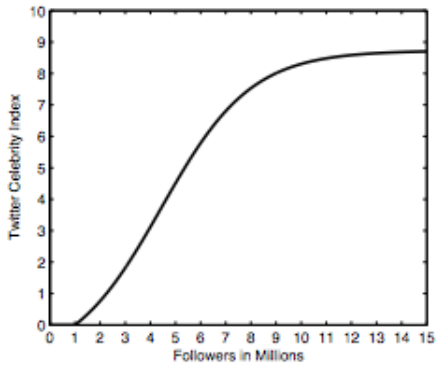
b. [3 points] Explain in practical terms what $T^{-1}(4.25) = 4.88$ means.

c. [3 points] Explain in practical terms what $T'(10) = 0.2278$ means.

d. [3 points] Explain in practical terms what $(T^{-1})'(7.238) = 0.71$ means.

Solution.

4. [12 points] The Twitter Celebrity Index (TCI) measures the celebrity of Twitter users; the function $T(x)$ takes the number of followers (in millions) of a given user and returns a TCI value from 0 to 10. Below is a graph of this function.



Use the graph above to help you answer the following questions.

a. [3 points] Explain in practical terms what $T(13.72) = 8.67$ means.

Solution: When a Twitter user has 13.72 million followers, their Twitter Celebrity index is 8.67.

b. [3 points] Explain in practical terms what $T^{-1}(4.25) = 4.88$ means.

Solution: When a user has a Twitter Celebrity index of 4.25, they have 4.88 million followers.

c. [3 points] Explain in practical terms what $T'(10) = 0.2278$ means.

Solution: When a Twitter user has 10 million followers, adding $100,000$ followers will increase their celebrity index by roughly .02278.

d. [3 points] Explain in practical terms what $(T^{-1})'(7.238) = 0.71$ means.

Solution: When a Twitter user has celebrity index 7.238, and increase of .1 to their index corresponds to gaining approximately .071 million (71,000) followers.

□

Exercise 2.4.2.

8. [9 points] A certain company's revenue R (in thousands of dollars) is given as a function of the amount of money a (in thousands of dollars) they spend on advertising by $R = f(a)$. Suppose that f is invertible.

a. [2 points] Which of the following is a valid interpretation of the equation $(f^{-1})'(75) = 0.5$? Circle one option.

- If the company spends \$75,000 more on advertising, their revenue will increase by about \$500.
- If the company increases their advertising expenditure from \$75,000 to \$76,000, their revenue will increase by about \$500.
- If the company wants a revenue of \$75,000, they should spend about \$500 on advertising.
- If the company wants to increase their revenue from \$75,000 to \$76,000, they should spend about \$500 more on advertising.

b. [2 points] The company plans to spend about \$100,000 on advertising. If $f'(100) = 0.5$, should the company spend more or less than \$100,000 on advertising? Justify your answer.

c. [5 points] The company's financial advisor claims that he has a formula for the dependence of revenue on advertising expenditure, and it is

$$f(a) = a\ln(a+1).$$

Using this formula, write the *limit definition* of $f'(100)$. You do not need to simplify or evaluate.

Solution.

8. [9 points] A certain company's revenue R (in thousands of dollars) is given as a function of the amount of money a (in thousands of dollars) they spend on advertising by $R = f(a)$. Suppose that f is invertible.

a. [2 points] Which of the following is a valid interpretation of the equation $(f^{-1})'(75) = 0.5$? Circle one option.

- If the company spends \$75,000 more on advertising, their revenue will increase by about \$500.
- If the company increases their advertising expenditure from \$75,000 to \$76,000, their revenue will increase by about \$500.
- If the company wants a revenue of \$75,000, they should spend about \$500 on advertising.
- If the company wants to increase their revenue from \$75,000 to \$76,000, they should spend about \$500 more on advertising.

Solution: The last option.

b. [2 points] The company plans to spend about \$100,000 on advertising. If $f'(100) = 0.5$, should the company spend more or less than \$100,000 on advertising? Justify your answer.

Solution:
They should spend less on advertising, because if they increase their advertising expenditure by \$1000, they will only gain about \$500 in revenue.

c. [5 points] The company's financial advisor claims that he has a formula for the dependence of revenue on advertising expenditure, and it is

$$f(a) = a\ln(a+1).$$

Using this formula, write the *limit definition* of $f'(100)$. You do not need to simplify or evaluate.

Solution:

$$f'(100) = \lim_{h\to 0} \frac{(100+h)\ln(100+h+1) - 100\ln(101)}{h}$$

□

Exercise 2.4.3.

2. [10 points] Louis owns a small soda company and is experimenting with new flavors. Let $b(p)$ model the number of thousands of bottles of bacon-flavored soda sold by his company per month if he charges p cents per bottle. You may assume $b(p)$ is differentiable and invertible.

a. [2 points] Give a practical interpretation of the statement $b^{-1}(8) = 150$.

b. [3 points] Give a practical interpretation of the statement $(b^{-1})'(4) = -10$.

c. [3 points] Write an expression that is equal to the price (in cents) that the company would have to charge per bottle in order to sell twice as many bottles of bacon-flavored soda as it sells at a price of 125 cents per bottle.

d. [2 points] Which of the following is a correct formula for a function $h(d)$ that gives the number of thousands of bottles sold per month at a price of d dollars per bottle? (Circle your answer.)

$$h(d) = 100b(d) \qquad h(d) = \frac{b(d)}{100} \qquad h(d) = b(100d) \qquad h(d) = b\left(\frac{d}{100}\right)$$

3. [5 points] Use the limit definition of the derivative to write an explicit expression for $r'(3)$ where $r(t) = (t+5)^{2t}$. Do not simplify or evaluate the limit. Your answer should not include the letter r.

$r'(3) =$ _____

Solution.

2. [10 points] Louis owns a small soda company and is experimenting with new flavors. Let $b(p)$ model the number of thousands of bottles of bacon-flavored soda sold by his company per month if he charges p cents per bottle. You may assume $b(p)$ is differentiable and invertible.

a. [2 points] Give a practical interpretation of the statement $b^{-1}(8) = 150$.

Solution: In order to sell 8000 bottles of bacon-flavored soda per month, the company should charge 150 cents per bottle.

b. [3 points] Give a practical interpretation of the statement $(b^{-1})'(4) = -10$.

Solution: In order to increase the number of bottles sold per month from 4000 to 5000, the company should lower the price about 10 cents.
If the company is currently selling 4000 bottles per month, lowering the price by 10 cents will increase sales by about 1000 bottles per month.
(There are other possible answers.)

c. [3 points] Write an expression that is equal to the price (in cents) that the company would have to charge per bottle in order to sell twice as many bottles of bacon-flavored soda as it sells at a price of 125 cents per bottle.

Solution: $b^{-1}(2b(125))$

d. [2 points] Which of the following is a correct formula for a function $h(d)$ that gives the number of thousands of bottles sold per month at a price of d dollars per bottle? (Circle your answer.)

$$h(d) = 100b(d) \qquad h(d) = \frac{b(d)}{100} \qquad \boxed{h(d) = b(100d)} \qquad h(d) = b\left(\frac{d}{100}\right)$$

3. [5 points] Use the limit definition of the derivative to write an explicit expression for $r'(3)$ where $r(t) = (t+5)^{2t}$. Do not simplify or evaluate the limit. Your answer should not include the letter r.

Solution:

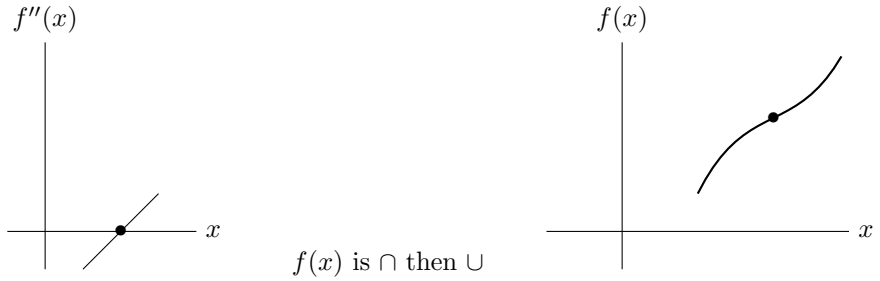
$$r'(3) = \lim_{h\to 0} \frac{(3+h+5)^{2(3+h)} - (3+5)^{2(3)}}{h}$$

□

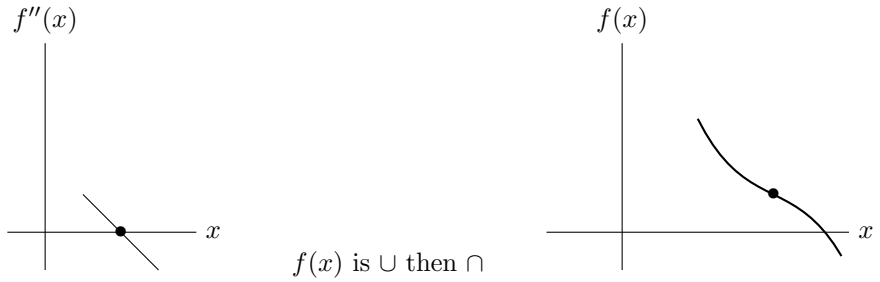
2.5 The second derivative

The second derivative is the derivative of the derivative.
Graphically:

If the graph of f is concave up and f'' exists on an interval, then $f'' \geq 0$ there.
If the graph of f is concave down and f'' exists on an interval, then $f'' \leq 0$ there.



$f(x)$ is $\cap$ then $\cup$

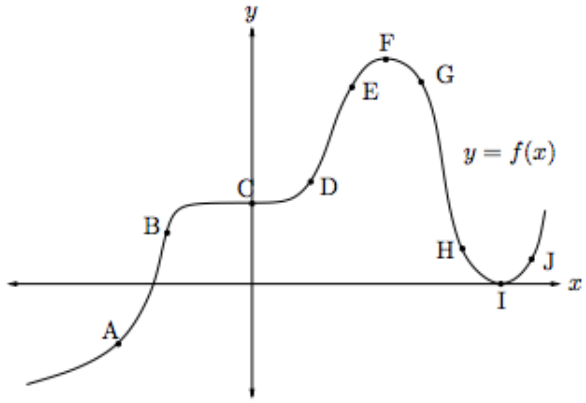


$f(x)$ is $\cup$ then $\cap$

In the context of position s as a function of time t, recall that $s'(t) = v(t)$ the velocity. Thus, $s''(t) = v'(t) = a(t)$ the acceleration.
(Note to instructor - the following example should be presented on the board. The next two exercises are group work).

Exercise 2.5.1.

6. (11 points) The graph of a continuous differentiable function f is given below. Use the graph to answer the following. No explanation necessary.



(a) List all labelled points (if any) where f' and f'' are both positive.

(b) List all labelled points (if any) where f' and f'' are both negative.

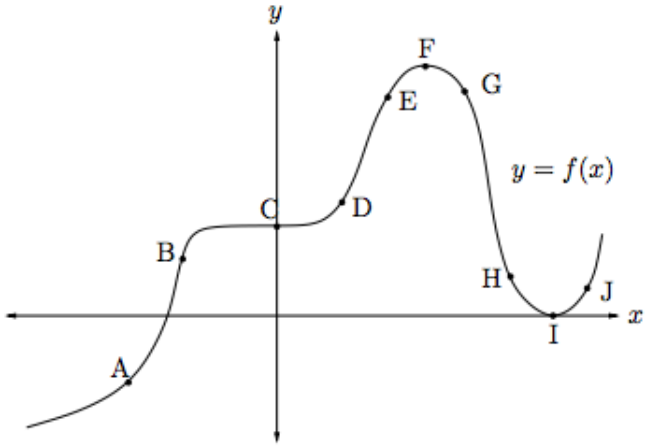
(c) List all labelled points (if any) where f and f' are both positive.

(d) List all labelled points (if any) where f and f' are both both negative.

(e) List all labelled points (if any) where at least two of f, f', f'' are zero.

Solution.

6. (11 points) The graph of a continuous differentiable function f is given below. Use the graph to answer the following. No explanation necessary.



(a) List all labelled points (if any) where f' and f'' are both positive.

A, D, J

(b) List all labelled points (if any) where f' and f'' are both negative.

G

(c) List all labelled points (if any) where f and f' are both positive.

B, D, E, J

(d) List all labelled points (if any) where f and f' are both both negative.

none

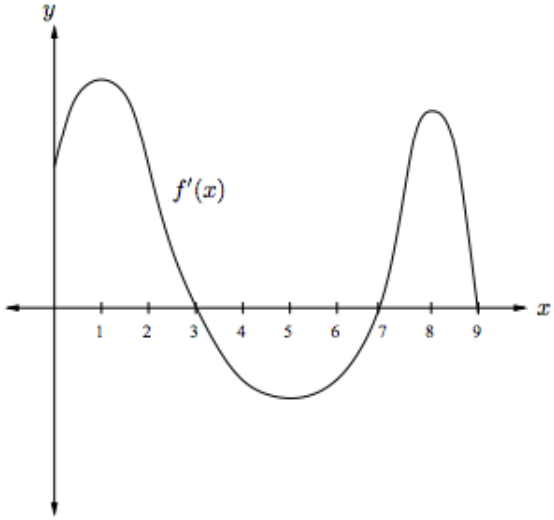
(e) List all labelled points (if any) where at least two of f, f', f'' are zero.

C, I

□

Exercise 2.5.2.

9. (10 points) The graph of $f'(x)$ (i.e., the *derivative* of f) is given below. Use the graph to answer the following questions:



(a) For which intervals is f increasing?

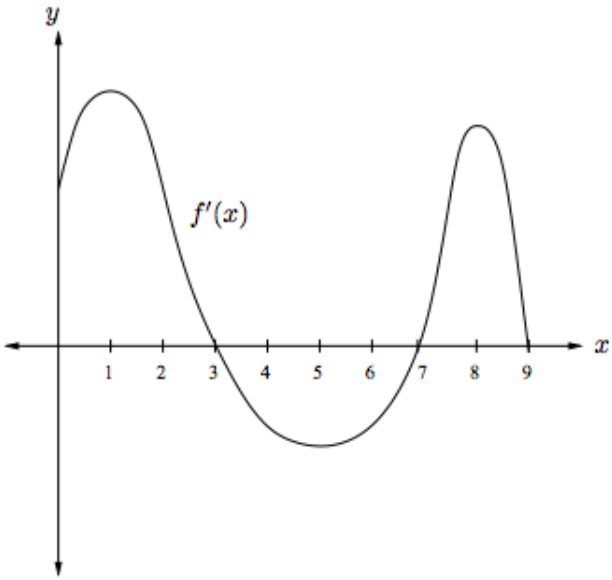
(b) For which intervals is f'' negative?

(c) For which value(s) of x (if any) does f have a local maximum?

(d) For which values of x (if any) does f switch from concave up to concave down?

Solution.

9. (10 points) The graph of $f'(x)$ (i.e., the *derivative* of f) is given below. Use the graph to answer the following questions:



(a) For which intervals is f increasing?

f is increasing when its derivative is positive, so for $0 < x < 3$ and $7 < x < 9$.

(b) For which intervals is f'' negative?

f'' is negative when f' is decreasing, so for $1 < x < 5$ and $8 < x < 9$.

(c) For which value(s) of x (if any) does f have a local maximum?

[Note: This was excluded from grading.]f has a local maximum at a value a when f' is positive for $x < a$ and negative for $x > a$. So f has a local maximum when $x = 3$.

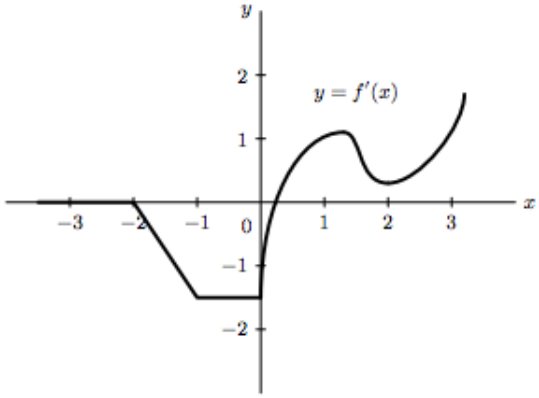
(d) For which value(s) of x (if any) does f switch from concave up to concave down?

f will switch from concave up to concave down when the second derivative switches from being positive to being negative, ie., when the derivative switches from increasing to decreasing, so at $x = 1$ and $x = 8$.

□

Exercise 2.5.3.

10. [10 points] Below is the graph of $f'(x)$, the **derivative** of the function $f(x)$.
Note that $f'(x)$ is zero for $x \leq -2$, linear for $-2 < x < -1$, and constant for $-1 < x < 0$.



For each of the following, circle all of the listed intervals for which the given statement is true over the **entire** interval. If there are no such intervals, circle NONE.
You do not need to explain your reasoning.

a. [2 points] $f'(x)$ is increasing.

$-2 < x < -1$ $\quad 0 < x < 1$ $\quad 1 < x < 2$ $\quad 2 < x < 3$ $\quad$ NONE

b. [2 points] $f'(x)$ is concave up.

$0 < x < 1$ $\quad 1 < x < 2$ $\quad 2 < x < 3$ $\quad$ NONE

c. [2 points] $f(x)$ is increasing.

$-2 < x < -1$ $\quad -1 < x < 0$ $\quad 0 < x < 1$ $\quad 1 < x < 2$ $\quad 2 < x < 3$ $\quad$ NONE

d. [2 points] $f(x)$ is linear but not constant.

$-3 < x < -2$ $\quad -2 < x < -1$ $\quad -1 < x < 0$ $\quad 0 < x < 1$ $\quad 1 < x < 2$ $\quad 2 < x < 3$ $\quad$ NONE

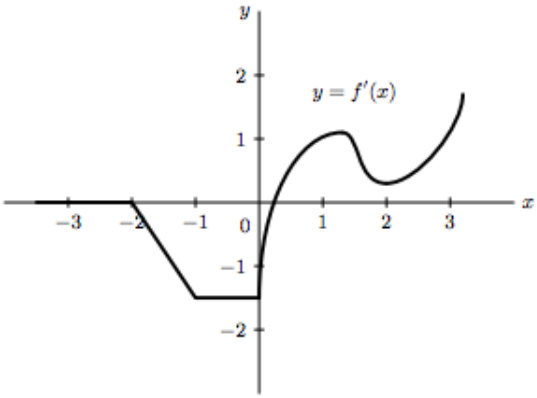
e. [2 points] $f(x)$ is constant.

$-3 < x < -2$ $\quad -2 < x < -1$ $\quad -1 < x < 0$ $\quad 0 < x < 1$ $\quad 1 < x < 2$ $\quad 2 < x < 3$ $\quad$ NONE

Solution.

10. [10 points] Below is the graph of $f'(x)$, the **derivative** of the function $f(x)$.

Note that $f'(x)$ is zero for $x \leq -2$, linear for $-2 < x < -1$, and constant for $-1 < x < 0$.



For each of the following, circle all of the listed intervals for which the given statement is true over the **entire** interval. If there are no such intervals, circle NONE.
You do not need to explain your reasoning.

a. [2 points] $f'(x)$ is increasing.

$-2 < x < -1$ $\boxed{0 < x < 1}$ $1 < x < 2$ $\boxed{2 < x < 3}$ NONE

b. [2 points] $f'(x)$ is concave up.

$0 < x < 1$ $1 < x < 2$ $\boxed{2 < x < 3}$ NONE

c. [2 points] $f(x)$ is increasing.

$-2 < x < -1$ $-1 < x < 0$ $0 < x < 1$ $\boxed{1 < x < 2}$ $\boxed{2 < x < 3}$ NONE

d. [2 points] $f(x)$ is linear but not constant.

$-3 < x < -2$ $-2 < x < -1$ $\boxed{-1 < x < 0}$ $0 < x < 1$ $1 < x < 2$ $2 < x < 3$ NONE

e. [2 points] $f(x)$ is constant.

$-3 < \boxed{x < -2}$ $-2 < x < -1$ $-1 < x < 0$ $0 < x < 1$ $1 < x < 2$ $2 < x < 3$ NONE

□

2.6 Differentiablity

At this section we will concentrate on what we already know: We say that a function $f(x)$ is differentiable at $x = a$ if $f'(a)$ exists (and finite).
If $f'(a)$ exists then $f(x)$ is continuous at $x = a$. So $\boxed{\text{differentiability implies continuity}}$ but NOT the other way around.

When a function is not differentiable?

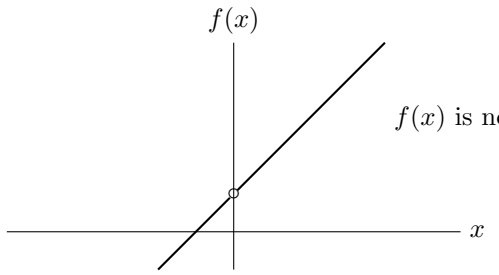
- Not defined at the point.
- Not continuous at the point.
- Not "smooth" at the point, i.e. has a sharp edge or V-like shape.

Conversely:

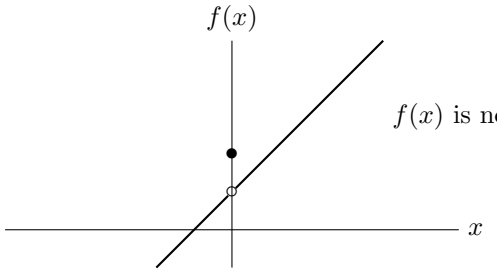
If $f(x)$ is differentiable at $x = a$, i.e. $f'(a)$ exists, then all must be true:

- $f(x)$ is defined at $x = a$.
- $f(x)$ is continuous at $x = a$.
- the graph of $f(x)$ is smooth at $x = a$.

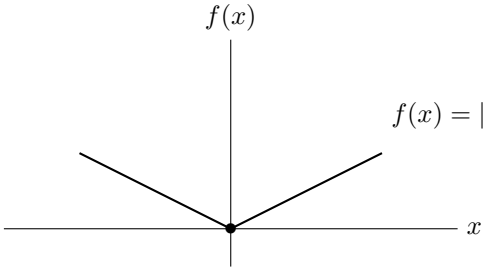
Here are a few examples of functions that are not differentiable at $x = 0$:



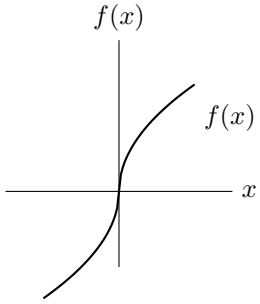
$f(x)$ is not defined at $x = 0$



$f(x)$ is not continuous at $x = 0$



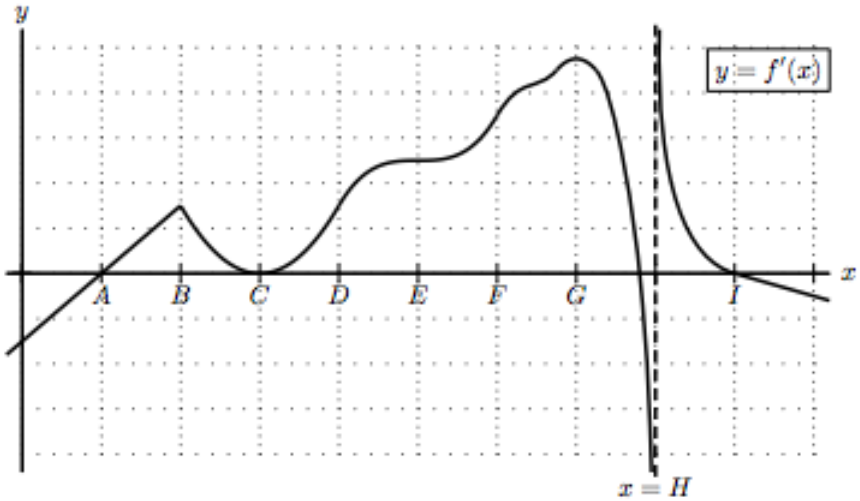
$f(x) = |x|$, has a sharp edge at $x = 0$



$f(x) = \sqrt{|x|}$, has a slop ∞ at $x = 0$

Exercise 2.6.1.

3. [12 points] The graph of a portion of $y = f'(x)$, the derivative of $f(x)$ is shown below. Note that there is a sharp corner at $x = B$ and that $x = H$ is a vertical asymptote.
The function $f(x)$ is continuous with domain $(-\infty, \infty)$.



For each of the questions below, circle all of the available correct answers.
(Circle NONE if none of the available choices are correct.)

a. [2 points] At which of the following six values of x is the function $f(x)$ not differentiable?

B C E F H I NONE

b. [2 points] At which of the following six values of x does the function $f'(x)$ appear to be not differentiable?

A B C D E F NONE

c. [2 points] At which of the following nine values of x does $f(x)$ have a critical point?

A B C D E F G H I NONE

d. [2 points] At which of the following nine values of x does $f(x)$ have a local minimum?

A B C D E F G H I NONE

e. [2 points] At which of the following nine values of x is $f''(x) = 0$?

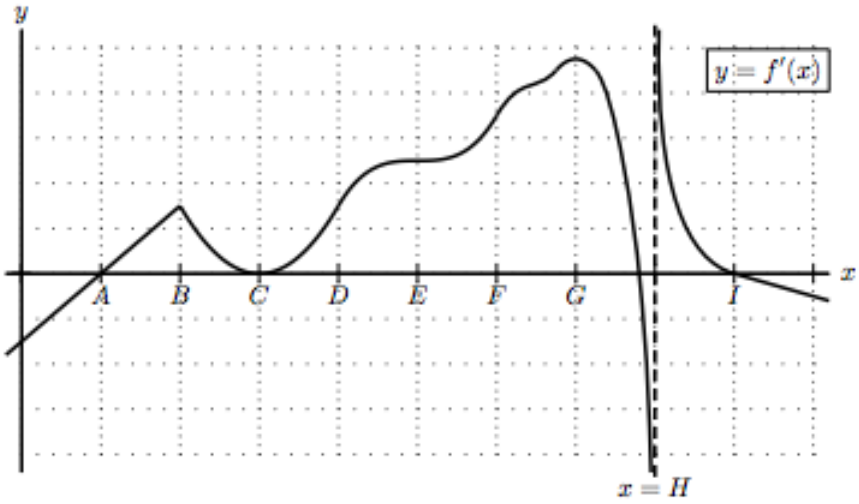
A B C D E F G H I NONE

f. [2 points] At which of the following nine values of x does $f(x)$ have an inflection point?

A B C D E F G H I NONE

Solution.

3. [12 points] The graph of a portion of $y = f'(x)$, the derivative of $f(x)$ is shown below. Note that there is a sharp corner at $x = B$ and that $x = H$ is a vertical asymptote.
The function $f(x)$ is continuous with domain $(-\infty, \infty)$.



For each of the questions below, circle all of the available correct answers.
(Circle NONE if none of the available choices are correct.)

a. [2 points] At which of the following six values of x is the function $f(x)$ not differentiable?

B C E F $\boxed{H}$ I NONE

b. [2 points] At which of the following six values of x does the function $f'(x)$ appear to be not differentiable?

A $\boxed{B}$ C D E F NONE

c. [2 points] At which of the following nine values of x does $f(x)$ have a critical point?

$\boxed{A}$ B $\boxed{C}$ D E F G $\boxed{H}$ $\boxed{I}$ NONE

d. [2 points] At which of the following nine values of x does $f(x)$ have a local minimum?

$\boxed{A}$ B C D E F G $\boxed{H}$ I NONE

e. [2 points] At which of the following nine values of x is $f''(x) = 0$?

A B $\boxed{C}$ D $\boxed{E}$ F $\boxed{G}$ H I NONE

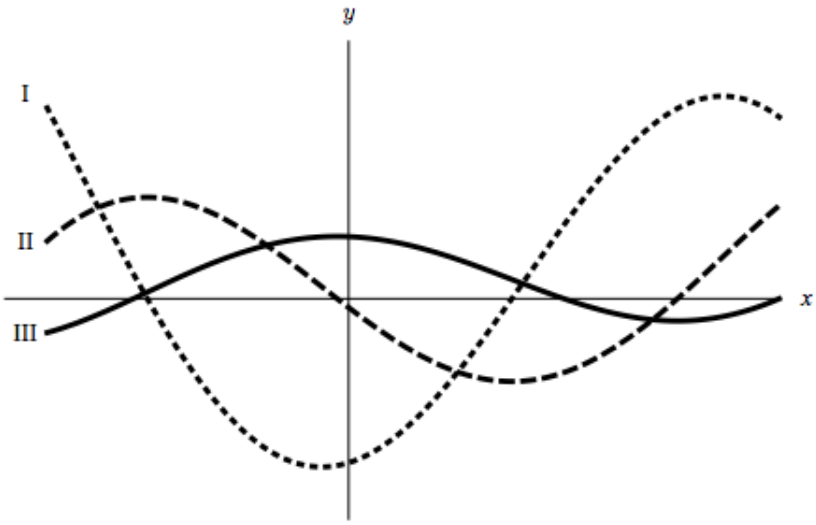
f. [2 points] At which of the following nine values of x does $f(x)$ have an inflection point?

A $\boxed{B}$ $\boxed{C}$ D E F $\boxed{G}$ H I NONE

□

Exercise 2.6.2.

7. [12 points] On the axes below are graphed f, f', and f''. Determine which is which, and justify your response with a brief explanation.



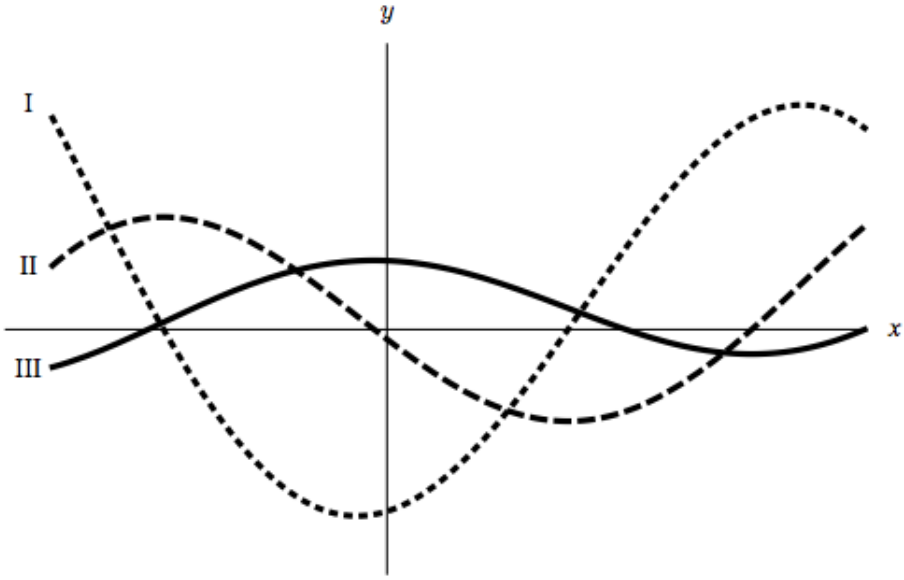
f:_____

f':_____

f'':_____

Solution.

7. [12 points] On the axes below are graphed f, f', and f''. Determine which is which, and justify your response with a brief explanation.



Solution: Looking to the far right of the graph, curve **I** has a critical point where it has a slope of zero. At this x-coordinate neither of the other graphs has a root. This means the derivative of **I** is not in this figure, so **I** must be f''. Looking to the far left of the graph, **II** has a local maximum where its derivative is zero. Although **III** has a root near the same x-value, **III** changes sign from negative to positive at this point. By the first derivative test, **III** cannot be the derivative of **II**. Thus, by process of elimination, **II** must be f' and **III** must be f.

f: **III**

f': **II**

f'': **I**

□

3 Shortcuts to differentiation

3.1 Polynomials - derivative formulas

So far, we were dealing with the definition of the derivative. It turns out that there are formulas that can help us with computing the derivative function.
For the power function, $f(x) = x^n$, n is any real number, we have:

$$f'(x) = \boxed{(x^n)' = nx^{n-1}}$$

For example: $(x^2)' = 2x$, $(1/x^3)' = (x^{-3})' = -3x^{-4}$. $(x^e)' = ex^{e-1}$.
The derivative of a constant is zero, because: $f(x) = c = cx^0 \Rightarrow f'(x) = c(x^0)' = c \cdot 0 \cdot x^{-1} = 0$.
The next property, is that the derivative commutes with multiplying by a constance c:

$$(c \cdot f(x))' = cf'(x).$$

For example: $(5x^2)' = 5(x^2)' = 10x$.
The derivative of a constant is zero, because: $f(x) = c = cx^0 \Rightarrow f'(x) = c(x^0)' = c \cdot 0 \cdot x^{-1} = 0$.
Another property, is "the derivative of the sum is the sum of the derivatives":

$$\boxed{(f(x) \pm g(x))' = f'(x) \pm g'(x)}$$

Now, we can take the derivative of any polynomial:

$$f(x) = x^3 + 4x^2 - 5x + 7 \Rightarrow f'(x) = 3x^2 + 6x - 5$$

Example 3.1.1. Let $f(x) = x + \sqrt{x} + 1$. Find the equation of the graph's tangent line at $x = 1$.
So the point is $(1, 3)$. The tangent line will have a slope of $f'(1)$. So

$$f'(x) = 1 + 0.5x^{-0.5} \Rightarrow f'(1) = 1.5$$

So we need the line with slope 1.5 passing through $(1, 3)$:

$$y - 3 = 1.5(x - 1) \Rightarrow y = 1.5x + 1.5 = 1.5(x + 1)$$

3.2 Exponential Function

Here are the formulas:

$$\boxed{(e^x)' = e^x}$$ [8]

$$\boxed{(a^x)' = \ln(a) \cdot a^x}$$

Note that $\ln(a)$ is just a number.

Example 3.2.1. What is the derivative of a^{kx}?

$$(a^{kx})' = ((a^k)^x)' = \ln(a^k)a^{kx} = \boxed{k\ln(a)a^{kx}}$$

Skip to Section 3.5 - Trig Functions

Remark 3.2.2.

$$\boxed{(\sin x)' = \cos x}$$
$$\boxed{(\cos x)' = -\sin x}$$
$$\boxed{(\tan x)' = 1/\cos^2 x = 1 + \tan^2 x}$$

Exercise 3.2.3. In groups:

1. $(x^2 + x^\pi)' = 2x + \pi x^{\pi-1}$
2. $(2\cos(x) - \sin(x))' = -2\sin(x) - \cos(x)$
3. $(x^2 + \cos(x))'' = 2 - \cos(x)$
4. $(2^x)'' = \ln(2)^2 2^x$
5. Find a, b, c such that $f(x) = ax^2 + bx + c$ and $g(x) = 2^x$ satisfy: $f(0) = g(0), f'(0) = g'(0), f'(0) = g'(0)$:
   We have $g(0) = 1, g'(0) = \ln(2), g''(0) = \ln^2(2)$.
   We have $f(0) = c, f'(0) = b, f''(0) = 2a \Rightarrow c = 1, b = \ln(2), a = \ln^2(2)/2$

[8] this is the reason why e is so special

3.3 Product Rule, Quotient Rule

$$(f(x)g(x))' = f' \cdot g + f \cdot g'$$

Example 3.3.1.

1. $(\sin(x)\cos(x)) = \sin(x)'\cos(x) + \sin(x)\cos(x)' = \cos^2(x) - \sin^2(x)$
2. $(x^2)' = (x \cdot x)' = 1x + x \cdot 1 = 2x$
3. $(f(x)^2)' = (f(x) \cdot f(x))' = f'f + ff' = 2f \cdot f'$
4. $\boxed{(f(x)^n)' = nf(x)^{n-1} \cdot f'(x)}$
5. $(\sin^2(x) + \cos^2(x))' = 2\sin(x)\cos(x) + 2\cos(x) \cdot -\sin(x) = 0$ This is not surprising since $\sin^2(x) + \cos^2(x) = 1$, it is a constant function.

Quotient rule:

$$\boxed{\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}}$$

Proof. $f/g = f \cdot g^{-1}$ so we use the product rule:

$$(f \cdot g^{-1})' = f'g^{-1} + f \cdot (g^{-1})' = f'g^{-1} + f \cdot -1 \cdot g^{-2} \cdot g = \frac{f'g - g'f}{g^2}$$

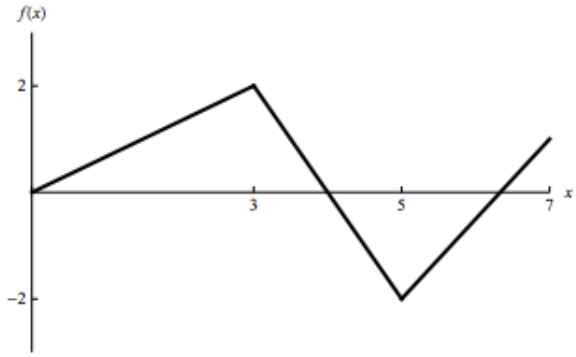
□

One more formula that you need:

$$\boxed{(\ln(x))' = \frac{1}{x} = x^{-1}}$$

Exercise 3.3.2.
Please IGNORE parts i and iv.

1. [10 points] Given below is a graph of a function $f(x)$ and a table for a function $g(x)$.



x	0	1	2	3	4
$g(x)$	4	3	1	2	$\frac{20}{3}$
$g'(x)$	-2	$-\frac{5}{2}$	$\frac{1}{2}$	3	$-\frac{1}{3}$

Give answers for the following or write "Does not exist." No partial credit will be given.

i) $\frac{d}{dx}f(g(x))$ at $x = 0$

ii) $\frac{d}{dx}[f(x)g(x)]$ at $x = 2$

iii) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$ at $x = 4$

iv) $\frac{d}{dx}[g(f(x))]$ at $x = 3$

v) $f(g'(3))$

Solution.

i) $\frac{d}{dx}f(g(x))$ at $x = 0$ By the chain rule $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$. At $x = 0$ we have

$$f'(g(0))g'(0) = f'(4) \cdot (-2) = 4.$$

ii) $\frac{d}{dx}[f(x)g(x)]$ at $x = 2$ By the product rule $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$. At $x = 2$ we have

$$f'(2)g(2) + f(2)g'(2) = (2/3)(1) + (4/3)(1/2) = 4/3.$$

iii) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$ at $x = 4$ By the quotient rule $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$. At $x = 4$ we have

$$\frac{f'(4)g(4) - f(4)g'(4)}{[g(4)]^2} = \frac{(-2)(20/3) - (0)(-1/3)}{(20/3)^2} = -3/10.$$

iv) $\frac{d}{dx}[g(f(x))]$ at $x = 3$ We know $g'(f(3)) = 1/2$, so for values of y near $f(3)$, $g(y)$ looks like a line with slope $1/2$. So $g(f(x))$ "looks like" $\frac{1}{2}f(x) + b$ for some constant b, for x near 3. Since $f(x)$ is "pointy" at $x = 3$, $\frac{1}{2}f(x) + b$ looks like a vertically compressed version of this pointy graph (near $x = 3$), which is still pointy. So $g(f(x))$ is also pointy at $x = 3$, hence not differentiable.

v) $f(g'(3))$ By the table, $g'(3) = 3$, so $f(g'(3)) = f(3) = 2$.

□

Exercise 3.3.3.

8. [12 points] In the following table, both f and g are differentiable functions of x. In addition, $g(x)$ is an invertible function. Write your answers in the blanks provided. You do not need to show your work.

x	2	3	4	5
$f(x)$	7	6	2	9
$f'(x)$	-2	1	3	2
$g(x)$	1	4	7	11
$g'(x)$	1	2	3	2

a. [3 points] If $h(x) = \frac{g(x)}{f(x)}$, find $h'(4)$.

$h'(4) =$ _____

b. [3 points] If $k(x) = f(x)g(x)$, find $k'(2)$.

$k'(2) =$ _____

Solution.

8. [12 points] In the following table, both f and g are differentiable functions of x. In addition, $g(x)$ is an invertible function. Write your answers in the blanks provided. You do not need to show your work.

x	2	3	4	5
$f(x)$	7	6	2	9
$f'(x)$	-2	1	3	2
$g(x)$	1	4	7	11
$g'(x)$	1	2	3	2

a. [3 points] If $h(x) = \frac{g(x)}{f(x)}$, find $h'(4)$.

$h'(4) =$ **-15/4**

b. [3 points] If $k(x) = f(x)g(x)$, find $k'(2)$.

$k'(2) =$ **5**

□

Quiz #4. Please write you name: _____ and email: _____

Let $f(x) = e^{2x} + \tan(x)$.

1. [20 pt] Where is $f(x)$ continuous?

2. [20 pt] Where is $f(x)$ differentiable?

3. [60 pt]Find the equation of the graph's tangent line at $x = 0$

Solution.
The function is differentiable and continuous as long as it is defined. e^{2x} is defined anywhere whereas $\tan(x) = \sin(x)/\cos(x)$ is defined anywhere expect where where $\cos(x) = 0$, that is for $x = \pi/2, \pi/2 + \pi, \pi/2 + 2\pi, ...$ and $x = -\pi/2, \pi/2 - \pi, -\pi/2 - 2\pi, ...$ We can write that $f(x)$ is defined on $(-\infty, \infty)$ expect for $x = \pi/2 + n\pi$ for any integer n.
$f'(x) = ((e^2)^x)' + \tan'(x) = ln(e^2)e^{2x} + 1 + \tan^2(x) = 2e^{2x} + 1 + \tan^2(x)$. So $f'(0) = 3$. Therefore, the tangent line has slope 3 and passes through $(0, 1)$, so the equation is:

$$y = 3x + 1$$

□

3.4 Chain Rule

The chain rule helps us the take the derivative of a composite function. Say:

$$h(x) = f(g(x))$$

Then $h'(x)$ is calculated in three steps. First, calculate the derivative of $f(\boxed{})$. Then of $g(x)$. Then plug in $g(x)$ in the box and get:

$$\boxed{\frac{d}{dx}f(g(x)) = f'(\boxed{g(x)}) \cdot g'(x)}$$

We can use this formula as long as $g(x) \neq 0$ and both $g'(x)$ and $f'(g(x))$ exist.

Example 3.4.1.

- $(\cos^2(x))' = 2\boxed{cos(x)} \cdot (-\sin(x)) = -2\cos(x)\sin(x)$
- $(e^{x^2})' = e^{\boxed{x^2}} \cdot 2x = 2xe^{x^2}$
- $(\tan(\sin(x)))' = (1 + \tan^2 \boxed{\sin(x)}) \cdot \cos(x)$

Group Work please complete parts i and iv in **3.3.2**

Remark 3.4.2. If $g(x)$ in not differentiable at some $x = a$ it does NOT mean that $f(g(x))$ is not differentiable at $g(a)$. Here is an example.
Let $g(x) = |x|$, the absolute value of x. Since it has a sharp edge at $x = 0$, it is not differentiable. Now, take $f(x) = x^2$. Notice: $f(g(x)) = (|x|)^2 = x^2$, which is differentiable everywhere.

3.5 Trig Functions

We already gave the relevant formulas here: **3.2.2**

3.6 Inverse Functions

When $f(x)$ is invertible we can define the inverse $g(x) = f^{-1}(x)$. Then $g(f(x)) = x$. Now take the derivative. we get:

$$1 = g'(f(x))f'(x) \Rightarrow \boxed{[f^{-1}(z)]'|_{z=f(x)} = \frac{1}{f'(x)}}$$

Example 3.6.1.

- $f(x) = \tan(x)$. $f^{-1}(1) =?$ ($f^{-1}(z)$ is called $\arctan(z)$). We have $[f^{-1}(z)]'|_{z=f(x)} = \frac{1}{f'(x)}$ so we just need to find x such that $f(x) = 1$. Then the answer will be $1/f'(x)$. $\tan(x) = 1$, so choose $x = \pi/4$. Then $f'(\pi/4) = 1 + \tan^2(\pi/4) = 1 + 1^2 = 2$. So the answer is $1/2$.
- We can do that in general: $\arctan'(z) = \frac{1}{\tan'(x)}$ with $\tan(x) = z$. $\tan'(x) = 1 + \tan^2(x) = 1 + z^2$. Thus

$$\boxed{\arctan'(z) = \frac{1}{1+z^2}}$$

- $\arcsin'(z) = 1/\cos(x)$ with $\sin(x) = z$. We can write $\cos(x) = \sqrt{1 - \sin^2(x)} = \sqrt{1 - z^2}$. So:

$$\boxed{\arcsin'(z) = \frac{1}{\sqrt{1-z^2}}}$$

Exercise 3.6.2.

1. [12 points] The table below gives several values of a differentiable function $f(x)$. Assume that both $f(x)$ and $f'(x)$ are invertible. Do not give approximations. If it is not possible to find the value exactly, write NOT POSSIBLE.

x	-3	-2	-1	0	1	2	3
$f(x)$	-8	-4	-1.2	0.5	1.4	1.8	2
$f'(x)$	5	3	2	1.2	0.5	0.3	0.1

a. [2 points] Let $g(x) = 3f(x) + 4$. Find $g'(1)$.

Answer: $g'(1) =$ _____

b. [2 points] Find $(f^{-1})'(2)$.

Answer: $(f^{-1})'(2) =$ _____

c. [2 points] Let $h(x) = f(e^x)$. Find $h'(\ln 2)$.

Answer: $h'(\ln 2) =$ _____

d. [2 points] Let $j(x) = e^{f(x)}$. Find $j'(-2)$.

Answer: $j'(-2) =$ _____

e. [2 points] Let $k(x) = f(x)f(x-2)$. Find $k'(1)$.

Answer: $k'(1) =$ _____

f. [2 points] Let $\ell(x) = \frac{f(x)}{f(x+3)}$. Find $\ell'(0)$.

Answer: $\ell'(0) =$ _____

Solution.

1. [12 points] The table below gives several values of a differentiable function $f(x)$. Assume that both $f(x)$ and $f'(x)$ are invertible. Do not give approximations. If it is not possible to find the value exactly, write NOT POSSIBLE.

x	-3	-2	-1	0	1	2	3
$f(x)$	-8	-4	-1.2	0.5	1.4	1.8	2
$f'(x)$	5	3	2	1.2	0.5	0.3	0.1

a. [2 points] Let $g(x) = 3f(x) + 4$. Find $g'(1)$.

Solution: $g'(x) = 3f'(x)$, so $g'(1) = 3 \cdot 0.5 = 1.5$

Answer: $g'(1) =$ 1.5

b. [2 points] Find $(f^{-1})'(2)$.

Solution: $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$, so $(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(3)} = \dfrac{1}{0.1} = 10.$

Answer: $(f^{-1})'(2) =$ 10

c. [2 points] Let $h(x) = f(e^x)$. Find $h'(\ln 2)$.

Solution: $h'(x) = f'(e^x) \cdot e^x$, so $h'(\ln 2) = f'(e^{\ln 2}) \cdot e^{\ln 2} = f'(2) \cdot 2 = 0.3 \cdot 2 = 0.6.$

Answer: $h'(\ln 2) =$ 0.6

d. [2 points] Let $j(x) = e^{f(x)}$. Find $j'(-2)$.

Solution: $j'(x) = e^{f(x)} \cdot f'(x)$, so $j'(-2) = e^{f(-2)} \cdot f'(-2) = e^{-4} \cdot 3.$

Answer: $j'(-2) =$ $3e^{-4}$

e. [2 points] Let $k(x) = f(x)f(x-2)$. Find $k'(1)$.

Solution: $k'(x) = f'(x)f(x-2) + f(x)f'(x-2)$, so
$k'(1) = f'(1)f(1-2) + f(1)f'(1-2) = f'(1)f(-1) + f(1)f'(-1) = 0.5 \cdot (-1.2) + 1.4 \cdot 2 = -0.6 + 2.8 = 2.2.$

Answer: $k'(1) =$ 2.2

f. [2 points] Let $\ell(x) = \frac{f(x)}{f(x+3)}$. Find $\ell'(0)$.

Solution: $\ell'(x) = \dfrac{f'(x)f(x+3) - f'(x+3)f(x)}{(f(x+3))^2}$, so
$\ell'(0) = \dfrac{f'(0)f(3) - f'(3)f(0)}{(f(3))^2} = \dfrac{1.2 \cdot 2 - 0.5 \cdot 0.1}{2^2} = \dfrac{2.4 - 0.05}{4} = \dfrac{2.35}{4} = 0.5875.$

Answer: $\ell'(0) =$ 0.5875

□

Exercise 3.6.3.

8. [12 points] In the following table, both f and g are differentiable functions of x. In addition, $g(x)$ is an invertible function. Write your answers in the blanks provided. You do not need to show your work.

x	2	3	4	5
$f(x)$	7	6	2	9
$f'(x)$	-2	1	3	2
$g(x)$	1	4	7	11
$g'(x)$	1	2	3	2

a. [3 points] If $h(x) = \frac{g(x)}{f(x)}$, find $h'(4)$.

$h'(4) =$ _____

b. [3 points] If $k(x) = f(x)g(x)$, find $k'(2)$.

$k'(2) =$ _____

c. [3 points] If $m(x) = g^{-1}(x)$, find $m'(4)$.

$m'(4) =$ _____

d. [3 points] If $n(x) = f(g(x))$, find $n'(3)$.

$n'(3) =$ _____

Solution.

8. [12 points] In the following table, both f and g are differentiable functions of x. In addition, $g(x)$ is an invertible function. Write your answers in the blanks provided. You do not need to show your work.

x	2	3	4	5
$f(x)$	7	6	2	9
$f'(x)$	-2	1	3	2
$g(x)$	1	4	7	11
$g'(x)$	1	2	3	2

a. [3 points] If $h(x) = \frac{g(x)}{f(x)}$, find $h'(4)$.

$h'(4) =$ **-15/4**

b. [3 points] If $k(x) = f(x)g(x)$, find $k'(2)$.

$k'(2) =$ **5**

c. [3 points] If $m(x) = g^{-1}(x)$, find $m'(4)$.

$m'(4) =$ **1/2**

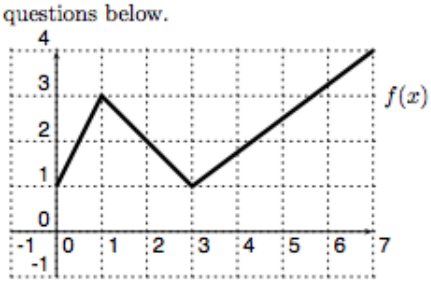
d. [3 points] If $n(x) = f(g(x))$, find $n'(3)$.

$n'(3) =$ **6**

□

Exercise 3.6.4.

2. [12 points]
Use the graph of the function f and the table of values for the function g to answer the questions below.



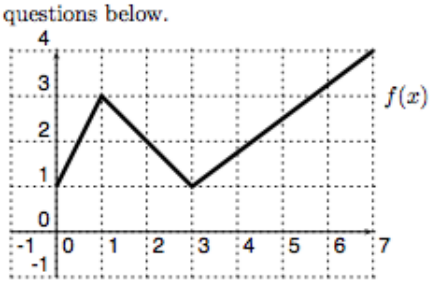
x	1	2	3	4	5	6
g(x)	0	4	0	-18	-56	-120
g'(x)	6	1	-10	-27	-50	-79
g"(x)	-2	-8	-14	-20	-26	-32

a. [6 points] Let $h(x) = \frac{g(x)}{f(2x+3)}$. Find $h'(1)$ or explain why it does not exist.

b. [6 points] Let $k(x) = g(g(x))$. Determine whether k is increasing or decreasing at $x = 2$.

Solution.

2. [12 points]
Use the graph of the function f and the table of values for the function g to answer the questions below.



x	1	2	3	4	5	6
g(x)	0	4	0	-18	-56	-120
g'(x)	6	1	-10	-27	-50	-79
g''(x)	-2	-8	-14	-20	-26	-32

a. [6 points] Let $h(x) = \frac{g(x)}{f(2x+3)}$. Find $h'(1)$ or explain why it does not exist.

Solution: Using the quotient rule and the chain rule, we get

$$\begin{aligned} h'(x) &= \frac{g'(x)f(2x+3) - g(x)f'(2x+3)\cdot 2}{(f(2x+3))^2} \\ h'(1) &= \frac{g'(1)f(5) - g(1)f'(5)\cdot 2}{(f(5))^2} \\ &= \frac{6\cdot 2.5 - 0\cdot 0.75\cdot 2}{(2.5)^2} \\ &= \frac{6}{2.5} = \frac{12}{5} = 2.4 \end{aligned}$$

b. [6 points] Let $k(x) = g(g(x))$. Determine whether k is increasing or decreasing at $x = 2$.

Solution: Using the chain rule, we get

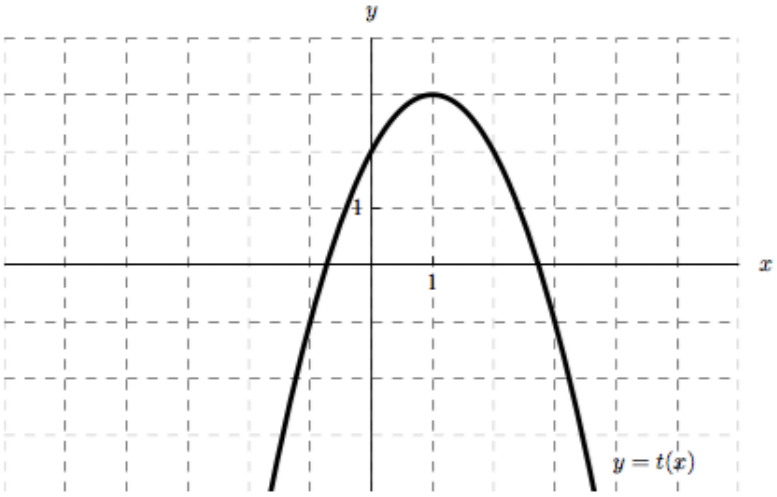
$$\begin{aligned} k'(x) &= g'(g(x))\cdot g'(x) \\ k'(2) &= g'(g(2))\cdot g'(2) \\ &= g'(4)\cdot g'(2) \\ &= (-27)\cdot 1 = -27 \end{aligned}$$

Since $k'(2) < 0$, we know that $k(x)$ is decreasing at $x = 2$.

□

Exercise 3.6.5.

2. [16 points]
Graphed below is a function $t(x)$. Define $p(x) = x^2 t(x)$, $q(x) = t(\sin(x))$, $r(x) = \frac{t(x)}{3x+1}$, and $s(x) = t(t(x))$. For this problem, do not assume $t(x)$ is quadratic.



Carefully estimate the following quantities.

a. [4 points] $p'(-1)$

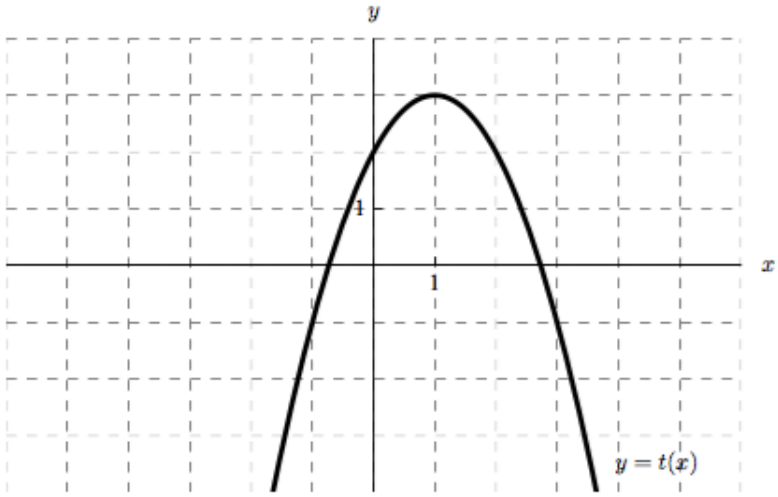
b. [4 points] $q'(0)$

c. [4 points] $r'(3)$

d. [4 points] $s'(0)$

Solution.

2. [16 points]
Graphed below is a function $t(x)$. Define $p(x) = x^2 t(x)$, $q(x) = t(\sin(x))$, $r(x) = \frac{t(x)}{3x+1}$, and $s(x) = t(t(x))$. For this problem, do not assume $t(x)$ is quadratic.



Carefully estimate the following quantities.

a. [4 points] $p'(-1)$

Solution: By the product rule, $p'(x) = 2xt(x) + x^2 t'(x)$. Estimating using the graph, we have

$$p'(-1) = 2(-1)t(-1) + (-1)^2 t'(-1) = (-2)(-1) + 4 = 6.$$

b. [4 points] $q'(0)$

Solution: By the chain rule, $q'(x) = t'(\sin x)\cos x$. Estimating using the graph, we have

$$q'(0) = t'(\sin 0)\cos 0 = t'(0) = 2.$$

c. [4 points] $r'(3)$

Solution: By the quotient rule, $r'(x) = \frac{(3x+1)t'(x) - 3t(x)}{(3x+1)^2}$. Estimating using the graph, we have

$$r'(3) = \frac{(3(3)+1)t'(3) - 3t(3)}{(3(3)+1)^2} = \frac{-40 - 3(-1)}{100} = -\frac{37}{100}$$

d. [4 points] $s'(0)$

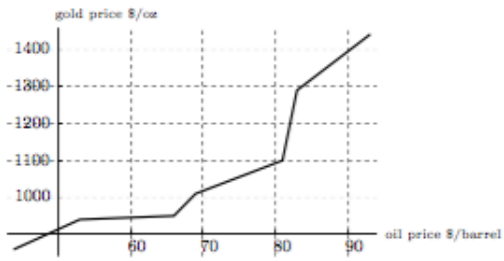
Solution: By the chain rule, $s'(x) = t'(t(x))t'(x)$. Estimating using the graph, we have

$$s'(0) = t'(t(0))t'(0) = t'(2) \cdot 2 = (-2)(2) = -4.$$

□

Exercise 3.6.6.

5. [15 points] The graph to the right shows a function $G(b)$ that approximates the price of an ounce of gold (in dollars) as a function of the cost of a barrel of oil for data between 2009 and 2011.[1]



a. [3 points] Estimate $G'(70)$.

b. [5 points] Recall that G^{-1} is defined to be a function such that $G^{-1}(G(b)) = b$ (or such that $G(G^{-1}(y)) = y$, where y is the price of an ounce of gold). Derive, using the chain rule, a formula for $(G^{-1})'$ in terms of G'.

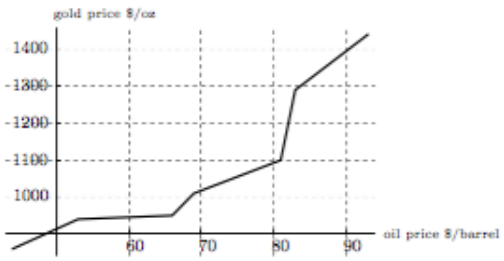
c. [4 points] Using parts (a) and (b), estimate $(G^{-1})'(G(70))$.

d. [3 points] Explain the practical meaning of your result in (c).

[1] Gold prices from <http://www.goldprice.org/>; oil from <http://en.wikipedia.org/wiki/Price_of_petroleum>.

Solution.

5. [15 points] The graph to the right shows a function $G(b)$ that approximates the price of an ounce of gold (in dollars) as a function of the cost of a barrel of oil for data between 2009 and 2011.[1]



a. [3 points] Estimate $G'(70)$.

Solution: From the graph, it appears that between $b = 70$ and $b = 80$, G increases by about 70 as b increases about 10. Thus we estimate that $G'(70) \approx 7$ \$/oz per \$/barrel.

b. [5 points] Recall that G^{-1} is defined to be a function such that $G^{-1}(G(b)) = b$ (or such that $G(G^{-1}(y)) = y$, where y is the price of an ounce of gold). Derive, using the chain rule, a formula for $(G^{-1})'$ in terms of G'.

Solution: We know that $G^{-1}(G(b)) = b$. Thus $\frac{d}{db}G^{-1}(G(b)) = 1$. Differentiating the left-hand side of this using the chain rule, we have $\frac{d}{db}G^{-1}(G(b)) = (G^{-1})'(G(b)) \cdot G'(b) = 1$. Thus $(G^{-1})'(G(b)) = 1/G'(b)$.
Alternately, if we start with $G(G^{-1}(y)) = y$, we have $\frac{d}{dy}G(G^{-1}(y)) = 1$. Applying the chain rule to the left-hand side, we have $G'(G^{-1}(y)) \cdot (G^{-1})'(y) = 1$, so that $(G^{-1})'(y) = 1/G'(G^{-1}(y))$. (Obviously, with $y = G(b)$, this is the same as the previous expression.)

c. [4 points] Using parts (a) and (b), estimate $(G^{-1})'(G(70))$.

Solution: Using part (b), we have $(G^{-1})'(G(70)) = 1/G'(70) = 1/7$ \$/barrel per \$/oz.

d. [3 points] Explain the practical meaning of your result in (c).

Solution: $(G^{-1})'(G(70)) = 0.14$ indicates that when the price of oil is 70 \$/barrel, the price of a barrel of oil goes up by about \$0.14 if the price of gold goes up by \$1.

[1] Gold prices from <http://www.goldprice.org/>; oil from <http://en.wikipedia.org/wiki/Price_of_petroleum>.

□

Exercise 3.6.7. Take a practice Gateway exam.

3.7 Implicit functions

Sometimes we are presented with an equation that includes functions. What we can is the take the derivative of both sides an thus get an equation for the derivative.

Example 3.7.1. Suppose we have

$$y^2x^3 = 4$$

And we need to find the derivative of $y(x)$ when $x = 1$. Then we take the derivative of both sides. Remember that $y(x)$ is a function so we need to use the chain rule.

$$2yy'x^3 + 3y^2x^2 = 0$$

This is an equation for the derivative. Plug in $x = 1$. We get:

$$2y(1)y'(1) + 3y^2(1) = 0 \Rightarrow y'(1) = -(3/2)y(1)$$

But what is $y(1)$? we have the original equation. We can plug in $x = 1$ and get:

$$y^2(1) = 4 \Rightarrow y(1) = \pm 2$$

So $y'(1) = \mp 3$. To get a specific value, we need more information. For example, whether $y(x)$ is increasing or decreasing when $x = 1$.

Exercise 3.7.2.

5. [13 points] The equation below implicitly defines a hyperbola.

$$x^2 - y^2 = 2x + xy + y + 2.$$

a. [5 points] Find $\frac{dy}{dx}$.

b. [4 points] Consider the two points $(4, 2)$ and $(2, -1)$. Show that one of these points lies on the hyperbola defined above, and one does not.

c. [4 points] For the point in part (b) which lies on the hyperbola, find the equation of the line which is tangent to the hyperbola at this point.

Solution.

5. [13 points] The equation below implicitly defines a hyperbola.

$$x^2 - y^2 = 2x + xy + y + 2.$$

a. [5 points] Find $\frac{dy}{dx}$.

Solution: We use implicit differentiation:

$$2x - 2y\frac{dy}{dx} = 2 + \left(1 \cdot y + x \cdot \frac{dy}{dx}\right) + \frac{dy}{dx} + 0.$$

Then solve for $\frac{dy}{dx}$:

$$\begin{aligned} 2x - 2 - y &= \frac{dy}{dx}(x + 2y + 1) \\ \frac{dy}{dx} &= \frac{2x - 2 - y}{x + 2y + 1} \end{aligned}$$

b. [4 points] Consider the two points $(4, 2)$ and $(2, -1)$. Show that one of these points lies on the hyperbola defined above, and one does not.

Solution: For the point $(4, 2)$, $x^2 - y^2 = 4^2 - 2^2 = 12$ and $2x + xy + y + 2 = 2(4) + 4(2) + 2 + 2 = 20$ are not equal, so $(4, 2)$ IS NOT on the hyperbola.

For the point $(2, -1)$, $x^2 - y^2 = 2^2 - (-1)^2 = 3$ and $2x + xy + y + 2 = 2(2) + 2(-1) - 1 + 2 = 3$ are equal, so $(2, -1)$ IS on the hyperbola.

c. [4 points] For the point in part (b) which lies on the hyperbola, find the equation of the line which is tangent to the hyperbola at this point.

Solution: From part (a),

$$\frac{dy}{dx} = \frac{2x - 2 - y}{x + 2y + 1},$$

so

$$\frac{dy}{dx}\Big|_{(x,y)=(2,-1)} = \frac{2(2) - 2 - (-1)}{2 + 2(-1) + 1} = \frac{3}{1} = 3.$$

Then the equation of the tangent line is $y = 3(x - 2) - 1$ or $y = 3x - 7$.

□

Exercise 3.7.3.

5. [12 points] Suppose a curve in the plane is given by the equation

$$\sin(\pi xy) = y - 1.$$

a. [3 points] Verify that the point $(x, y) = (1, 1)$ is on the curve.

b. [5 points] Calculate $\frac{dy}{dx}$.

c. [4 points] Find the equation for the tangent line to the curve at the point $(1, 1)$.

Solution.

5. [12 points] Suppose a curve in the plane is given by the equation

$$\sin(\pi xy) = y - 1.$$

a. [3 points] Verify that the point $(x, y) = (1, 1)$ is on the curve.

Solution: At $(1, 1)$, the right hand side is $\sin(\pi) = 0$ and the left hand side is $1 - 1 = 0$. Therefore the point is on the curve since the right and left hand sides are equal.

b. [5 points] Calculate $\frac{dy}{dx}$.

Solution: Taking the derivative with respect to x of the equation, we have

$$\pi \cos(\pi xy) \cdot (y + x\frac{dy}{dx}) = \frac{dy}{dx}.$$

Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{\pi y \cos(\pi xy)}{1 - \pi x \cos(\pi xy)}.$$

c. [4 points] Find the equation for the tangent line to the curve at the point $(1, 1)$.

Solution: The slope of the tangent line to the curve is

$$\frac{dy}{dx}(1, 1) = \frac{\pi \cos(\pi)}{1 - \pi \cos(\pi)} = \frac{-\pi}{1 + \pi}.$$

The equation for the tangent line is

$$y - 1 = \frac{-\pi}{1 + \pi}(x - 1).$$

□

Exercise 3.7.4.

1. [12 points] The following questions relate to the implicit curve $2x^2 + 4x - x^2y^2 + 3y^4 = -1$.

a. [6 points] Calculate $\frac{dy}{dx}$.

b. [2 points] Q is the only point on the curve that has a y-coordinate of 1. Find the x-coordinate of Q.

c. [4 points] Find the equation of the tangent line to the curve at Q.

Solution.

1. [12 points] The following questions relate to the implicit curve $2x^2 + 4x - x^2y^2 + 3y^4 = -1$.

a. [6 points] Calculate $\frac{dy}{dx}$.

Solution: Differentiating both sides with respect to x, we get

$$4x + 4 - 2xy^2 - 2x^2y\frac{dy}{dx} + 12y^3\frac{dy}{dx} = 0.$$

Moving all terms with no $\frac{dy}{dx}$ to the other side and factoring out $\frac{dy}{dx}$ gives us

$$\frac{dy}{dx}(12y^3 - 2x^2y) = 2xy^2 - 4x - 4.$$

So

$$\frac{dy}{dx} = \frac{2xy^2 - 4x - 4}{12y^3 - 2x^2y} = \frac{xy^2 - 2x - 2}{6y^3 - x^2y}.$$

b. [2 points] Q is the only point on the curve that has a y-coordinate of 1. Find the x-coordinate of Q.

Solution: Plugging $y = 1$ into the equation for the curve gives us

$$2x^2 + 4x - x^2 + 3 = -1.$$

Moving all the terms to the left, we get

$$x^2 + 4x + 4 = 0.$$

This factors as $(x + 2)^2 = 0$, so $x = -2$.

c. [4 points] Find the equation of the tangent line to the curve at Q.

Solution: To find the slope, we plug in $x = -2$ and $y = 1$ to $\frac{dy}{dx}$.

$$\text{slope} = \frac{-2 + 4 - 2}{6 - 4} = 0.$$

Thus, the tangent line is the horizontal line passing through Q, which has equation $y = 1$.

□

3.8 SKIP

3.9 Linear Approximation

Let $y = f(x)$ be a function. The tangent line approximation of $f(x)$ at the point $(a, f(a)$ is:

$$\boxed{y = \ell(x) = f(a) + f'(a)(x - a)}$$

This is also the tangent line. We call it "approximation" because we can use the tanget line to approximate the values of $f(x)$ in the vicinity of $x = a$. For a point b near a we have:

$$f(b) \approx f(a) + f'(a)(b - a)$$

Example 3.9.1. Suppose that we are given that $f(2) = 5$ and $f'(2) = 3$. Approximate $f(2.1)$.
We use the tangent line approximation.

$$f(2.1) \approx f(2) + f'(2)(2.1 - 2) = 5 + 3(0.1) = 5.3$$

Graphically:

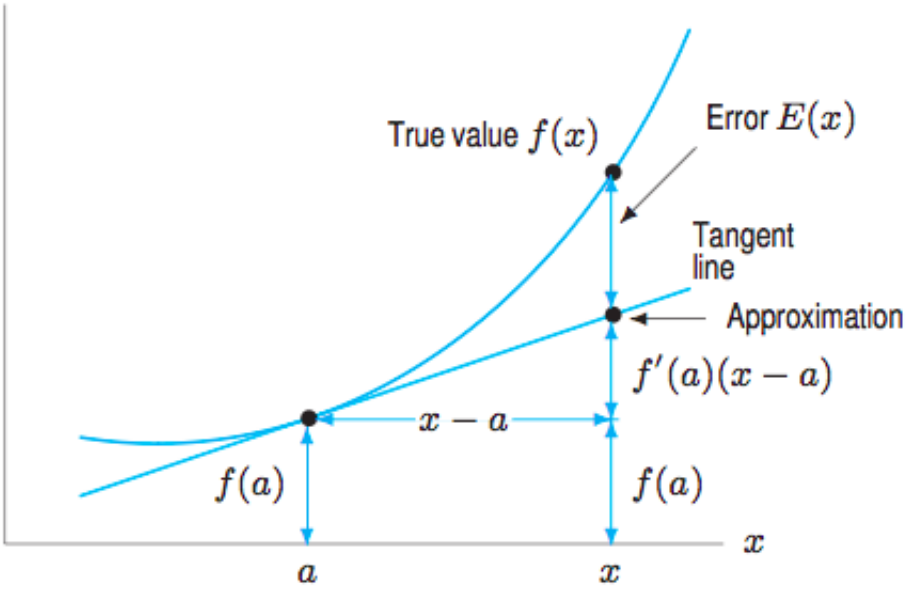


Figure 3.40: The tangent line approximation and its error

The error is defined as:

$$\boxed{E(x) = f(x) - \ell(x) = f(x) - f(a) - f'(a)(a - x)}$$

Remark 3.9.2.

- In order to to give the value of the error at the point $x = b$ we need to know the actual value of $f(x)$ at $x = b$.
- We have a way to evaluate the error if we know the second derivative at the point $x = a$

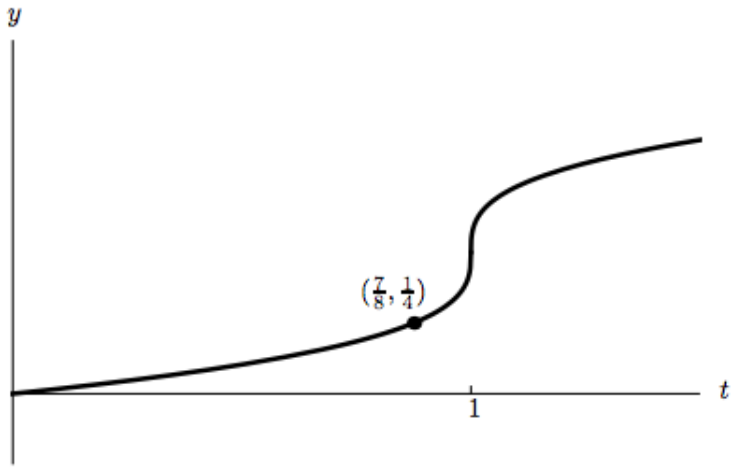
$$E(b) \approx \frac{f''(a)}{2}(b - a)^2$$

- Graphically we can see that if $f''(a) > 0$ and the function concave up, then the error is positive. So we underestimate using the tanget line approximation. Otherwise - we have an overestimate.

$f''(a) > 0$	underestimation
$f''(a) < 0$	overestimation
$f''(a) = 0$	can't tell

Exercise 3.9.3.

6. [15 points] Given below is the graph of a function $h(t)$. Suppose $j(t)$ is the local linearization of $h(t)$ at $t = \frac{7}{8}$.



a. [5 points] Given that $h'(\frac{7}{8}) = \frac{2}{3}$, find an expression for $j(t)$.

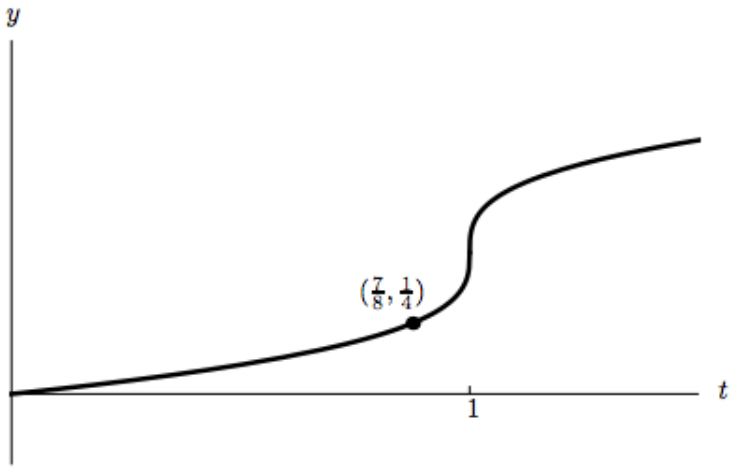
b. [4 points] Use your answer from (a) to approximate $h(1)$.

c. [3 points] Is the approximation from (b) an over- or under-estimate? Explain.

d. [3 points] Using $j(t)$ to estimate values of $h(t)$, will the estimate be more accurate at $t = 1$ or at $t = \frac{3}{4}$? Explain.

Solution.

6. [15 points] Given below is the graph of a function $h(t)$. Suppose $j(t)$ is the local linearization of $h(t)$ at $t = \frac{7}{8}$.



a. [5 points] Given that $h'(\frac{7}{8}) = \frac{2}{3}$, find an expression for $j(t)$.

Solution: The local linearization is the tangent line to the curve. We know this line has slope $h'(\frac{7}{8}) = \frac{2}{3}$ and it goes through the point $(\frac{7}{8}, \frac{1}{4})$, so it has equation

$$y - \frac{1}{4} = \frac{2}{3}(t - \frac{7}{8})$$

using point slope form. Solving for y we have $y = \frac{2}{3}t - \frac{1}{3}$. So $j(t) = \frac{2}{3}t - \frac{1}{3}$. stuff

b. [4 points] Use your answer from (a) to approximate $h(1)$.

Solution: Since $j(t)$ approximates $h(t)$ for t-values near $\frac{7}{8}$, we have

$$h(1) \approx j(1) = \frac{2}{3}(1) - \frac{1}{3} = \frac{1}{3}.$$

c. [3 points] Is the approximation from (b) an over- or under-estimate? Explain.

Solution: The approximation in (b) is an underestimate. The function $h(t)$ is concave up at $t = 7/8$ which means the graph lies above the local linearization for t-values near $7/8$. Since we are using the local linearization to estimate the function value, our estimate will be less than the actual function value.

d. [3 points] Using $j(t)$ to estimate values of $h(t)$, will the estimate be more accurate at $t = 1$ or at $t = \frac{3}{4}$? Explain.

Solution: The estimate at $t = 3/4$ will be more accurate. This can be seen by drawing the tangent line and measuring the vertical distance between the estimated value and the function value at the t values $3/4$ and 1. The line is much closer to the function at $t = 3/4$ than it is at $t = 1$.

□

Exercise 3.9.4.

6. [10 points] Calvin is stuck in the desert, and he needs to build a cube out of cactus skins to hold various supplies. He wants his cube to have a volume of 8.1 cubic feet, but he needs to figure out the side length to cut the cactus skins the right size. He has forgotten his trusty calculator, so he decides to figure out the side length of his cube using calculus.

a. [5 points] Find a local linearization of the function $f(x) = (x+8)^{1/3}$ at $x = 0$.

b. [3 points] Use your linearization to approximate $(8.1)^{1/3}$.

c. [2 points] Should your approximation from part **b.** be an over-estimate or an under-estimate? Why?

Solution.

6. [10 points] Calvin is stuck in the desert, and he needs to build a cube out of cactus skins to hold various supplies. He wants his cube to have a volume of 8.1 cubic feet, but he needs to figure out the side length to cut the cactus skins the right size. He has forgotten his trusty calculator, so he decides to figure out the side length of his cube using calculus.

a. [5 points] Find a local linearization of the function $f(x) = (x+8)^{1/3}$ at $x = 0$.

Solution: The derivative is $f'(x) = \frac{1}{3}(x+8)^{-2/3}$. To find the local linearization we compute $f'(0) = \frac{1}{3}(8)^{-2/3} = \frac{1}{12}$ and $f(0) = 2$. The equation for the tangent line to f at $x = 0$ is $y - 2 = \frac{1}{12}x$. So the local linearization of f near $x = 0$ is

$$L(x) = \frac{1}{12}x + 2.$$

b. [3 points] Use your linearization to approximate $(8.1)^{1/3}$.

Solution: We need to approximate $(8.1)^{1/3} = f(0.1)$. According to our local linearization,

$$f(0.1) \approx L(0.1) = \frac{1}{12}(0.1) + 2 = \frac{241}{120}.$$

c. [2 points] Should your approximation from part **b.** be an over-estimate or an under-estimate? Why?

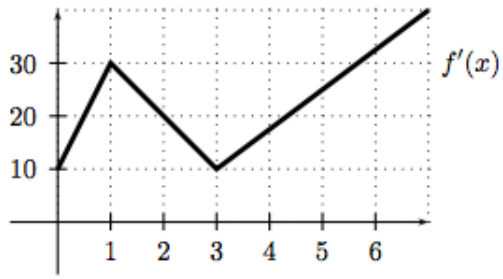
Solution:
The second derivative of f is $f''(x) = -\frac{2}{9}(x+8)^{-5/3}$. For values of x near 0, the second derivative will be negative which means f is concave down near 0. This means our estimate is an overestimate.

□

Please ignore part c in the next one.

Exercise 3.9.5.

2. [12 points]
Use the graph of the function f' and the table of values for the function g to answer the questions below. Each problem requires only a small amount of work, but you must show it.



x	-20	-10	0	10	20	30
g(x)	0	4	0	-18	-56	-120
g'(x)	6	1	-10	-27	-50	-79

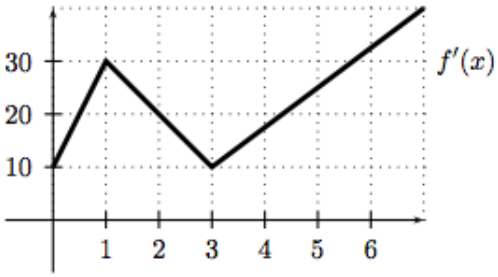
a. [3 points] Write a formula for the local linearization of g near $x = 10$ and use it to approximate $g(10.1)$.

b. [3 points] Using the table, estimate $g''(-10)$.

c. [3 points] If $f(3) = 30$, find the exact value of $f(1)$.

Solution.

2. [12 points]
Use the graph of the function f' and the table of values for the function g to answer the questions below. Each problem requires only a small amount of work, but you must show it.



x	-20	-10	0	10	20	30
g(x)	0	4	0	-18	-56	-120
g'(x)	6	1	-10	-27	-50	-79

a. [3 points] Write a formula for the local linearization of g near $x = 10$ and use it to approximate $g(10.1)$.

Solution:

$$\begin{aligned} g(x) &\approx g(10) + g'(10)(x - 10) \\ g(10.1) &\approx g(10) + g'(10)(10.1 - 10) \\ &= -18 + (-27)(0.1) = -20.7 \end{aligned}$$

b. [3 points] Using the table, estimate $g''(-10)$.

Solution: Note that $g''(x) \approx \dfrac{\Delta g'(x)}{\Delta x}$. We will use the two x-values closest to -10: $x = -20$ and $x = 0$.

$$g''(10) \approx \frac{g'(0) - g'(-20)}{0 - (-20)} = \frac{-10 - 6}{0 - (-20)} = \frac{-16}{20} = \frac{-4}{5} = -0.8$$

c. [3 points] If $f(3) = 30$, find the exact value of $f(1)$.

Solution: By the Fundamental Theorem of Calculus,

$$f(3) - f(1) = \int_1^3 f'(x)dx.$$

This integral represents the area under $f'(x)$ and above the x-axis between $x = 1$ and $x = 3$. Using basic geometry, this area is 40. Thus, $f(3) - f(1) = 40$, so $f(1) = f(3) - 40 = 30 - 40 = -10$.

$\int^3$

□

Quadratic Approximation - Not in the book, only in class
Suppose $f(x) = \cos(x)$ and we would like to use a linearization at $x = 0$. Notice that $\cos(0) = 1$ and $\cos'(0) = -\sin(x)|_{x=0} = 0$ so the linear approximation is just $y = 1$. This is not good. So the have the quadratic approximation at $x = a$:

$$\boxed{Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2}$$

Notice: $Q(x)$ is the best quadratic polynomial one can fit to $f(x)$ at $x = a$. Also:

- $f(a) = Q(a)$
- $f'(a) = Q'(a)$
- $f''(a) = Q''(a)$

Example 3.9.6. Let $f(x) = \cos(x)$. Find the quadratic approximation at $x = 0$.
So we have $\cos'(0) = -\cos(x)|_{x=0} = -1$. We get:

$$Q(x) = 1 - \frac{1}{2}x^2$$

Approximate $\cos(\pi/6)$. We get $1 - 0.5(\pi/6)^2 = 0.863$. The real value is $\sqrt{3}/2 = 0.866$.

Exercise 3.9.7.

5. [12 points] In Srebmun Foyoj, Maddy and Cal are eating lava cake. Let $T(v)$ be the time (in seconds) it takes Maddy to eat a v cm^3 serving of lava cake. Assume $T(v)$ is invertible and differentiable for $0 < v < 1000$. Several values of $T(v)$ and its first and second derivatives are given in the table below.

v	10	15	60	100	150	200	300
$T(v)$	11	22	84	194	393	513	912
$T'(v)$	2.4	1.9	1.8	3.6	3.7	0.9	17.5
$T''(v)$	-0.11	-0.08	0.05	0.04	-0.04	-0.05	0.59

Remember to show your work carefully.

a. [4 points] Use an appropriate linear approximation to estimate the amount of time it takes Maddy to eat a 64 cm^3 serving of lava cake. *Include units.*

Answer: _____

b. [4 points] Use the quadratic approximation of $T(v)$ at $v = 200$ to estimate $T(205)$. (Recall that a formula for the quadratic approximation $Q(x)$ of a function $f(x)$ at $x = a$ is $Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$.)

Answer: $T(205) \approx$ _____

c. [4 points] Let $C(v)$ be the time (in seconds) it takes Cal to eat a v cm^3 serving of lava cake, and suppose $C(v) = T(\sqrt{v})$. Let $L(v)$ be the local linearization of $C(v)$ at $v = 100$. Find a formula for $L(v)$. Your answer should <u>not</u> include the function names T or C.

Answer: $L(v) =$ _____

Solution.

5. [12 points] In Srebmun Foyoj, Maddy and Cal are eating lava cake. Let $T(v)$ be the time (in seconds) it takes Maddy to eat a v cm^3 serving of lava cake. Assume $T(v)$ is invertible and differentiable for $0 < v < 1000$. Several values of $T(v)$ and its first and second derivatives are given in the table below.

v	10	15	60	100	150	200	300
$T(v)$	11	22	84	194	393	513	912
$T'(v)$	2.4	1.9	1.8	3.6	3.7	0.9	17.5
$T''(v)$	-0.11	-0.08	0.05	0.04	-0.04	-0.05	0.59

Remember to show your work carefully.

a. [4 points] Use an appropriate linear approximation to estimate the amount of time it takes Maddy to eat a 64 cm^3 serving of lava cake. *Include units.*

Solution: The closest point in the table to $v = 64$ is $v = 60$, so this is the appropriate choice for the tangent line approximation. Based on the table, the line will go through $(60, 84)$ and have slope 1.8, so it must be $L(v) = 84 + 1.8(v - 60)$. Plugging in 64 for v, we get an estimate of 91.2 seconds.

Answer: 91.2 seconds

b. [4 points] Use the quadratic approximation of $T(v)$ at $v = 200$ to estimate $T(205)$. (Recall that a formula for the quadratic approximation $Q(x)$ of a function $f(x)$ at $x = a$ is $Q(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$.)

Solution: Let $Q(v)$ be the quadratic approximation of $T(v)$ at $v = 200$. Then

$$Q(v) = T(200)+T'(200)(v-200)+\frac{T''(200)}{2}(v-200)^2 = 513+0.9(v-200)+\frac{-0.05}{2}(v-200)^2.$$

So the resulting approximation of $T(205)$ is given by

$$T(205) \approx Q(205) = 513+0.9(205-200)-\frac{0.05}{2}(205-200)^2 = 513+4.5-0.625 = 516.875.$$

Answer: $T(205) \approx$ 516.875

c. [4 points] Let $C(v)$ be the time (in seconds) it takes Cal to eat a v cm^3 serving of lava cake, and suppose $C(v) = T(\sqrt{v})$. Let $L(v)$ be the local linearization of $C(v)$ at $v = 100$. Find a formula for $L(v)$. Your answer should <u>not</u> include the function names T or C.

Solution: We know $L(v) = C(100)+C'(100)(v-100)$. We also know $C(100) = T(10) = 11$. So we need to find $C'(100)$.

Since $C(v) = T(\sqrt{v})$, we apply the chain rule and see that $C'(v) = \frac{1}{2\sqrt{v}}T'(\sqrt{v})$. Using the table above, we then find that $C'(100) = \frac{1}{20}T'(10) = \frac{2.4}{20} = 0.12$.

So $L(v) = 11 + 0.12(v - 100)$.

Answer: $L(v) =$ $11 + 0.12(v - 100)$

□

3.10 MVT

Theorem 3.7: The Mean Value Theorem

If f is continuous on $a \leq x \leq b$ and differentiable on $a < x < b$, then there exists a number c, with $a < c < b$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In other words, $f(b) - f(a) = f'(c)(b - a)$.

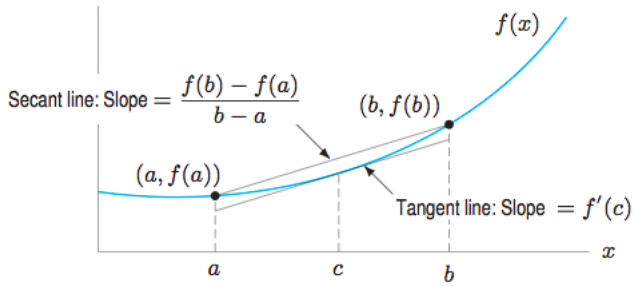
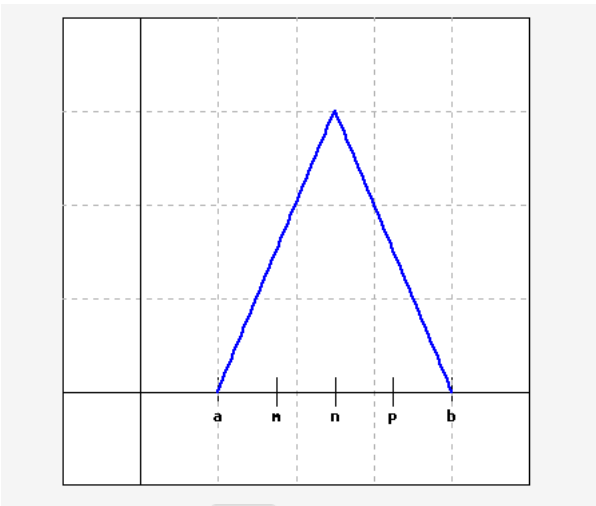


Figure 3.44: The point c with $f'(c) = \frac{f(b)-f(a)}{b-a}$

This theorem tells us that we can take a CTS and differentiable function, draw a secant line, and then it is guaranteed that for some middle point we can find a tangent line with the same slope as the secant line.

Example 3.10.1. Consider the following graph:

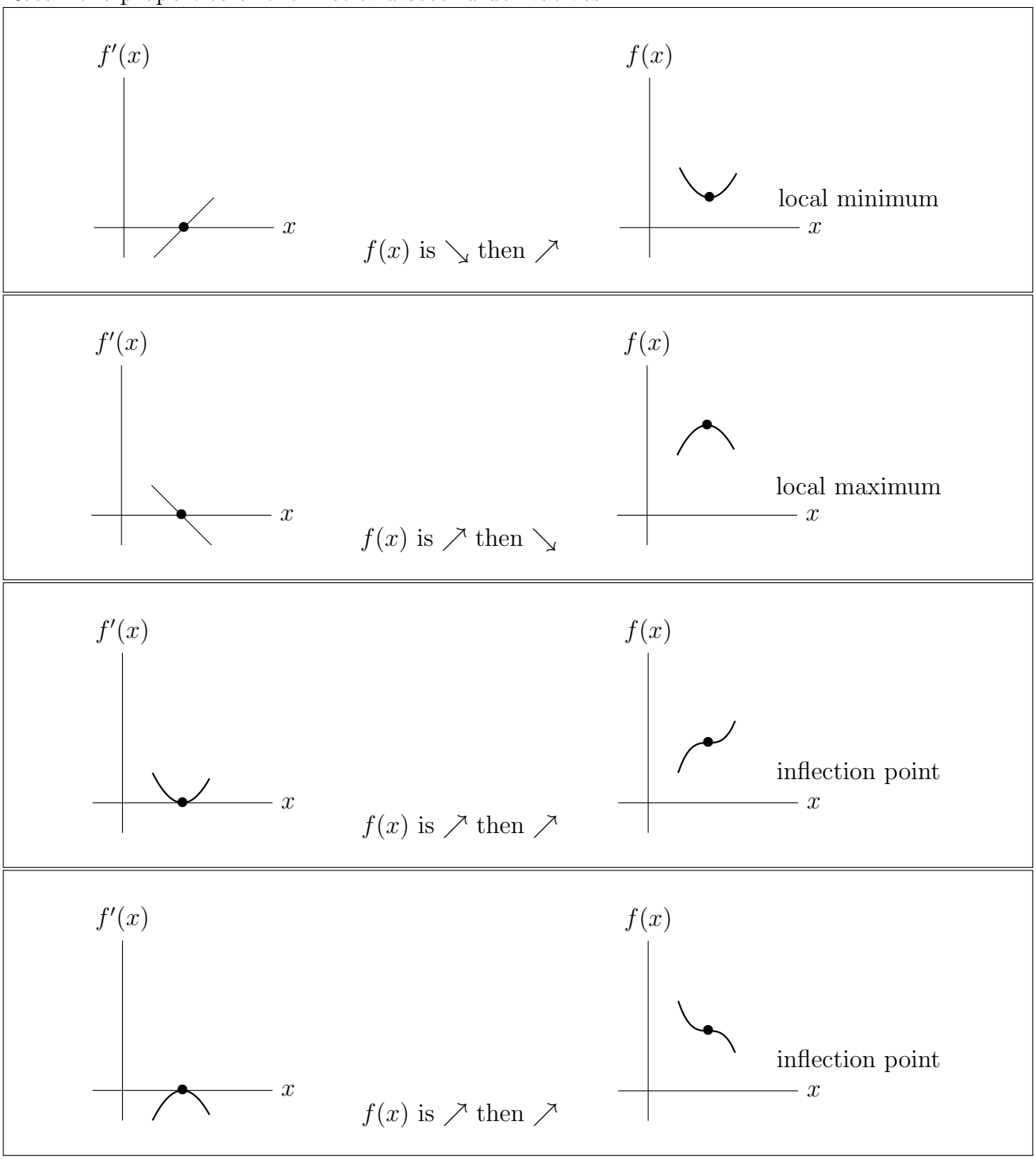


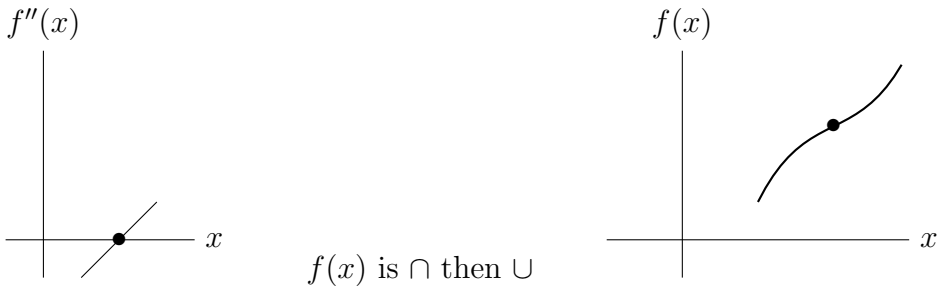
Dose this function satisfy the hypotheses of the MVT? NO!
Indeed, the secant line between a and b has a slope 0, but no point on the graph has derivative 0.
On the other hand, if we "smooth" the function up, then MVT applies. The tip will have a derivative of 0 — similar to downward facing parabola.

4 Using The Derivative

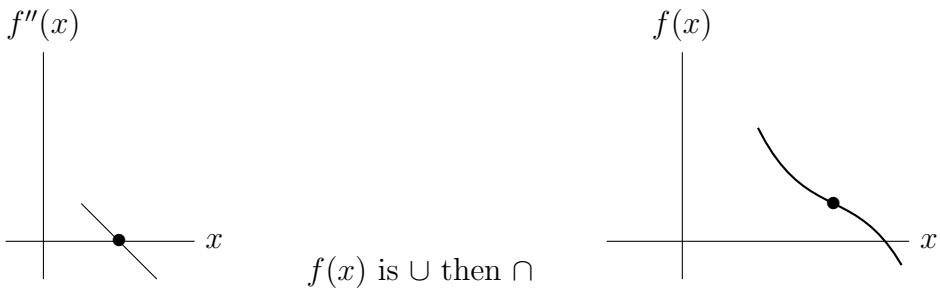
4.1 Using the first and second derivative

Recall the properties of the first and second derivatives:





$f(x)$ is $\cap$ then $\cup$



$f(x)$ is $\cup$ then $\cap$

We are in the business of detecting those local extreme for **CTS** functions.
We first define a **Critical Point**: $x = p$ is a critical point of $f(x)$ if $f'(p) = 0$ or undefined. These points are the candidates for local extrema. We can check each using the **First Derivative test:**

- If $f'(x) > 0$ to the left of p and $f'(x) < 0$ to the right of it then it is a local maximum.
- If $f'(x) < 0$ to the left of p and $f'(x) > 0$ to the right of it then it is a local minimum.

Or using the **Second derivative test:**

- $f'(p) = 0$ and $f''(p) > 0$ then minimum.
- $f'(p) = 0$ and $f''(p) < 0$ then maximum.
- $f'(p) = 0$ and $f''(p) = 0$ then inconclusive.

An inflection point is when $f(x)$ is CTS and changes from being concave up to concave down or the other way around. So if $x = p$ is an inflection point: $f(x)$ CTS at $x = p$, and $f''(p) = 0$ or undefined, and $f''(x)$ changes signs around it.

To sum up - how to find local extrema:

- Find all critical points.
- Sketch a number line and mark where $f'(x)$ is positive or negative.
- Draw a conclusion.

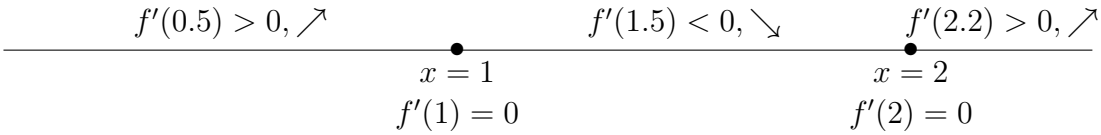
To sum up - how to find inflection points:

- Find all values where $f''(x) = 0$ or undefined.
- Sketch a number line and mark where $f''(x)$ is positive or negative.
- Draw a conclusion.

Example 4.1.1. Find the local extrema of $2x^3 - 9x^2 + 12x - 4$ and inflection points:
Find all critical points:

$$f'(x) = 6x^2 - 18x + 12 = 6(x-1)(x-2) = 0 \Rightarrow x = 1, 2$$

Draw a number line to get the types of the local points:

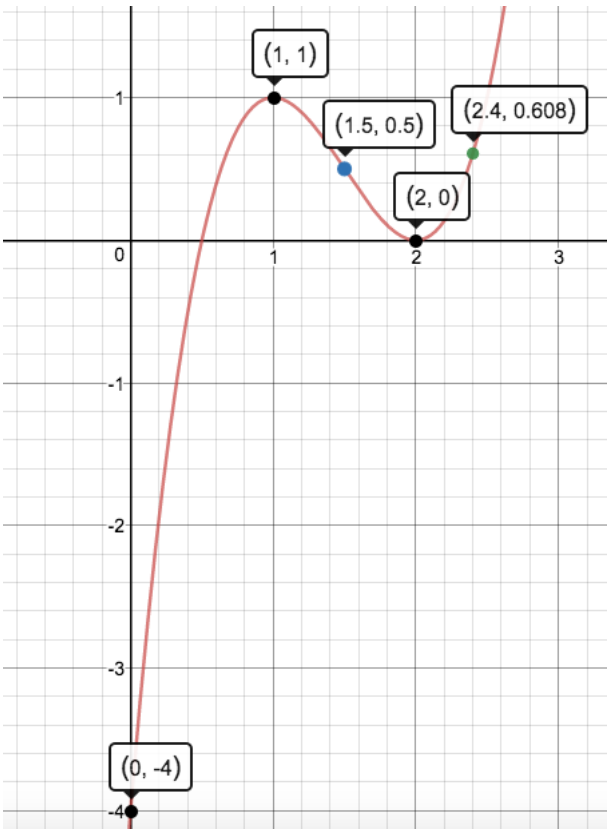


So we conclude:

- $f(1) = 1$, local max
- $f(2) = 0$, local min

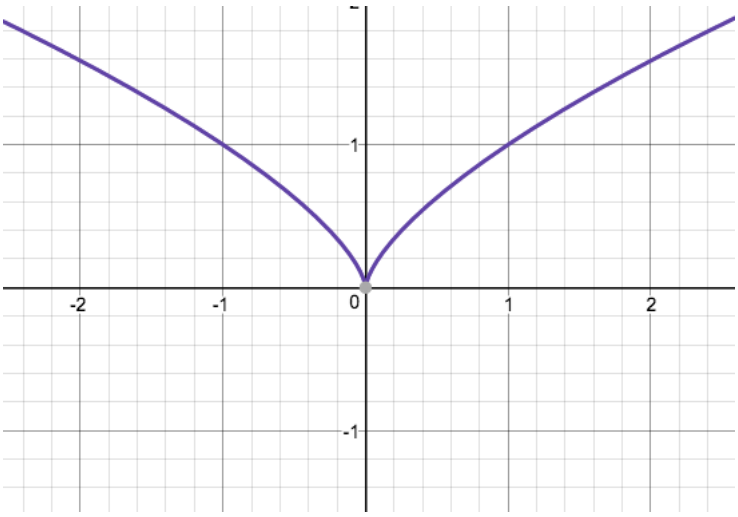
For the inflection points, $f''(x) = 12x - 18$ so the only candidate for an inflection point is $x = 18/12 = 1.5$.

So indeed it is an inflection point. Together we get:



Example 4.1.2. Find local extreme of $f(x) = x^{2/3}$.
$f'(x) = (2/3)x^{-1/3}$, it is never 0, but it is undefined for $x = 0$. So this is the only critical point.

$$f'(-1) < 0, f'(1) > 0 \Rightarrow x = 0 \text{ is a local min}$$


1
-2
-1
0
1
2
-1

Exercise 4.1.3.

3. [10 points]
Find the x- and y-coordinates of all local minima, local maxima, and inflection points of the function $f(x)$ defined below. Your answers may involve the positive constant B. You must clearly mark your answers and provide justification to receive credit.

$$f(x) = e^{-18x^2+B}$$

Solution.

3. [10 points]
Find the x- and y-coordinates of all local minima, local maxima, and inflection points of the function $f(x)$ defined below. Your answers may involve the positive constant B. You must clearly mark your answers and provide justification to receive credit.

$$f(x) = e^{-18x^2+B}$$

Solution: We will need both $f'(x)$ (chain rule) and $f''(x)$ (product rule and chain rule).

$$\begin{aligned} f'(x) &= e^{-18x^2+B}(-36x) = -36xe^{-18x^2+B} \\ f''(x) &= -36e^{-18x^2+B} + (-36x)e^{-18x^2+B}(-36x) \\ &= -36e^{-18x^2+B}(1-36x^2) \end{aligned}$$

First we will find the critical points. We know f' is never undefined, and $f'(x) = 0$ only when $x = 0$, since e^{-18x^2+B} is always positive. The y-value for $x = 0$ is $e^{-18\cdot 0^2+B} = e^B$ Thus, $(0, e^B)$ is the only critical point. Since $f''(0) = -36e^B(1) = -36e^B$ is negative, we know this point is a local maximum.

LOCAL MAXIMUM AT $(0, e^B)$.

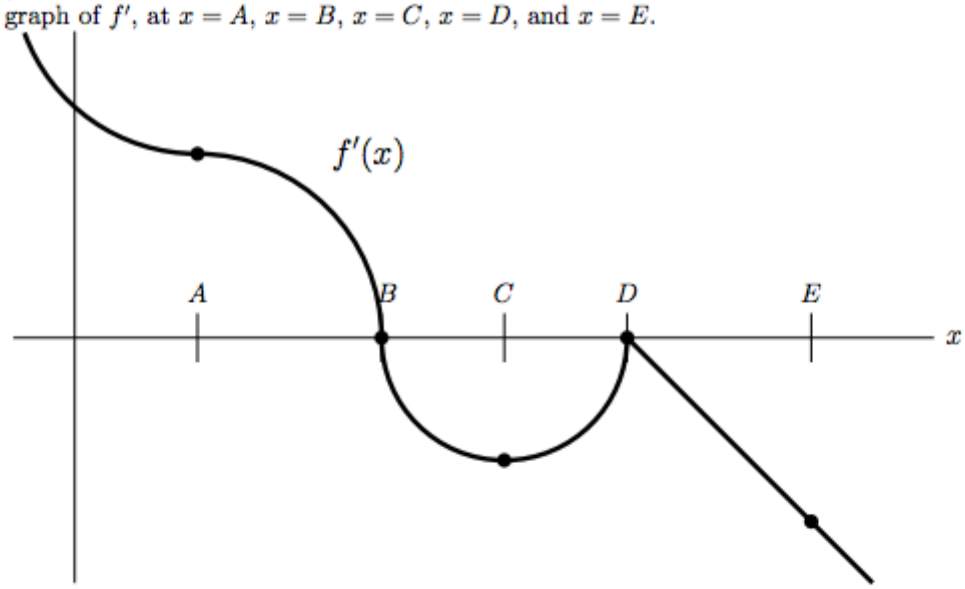
Next we will find potential inflection points. We know f'' is never undefined, and $f''(x) = 0$ when $1 - 36x^2 = 0$, since e^{-18x^2+B} is always positive. Solving, we find that $f''(x) = 0$ when $x = \pm\frac{1}{6}$. Both of these points have a y-value of $e^{-18(\frac{1}{6})^2+B} = e^{-\frac{1}{2}+B}$. We need to test f'' near these x-values to check whether we actually have inflection points.
When $x < -\frac{1}{6}$, $f''(x) > 0$.
When $-\frac{1}{6} < x < \frac{1}{6}$, $f''(x) < 0$.
When $x > \frac{1}{6}$, $f''(x) > 0$.
Since f'' changes sign at both of these points, f changes concavity at both points, so both are inflection points.

INFLECTION POINTS AT $(-\frac{1}{6}, e^{-\frac{1}{2}+B})$ and $(\frac{1}{6}, e^{-\frac{1}{2}+B})$.

□

Exercise 4.1.4.

6. [12 points] The *derivative* of a function f is graphed below. Five points are marked on the graph of f', at $x = A$, $x = B$, $x = C$, $x = D$, and $x = E$.



For each of the following, circle ALL answers which are correct. Each part has at least one correct answer. Pay careful attention to whether each question is asking about f, f', or f''.

a. [2 points] The function f' has a local minimum when _____.

$x = A$ $x = B$ $x = C$ $x = D$ $x = E$

b. [2 points] The function f is increasing when _____.

$x = A$ $x = B$ $x = C$ $x = D$ $x = E$

c. [2 points] The function f has a critical point when _____.

$x = A$ $x = B$ $x = C$ $x = D$ $x = E$

d. [2 points] The global maximum of f on the interval $A \leq x \leq E$ occurs when _____.

$x = A$ $x = B$ $x = C$ $x = D$ $x = E$

e. [2 points] The function f has an inflection point when _____.

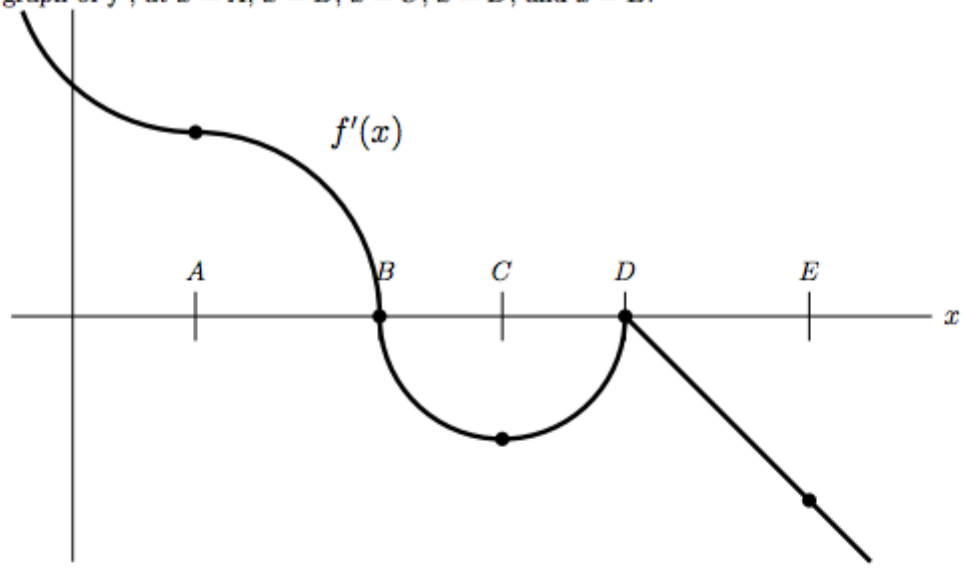
$x = A$ $x = B$ $x = C$ $x = D$ $x = E$

f. [2 points] The function f'' is undefined when _____.

$x = A$ $x = B$ $x = C$ $x = D$ $x = E$

Solution.

6. [12 points] The *derivative* of a function f is graphed below. Five points are marked on the graph of f', at $x = A$, $x = B$, $x = C$, $x = D$, and $x = E$.



For each of the following, circle ALL answers which are correct. Each part has at least one correct answer. Pay careful attention to whether each question is asking about f, f', or f''.

a. [2 points] The function f' has a local minimum when _____.

$x = A$ $\quad$ $x = B$ $\quad$ $\boxed{x = C}$ $\quad$ $x = D$ $\quad$ $x = E$

b. [2 points] The function f is increasing when _____.

$\boxed{x = A}$ $\quad$ $x = B$ $\quad$ $x = C$ $\quad$ $x = D$ $\quad$ $x = E$

c. [2 points] The function f has a critical point when _____.

$x = A$ $\quad$ $\boxed{x = B}$ $\quad$ $x = C$ $\quad$ $\boxed{x = D}$ $\quad$ $x = E$

d. [2 points] The global maximum of f on the interval $A \leq x \leq E$ occurs when _____.

$x = A$ $\quad$ $\boxed{x = B}$ $\quad$ $x = C$ $\quad$ $x = D$ $\quad$ $x = E$

e. [2 points] The function f has an inflection point when _____.

$x = A$ $\quad$ $x = B$ $\quad$ $\boxed{x = C}$ $\quad$ $\boxed{x = D}$ $\quad$ $x = E$

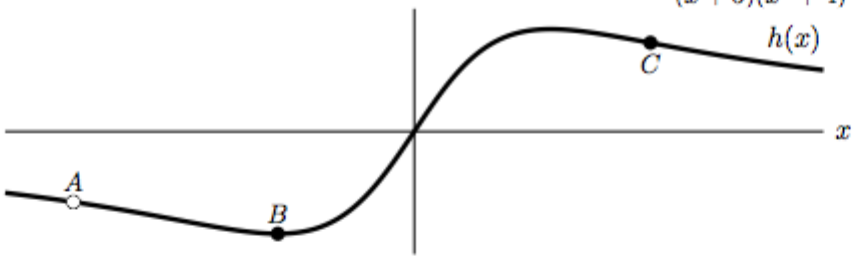
f. [2 points] The function f'' is undefined when _____.

$x = A$ $\quad$ $\boxed{x = B}$ $\quad$ $x = C$ $\quad$ $\boxed{x = D}$ $\quad$ $x = E$

□

Exercise 4.1.5.

8. [13 points] Below, there is a graph of the function $h(x) = \dfrac{2x^2 + 10x}{(x+5)(x^2+4)}$.



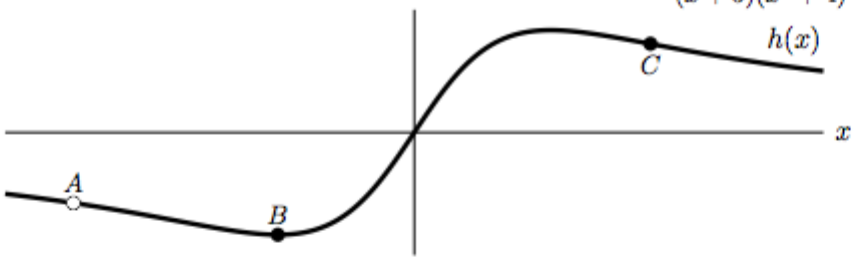
a. [3 points] The point A is a hole in the graph of h. Find the x- and y-coordinates of A.

b. [5 points] The point B is a local minimum of h. Find the x- and y-coordinates of B.

c. [5 points] The point C is an inflection point of h. Find the x- and y-coordinates of C.

Solution.

8. [13 points] Below, there is a graph of the function $h(x) = \dfrac{2x^2+10x}{(x+5)(x^2+4)}$.



a. [3 points] The point A is a hole in the graph of h. Find the x- and y-coordinates of A.

Solution: Simplifying $h(x)$, we have $h(x) = \dfrac{2x(x+5)}{(x+5)(x^2+4)}$. Since the factor $(x+5)$ cancels, the hole occurs when $x = -5$. We look at the limit as x approaches -5 on the cancelled form to get the y-coordinate:

$$\lim_{x\to -5} h(x) = \lim_{x\to -5} \frac{2x}{x^2+4} = \frac{-10}{29},$$

Thus, $A = (-5, \frac{-10}{29})$.

b. [5 points] The point B is a local minimum of h. Find the x- and y-coordinates of B.

Solution: Using the quotient rule on the simplified form of h, we have $h'(x) = \dfrac{4-x^2}{(x^2+4)^2}$. This is never undefined, and it is equal to zero when $4 - x^2 = 0$ or $x = \pm 2$. From the graph, we can see that the local minimum occurs at $x = -2$. The y-coordinate here is $y = \frac{-4}{8} = -\frac{1}{2}$, so $B = (-2, -\frac{1}{2})$.

c. [5 points] The point C is an inflection point of h. Find the x- and y-coordinates of C.

Solution: We use the quotient rule again to find $h''(x) = \dfrac{2x^3-24x}{(x^2+4)^3} = \dfrac{2x(x^2-12)}{(x^2+4)^3}$. This is never undefined, and it is zero when $2x(x^2-12) = 0$, i.e. when $x = 0, \pm 2\sqrt{3}$. From the graph, we see that our x-coordinate must be $+2\sqrt{3}$, and then $y = \frac{4\sqrt{3}}{16} = \frac{\sqrt{3}}{4}$, so $C = (2\sqrt{3}, \frac{\sqrt{3}}{4})$.

□

Exercise 4.1.6.

2. [14 points] The table for the *derivative* of a function h with continuous first derivative is given below. Assume that between each consecutive value of x, the derivative h' is either increasing or decreasing. For each statement below, indicate whether the statement is true, false, or cannot be determined from the information given. No partial credit will be given.

x	-4	-3	-2	-1	0	1	2	3	4
$h'(x)$	2	3	1	-3	-4	-2	0	2	1

a.) The function h has a local maximum on the interval $-2 < x < -1$.

True False Not enough information

b.) The function h is negative on the interval $-1 < x < 1$.

True False Not enough information

c.) The function h is concave up on the interval $0 < x < 4$.

True False Not enough information

d.) The function h is decreasing on the interval $-3 < x < -2$.

True False Not enough information

e.) The function h has an inflection point on the interval $-1 < x < 1$.

True False Not enough information

f.) The derivative function, h', has a critical point at $x = 2$.

True False Not enough information

g.) The second derivative function, h'', is positive on the interval $0 < x < 3$.

True False Not enough information

Solution.

2. [14 points] The table for the *derivative* of a function h with continuous first derivative is given below. Assume that between each consecutive value of x, the derivative h' is either increasing or decreasing. For each statement below, indicate whether the statement is true, false, or cannot be determined from the information given. No partial credit will be given.

x	-4	-3	-2	-1	0	1	2	3	4
$h'(x)$	2	3	1	-3	-4	-2	0	2	1

a.) The function h has a local maximum on the interval $-2 < x < -1$.

$\boxed{\text{True}}$ False Not enough information

b.) The function h is negative on the interval $-1 < x < 1$.

True False $\boxed{\text{Not enough information}}$

c.) The function h is concave up on the interval $0 < x < 4$.

True $\boxed{\text{False}}$ Not enough information

d.) The function h is decreasing on the interval $-3 < x < -2$.

True $\boxed{\text{False}}$ Not enough information

e.) The function h has an inflection point on the interval $-1 < x < 1$.

$\boxed{\text{True}}$ False Not enough information

f.) The derivative function, h', has a critical point at $x = 2$.

True False $\boxed{\text{Not enough information}}$

g.) The second derivative function, h'', is positive on the interval $0 < x < 3$.

True False $\boxed{\text{Not enough information}}$

□

4.2 Global Optimization

Given a function $f(x)$ we would like to find its global max/min - the points where $f(x)$ attain the biggest/lowest values. Those values can be on local extrema, on endpoints, or DNE.
Case 1: domain is a closed interval and the functions is **CTS**. Then we have the **Extreme Value Theorem** (EVT) that tells us that the global extrema exists and it is one of the local extrema or the endpoints.

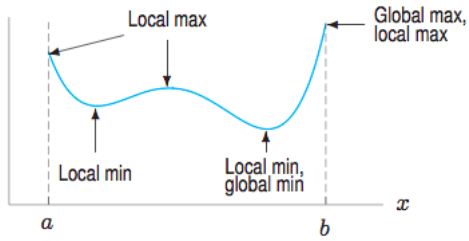


Figure 4.18: Global maximum and minimum on a closed interval $a \leq x \leq b$

In this case:

- Find all all critical points.
- Evaluate $f(x)$ at all critical points and endpoints.
- Take the biggest/lowest value

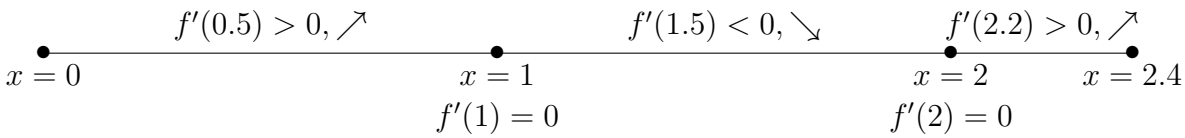
Example 4.2.1. Find the global extrema of $2x^3 - 9x^2 + 12x - 4$ on the domain [0,2.4] and graph the function:
Closed interval so we can use the EVT:
Find all critical points:

$$f'(x) = 6x^2 - 18x + 12 = 6(x-1)(x-2) = 0 \Rightarrow x = 1, 2$$

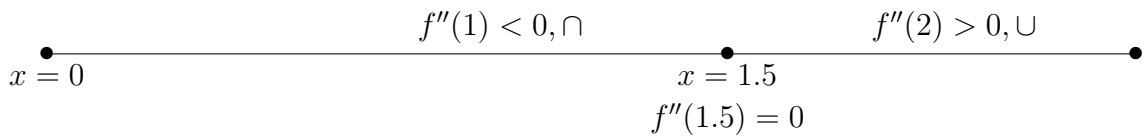
So we need to compare 4 values:

- $f(0) = -4$, end point, global min
- $f(1) = 1$, local max and global max
- $f(2) = 0$, local min
- $f(2.4) = 0.608$, end point.

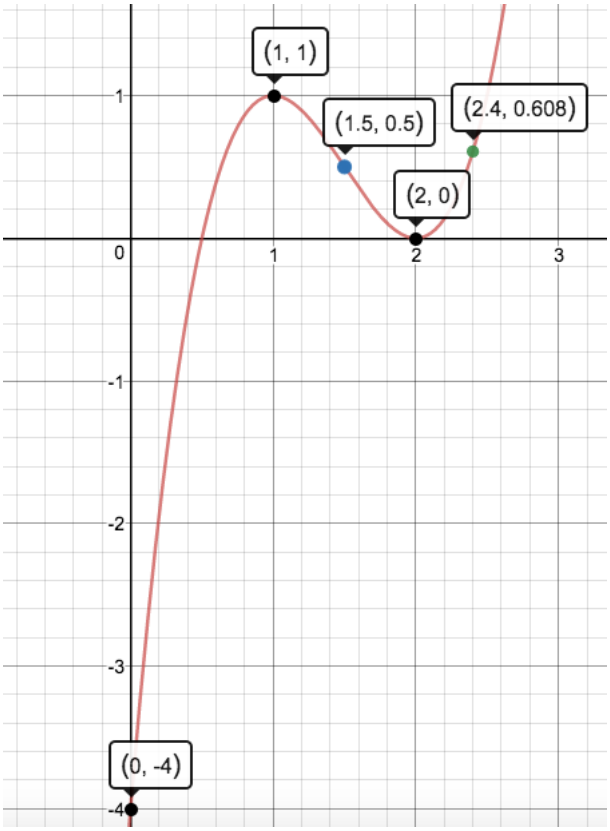
Draw a number line to get the types of the local points (not required for the global extrema but useful for the graph):



Also, $f''(x) = 12x - 18$ so the only candidate for an inflection point is $x = 18/12 = 1.5$.



Together we get:



Case 2: If either the function is not CTS or the domain is not a closed interval we have to:

- Test all critical points.
- Sketch the graph and check:
  - behavior at endpoints and points of discontinuity.
  - behavior at vertical asymptotes, if any,
  - behavior at $\pm\infty$ if part of the domain.

Example 4.2.2. $f(x) = xe^{-x}$. Find global Extrema on $[0, \infty)$.
Since the domain is not a closed interval, we cannot use the EVT. Let us take the derivatives and sketch the graph:

$$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$$

So only $x = 1$ is a critical point.

$$f''(x) = -e^{-x} - (1-x)e^{-x} = e^{-x}(x-2)$$

So $x = 2$ is a candidate for an inflection point.

$f'(0.5) > 0, \overset{\text{max}}{\nearrow}$ $\qquad$ $f'(2) < 0, \searrow$

$x = 1$
$f'(1) = 0$

$f''(0.5) < 0, \cap$ $\qquad$ $f'(2.5) > 0, \cup$

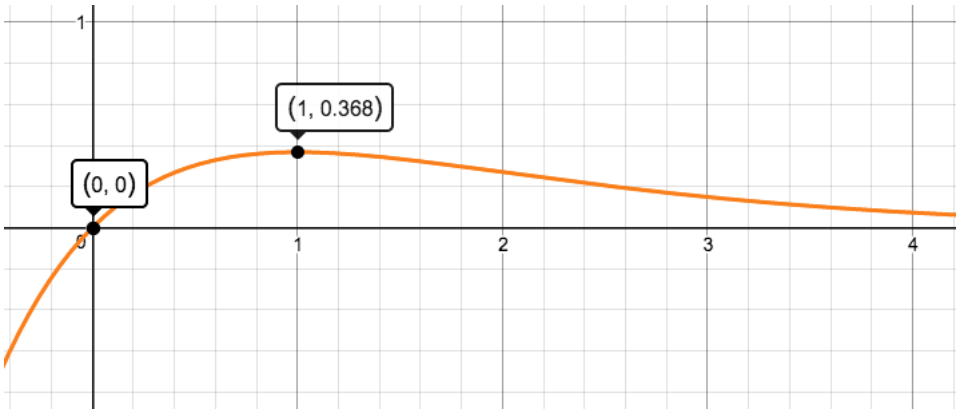
$x = 2$
$f''(2) = 0$

since e^x dominates x, as $x \to \infty$, $f(x) \to 0$, while $f(x) > 0$ for $x > 0$. To sum up the relevant information:

- $(1, f(1)) = (1, e^{-1})$ is a local max, and a global max.

- $(0, f(0)) = (0, 0)$ is a global min.
- $f(x \to \infty) = 0$ but $f(x)$ is positive while doing so.

This is the graph:



Exercise 4.2.3.

7. [15 points] Suppose a is a positive constant and

$$f(x) = 2x^3 - 3ax^2.$$

a. [10 points] Find the absolute maximum and minimum values of $f(x)$ on the closed interval $[-a, \frac{3}{2}a]$. Specify all x values where the maximum and minimum value are achieved.

b. [5 points] Find all inflection points of $f(x)$.

Solution.

7. [15 points] Suppose a is a positive constant and

$$f(x) = 2x^3 - 3ax^2.$$

a. [10 points] Find the absolute maximum and minimum values of $f(x)$ on the closed interval $[-a, \frac{3}{2}a]$. Specify all x values where the maximum and minimum value are achieved.

Solution: Seeking critical points, we take the derivative of f and set it equal to zero.

$$f'(x) = 6x^2 - 6ax = 6x(x-a) = 0.$$

Using this equation we find the critical points to be $x = 0, a$. Now we put the critical points and the endpoints of the interval back into the orginal function and compare the values. We compute $f(-a) = -5a^3$, $f(0) = 0$, $f(a) = -a^3$, $f(\frac{3}{2}a) = 0$.
This means the absolute max of f on this interval is 0 and this value is achieved at $x = 0, \frac{3}{2}a$. The absolute min is $-5a^3$ and this value is achieved at $x = -a$.

b. [5 points] Find all inflection points of $f(x)$.

Solution: Seeking inflection points, we find $f''(x) = 12x - 6a$. Setting this equal to zero we find $x = \frac{a}{2}$. To verify this is an inflection point we test $f''(0) = -6a < 0$ and $f''(a) = 6a > 0$. This means f'' changes sign at $x = \frac{a}{2}$, so it is an inflection point.

□

Exercise 4.2.4.

7. [11 points] Consider the continuous function

$$f(x) = \begin{cases} x \cdot 2^{-x} & 1 \leq x < 3, \\ \frac{1}{2-x} + \frac{11}{8} & 3 \leq x \leq 5. \end{cases}$$

Note that the domain of f is $[1, 5]$.

a. [7 points] Find the x-values of the critical points of f.

b. [4 points] Find the y-values of the global maximum and global minimum of f if they exist, or explain why they don't exist.

Solution.

7. [11 points] Consider the continuous function

$$f(x) = \begin{cases} x \cdot 2^{-x} & 1 \leq x < 3, \\ \frac{1}{2-x} + \frac{11}{8} & 3 \leq x \leq 5. \end{cases}$$

Note that the domain of f is $[1, 5]$.

a. [7 points] Find the x-values of the critical points of f.

Solution: To find the critical points, we first take the derivatives of the two functions and set them equal to zero.

$$\frac{d}{dx} x2^{-x} = 2^{-x} - x \ln 2 \cdot 2^{-x} = 2^{-x}(1 - x \ln 2) = 0.$$

2^{-x} is never 0 so $1 = x \ln 2$ so $x = \frac{1}{\ln 2} \approx 1.44$, which is between 1 and 3, so it is a critical point.

$$\frac{d}{dx}\left(\frac{1}{2-x} + \frac{11}{8}\right) = \frac{1}{(2-x)^2},$$

which is never 0, but undefined at 2, which is not between 3 and 5 and therefore not a critical point.
To check if there is a critical point at $x = 3$, we can either graph the function on a calculator and see that there is a sharp corner there, or we can check and see that the derivatives of the two functions are not equal there:

$$2^{-3}(1 - 3 \ln 2) \approx -0.13 \neq 1 = \frac{1}{(2-3)^2}.$$

Thus, the critical points are at $x = \frac{1}{\ln 2}$ and $x = 3$.

b. [4 points] Find the y-values of the global maximum and global minimum of f if they exist, or explain why they don't exist.

Solution: Since this is a closed interval, we can just test the critical points and endpoints.
$f(1) = 1 \cdot 2^{-1} = 0.5$
$f\left(\frac{1}{\ln 2}\right) = \frac{1}{\ln 2} \cdot 2^{-\frac{1}{\ln 2}} \approx 0.53$
$f(3) = 3 \cdot 2^{-3} = \frac{3}{8} = 0.375$
$f(5) = \frac{1}{2-5} + \frac{11}{8} = \frac{25}{24} \approx 1.042$
So the global maximum is $\frac{25}{24} \approx 1.042$ and the global minimum is $\frac{3}{8} = 0.375$.

□

Exercise 4.2.5.

8. [11 points] A function $g(t)$ and its derivative are given by

$$g(t) = 10e^{-0.5t}(t^2 - 2t + 2) \qquad \text{and} \qquad g'(t) = -10e^{-0.5t}(0.5t^2 - 3t + 3).$$

a. [2 points] Find the t-coordinates of all critical points of $g(t)$. If there are none, write NONE. For full credit, you must find the exact t-coordinates.

Answer: Critical point(s) at $t =$ _____

b. [6 points] For each of the following, find the values of t that maximize and minimize $g(t)$ on the given interval. Be sure to show enough evidence that the points you find are indeed global extrema. For each answer blank, write NONE in the answer blank if appropriate.

(i) Find the values of t that maximize and minimize $g(t)$ on the interval $[0, 8]$.

Answer: Global max(es) at $t =$ _____ Global min(s) at $t =$ _____

(ii) Find the values of t that maximize and minimize $g(t)$ on the interval $[4, \infty)$.

Answer: Global max(es) at $t =$ _____ Global min(s) at $t =$ _____

c. [3 points] Let $G(t)$ be the antiderivative of $g(t)$ with $G(0) = -5$. Find the t-coordinates of all critical points and inflection points of $G(t)$. For each answer blank, write NONE if appropriate. You do not need to justify your answers.

Answer: Critical point(s) at $t =$ _____

Answer: Inflection point(s) at $t =$ _____

Solution.

8. [11 points] A function $g(x)$ and its derivative are given by

$$g(t) = 10e^{-0.5t}(t^2 - 2t + 2) \qquad \text{and} \qquad g'(t) = -10e^{-0.5t}(0.5t^2 - 3t + 3).$$

a. [2 points] Find the t-coordinates of all critical points of $g(t)$. If there are none, write NONE. For full credit, you must find the exact t-coordinates.

Solution: Since $g'(t)$ is defined for all t, the only critical points occur where $g'(t) = 0$. To find these t values, we use the Quadratic Formula:

$$t = 3 \pm \sqrt{9-6} = 3 \pm \sqrt{3}.$$

Answer: Critical point(s) at $t =$ $3-\sqrt{3},\ 3+\sqrt{3}$

b. [6 points] For each of the following, find the values of t that maximize and minimize $g(t)$ on the given interval. Be sure to show enough evidence that the points you find are indeed global extrema. For each answer blank, write NONE in the answer blank if appropriate.

(i) Find the values of t that maximize and minimize $g(t)$ on the interval $[0, 8]$.

Solution: Since $[0, 8]$ is a closed interval, by the Extreme Value Theorem, $g(t)$ must have both a global max and a global min, occuring either at a critical point or at an endpoint. We therefore make a table of values at $t = 0, t = 3 - \sqrt{3} \approx 1.27$, $t = 3 + \sqrt{3} \approx 4.73$, and $t = 8$:

t	0	1.27	4.73	8
$g(t)$	20	5.69	14.01	9.16

From the table, we see the global max at $t = 0$ and the global min at $t = 3 - \sqrt{3} \approx 1.27$.

Answer: Global max(es) at $t =$ 0 Global min(s) at $t =$ $3-\sqrt{3}$

(ii) Find the values of t that maximize and minimize $g(t)$ on the interval $[4, \infty)$.

Solution: Note that only one of our critical points, $t = 3 + \sqrt{3} \approx 4.73$, lies in this interval. Global extrema, if they exist, then, can only occur at $t = 4$ and $t = 3 + \sqrt{3}$, so we make a table of these values:

t	4	4.73
$g(t)$	13.53	14.01

We must also consider the behavior as $t \to \infty$, the open endpoint of our interval. Note that $g'(t) < 0$ for $t > 3+\sqrt{3}$, so $g(t)$ is decreasing for $t > 3 + \sqrt{3}$. So the global max occurs at the largest value in the table, at $t = 3 + \sqrt{3} \approx 4.73$. As t gets larger and larger, $g(t) = \frac{10(t^2-2t+2)}{e^{0.5t}}$ tends to 0, as $e^{0.5t}$ grows faster than any polynomial in the long run. Since this limiting value of 0 is smaller than every value in our table, there is no global min.

Answer: Global max(es) at $t =$ $3+\sqrt{3}$ Global min(s) at $t =$ NONE

c. [3 points] Let $G(t)$ be the antiderivative of $g(t)$ with $G(0) = -5$. Find the t-coordinates of all critical points and inflection points of $G(t)$. For each answer black, write NONE if appropriate. You do not need to justify your answers.

Solution: Critical points of $G(t)$ are zeros of $g(t)$, of which there are none. Inflection points of $G(t)$ are local extrema of $g(t)$, which occur at $t = 3 \pm \sqrt{3}$ (which we know to be local extrema because they are in fact global extrema in the interiors of some intervals).

Answer: Critical point(s) at $t =$ NONE

Answer: Inflection point(s) at $t =$ $3-\sqrt{3},\ 3+\sqrt{3}$

□

Usually one can see question of optimizing functions outside some extra context, this is what the next section is all about.

4.3 Optimization and Modeling

In a specific context, one should create a function that represent the scenario, identify the domain and the find global extrema.

Exercise 4.3.1.

2. [13 points] The U-value of a wall of a building is a positive number related to the rate of energy transfer through the wall. Walls with a lower U-value keep more heat in during the winter than ones with a higher U-value. Consider a wall which consists of two materials, material A with U-value a and material B with U-value b. The U-value of the wall w is given by

$$w = \frac{ab}{b+a}.$$

Considering a as a constant, we can think of w as a function of b, $w = u(b)$.

a. [4 points] Write the limit definition of the derivative of $u(b)$.

b. [4 points] Calculate $u'(b)$. (You do **not** need to use the limit definition of the derivative for your calculation.)

c. [5 points] Find the x- and y-coordinates of the global minimum and maximum of $u(b)$ for b in the interval $[1, 2]$. Your answer may involve the parameter a. [Recall that $a, b > 0$.]

Global minimum on $[1, 2]$: _____

Global maximum on $[1, 2]$: _____

Solution.

2. [13 points] The U-value of a wall of a building is a positive number related to the rate of energy transfer through the wall. Walls with a lower U-value keep more heat in during the winter than ones with a higher U-value. Consider a wall which consists of two materials, material A with U-value a and material B with U-value b. The U-value of the wall w is given by

$$w = \frac{ab}{b+a}.$$

Considering a as a constant, we can think of w as a function of b, $w = u(b)$.

a. [4 points] Write the limit definition of the derivative of $u(b)$.

Solution: The derivative of $u(b)$ is defined to be

$$u'(b) = \lim_{h\to 0} \frac{u(b+h)-u(b)}{h} = \lim_{h\to 0} \frac{a(b+h)/(b+h+a) - ab/(b+a)}{h}$$

b. [4 points] Calculate $u'(b)$. (You do **not** need to use the limit definition of the derivative for your calculation.)

Solution: By the quotient rule,

$$u'(b) = \frac{(b+a)(a)-ab}{(b+a)^2} = \frac{a^2}{(b+a)^2}.$$

c. [5 points] Find the x- and y-coordinates of the global minimum and maximum of $u(b)$ for b in the interval $[1,2]$. Your answer may involve the parameter a. [Recall that $a, b > 0$.]

Solution: The derivative $u'(b)$ is strictly positive for all $b > 0$ by part **b.** This means there are not any critical points on $[1,2]$ and u is strictly increasing so $b = 1$ is the global minimum while $b = 2$ is the global maximum. Now we compute $u(1) = \frac{a}{1+a}$ and $u(2) = \frac{2a}{2+a}$.

Global minimum on $[1,2]$: $(1, \frac{a}{1+a})$

Global maximum on $[1,2]$: $(2, \frac{a}{2+a})$

□

Exercise 4.3.2.

4. [8 points] A ship's captain is standing on the deck while sailing through stormy seas. The rough waters toss the ship about, causing it to rise and fall in a sinusoidal pattern. Suppose that t seconds into the storm, the height of the captain, in feet above sea level, is given by the function

$$h(t) = 15\cos(kt) + c$$

where k and c are nonzero constants.

a. [3 points] Find a formula for $v(t)$, the vertical velocity of the captain, in feet per second, as a function of t. The constants k and c may appear in your answer.

Answer: $v(t) =$ _____

b. [2 points] Find a formula for $v'(t)$. The constants k and c may appear in your answer.

Answer: $v'(t) =$ _____

c. [3 points] What is the maximum vertical acceleration experienced by the captain? The constants k and c may appear in your answer. You do not need to justify your answer or show work. *Remember to include units.*

Answer: Max vertical acceleration: _____

Solution.

4. [8 points] A ship's captain is standing on the deck while sailing through stormy seas. The rough waters toss the ship about, causing it to rise and fall in a sinusoidal pattern. Suppose that t seconds into the storm, the height of the captain, in feet above sea level, is given by the function

$$h(t) = 15\cos(kt) + c$$

where k and c are nonzero constants.

a. [3 points] Find a formula for $v(t)$, the vertical velocity of the captain, in feet per second, as a function of t. The constants k and c may appear in your answer.

Solution: The velocity is the derivative of the height function, so we compute

$$v(t) = h'(t) = -15k\sin(kt).$$

Notice that the Chain Rule gives us a factor of k out front, and since c is an additive constant, it disappears when we take the derivative.
Notice also that $v(t) = \frac{dh}{dt}$ does indeed have units of feet per second, as required.

Answer: $v(t) =$ $-15k\sin(kt)$

b. [2 points] Find a formula for $v'(t)$. The constants k and c may appear in your answer.

Answer: $v'(t) =$ $-15k^2\cos(kt)$

c. [3 points] What is the maximum vertical acceleration experienced by the captain? The constants k and c may appear in your answer. You do not need to justify your answer or show work. *Remember to include units.*

Solution: The acceleration is just the derivative of the velocity function, which was just computed in the previous part.
Since $v'(t) = -15k^2\cos(kt)$ is sinusoidal with midline 0 and amplitude $15k^2$, the maximum value it achieves is $15k^2$.
Since $v'(t) = \frac{dv}{dt}$, the units on the acceleration are feet per second per second, or feet per second squared.

Answer: Max vertical acceleration: $15k^2$ **ft/s^2**

□

Exercise 4.3.3.

9. [10 points] Our friend Oren, the Math 115 student, wants to minimize how long it will take him to complete his upcoming web homework assignment. Before starting the assignment, he buys a cup of tea containing 55 milligrams of caffeine.
Let $H(x)$ be the number of minutes it will take Oren to complete tonight's assignment if he consumes x milligrams of caffeine. For $10 \leq x \leq 55$

$$H(x) = \frac{1}{120}x^2 - \frac{4}{3}x + 20\ln(x)\,.$$

Instead of immediately starting the assignment, he solves a calculus problem to determine how much caffeine he should consume.

a. [8 points] Find all the values of x at which $H(x)$ attains global extrema on the interval $10 \leq x \leq 55$. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema.

(For each answer blank below, write NONE in the answer blank if appropriate.)

Answer: global min(s) at $x =$ _____

Answer: global max(es) at $x =$ _____

b. [2 points] Assuming Oren consumes at least 10 milligrams and at most 55 milligrams of caffeine, what is the shortest amount of time it could take for him to finish his assignment? *Remember to include units.*

Answer: _____

Solution.

9. [10 points] Our friend Oren, the Math 115 student, wants to minimize how long it will take him to complete his upcoming web homework assignment. Before starting the assignment, he buys a cup of tea containing 55 milligrams of caffeine.
Let $H(x)$ be the number of minutes it will take Oren to complete tonight's assignment if he consumes x milligrams of caffeine. For $10 \leq x \leq 55$

$$H(x) = \frac{1}{120}x^2 - \frac{4}{3}x + 20\ln(x)\,.$$

Instead of immediately starting the assignment, he solves a calculus problem to determine how much caffeine he should consume.

a. [8 points] Find all the values of x at which $H(x)$ attains global extrema on the interval $10 \leq x \leq 55$. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema.

Solution: Since $H(x)$ is continuous on the interval $10 \leq x \leq 55$, by the Extreme Value Theorem, $H(x)$ attains both a global minimum and a global maximum on this interval. These will occur at either endpoints or critical points.
Now,

$$H'(x) = \frac{x}{60} - \frac{4}{3} + \frac{20}{x} = \frac{x^2 - 80x + 1200}{60x} = \frac{(x-60)(x-20)}{60x}\,.$$

Thus, $H(x)$ has exactly one critical point on the interval $10 \leq x \leq 55$, and it is at $x = 20$.
To determine the global extrema, we compare the values of $H(x)$ at all critical points and endpoints

x	10	20	55
$H(x)$	≈ 33.55	≈ 36.58	≈ 32.02

Thus, the global minimum is at $x = 55$, and the global maximum is at $x = 20$.

(For each answer blank below, write NONE in the answer blank if appropriate.)

Answer: global min(s) at $x =$ 55

Answer: global max(es) at $x =$ 20

b. [2 points] Assuming Oren consumes at least 10 milligrams and at most 55 milligrams of caffeine, what is the shortest amount of time it could take for him to finish his assignment? *Remember to include units.*

Solution: The minimum of $H(x)$ occurs at $x = 55$, where $H(55) \approx 32.02$.

Answer: ≈ 32 **minutes**

□

Quiz #6. Please write you name: _____ and email: _____

1. [50 pt] Pass the Gateway by Nov-3-2016, 4pm.

2. [50 pt] Let $f(x) = xe^{-x}$. Find global extrema on $[0, 3]$.

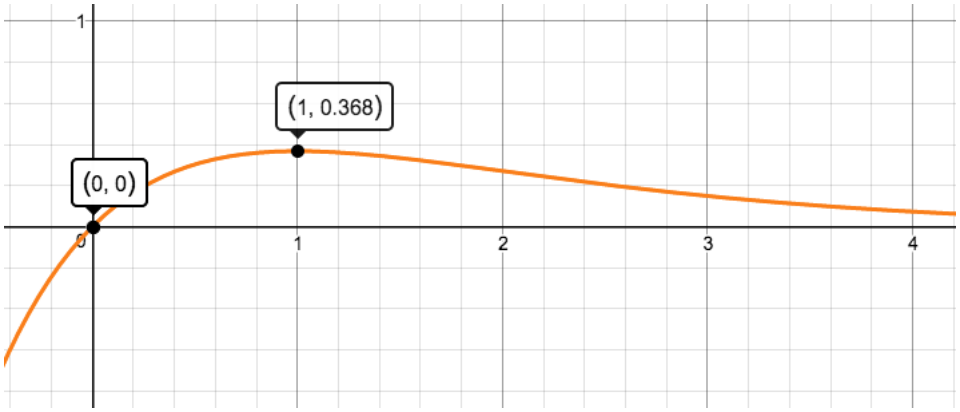
Solution. This is very similar to the example we did in class. See **4.2.2**. Since the domain is a closed interval, and the function is CTS, we can use the EVT. Let us take the derivative and find critical points:

$$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$$

So only $x = 1$ is a critical point. Therefore, by the EVT, all the candidates for global extrema are $x = 0, 1, 3$:

- $(0, f(0)) = (0, 0)$, global min
- $(1, f(1)) = (1, e^{-1})$, global max
- $(3, f(3) = (3, e^{-3})$
- $0 < e^{-3} < e^{-1}$

This is the graph:



□

Remark on Exam2:
Geometry Facts You Should Know (or have on your notecard)

- Perimeter of
  - polygon (sum of side lengths)
  - circle ($C = 2\pi r$)
- Area of
  - rectangle ($A = bh$)
  - triangle ($A = bh/2$)
  - circle ($A = \pi r^2$)
  - trapezoid ($A = \frac{(b_1+b_2)}{2} \cdot h$)
- Volume of
  - box ($V = \ell wh$)
  - cylinder ($V = \pi r^2 h$)
- Surface area of a box
- Pythagorean Theorem and Distance Formula
- Trigonometry
  - Triangle Trigonometry (e.g. $\sin(\theta) =$"opposite/hypotenuse")
  - Pythagorean Identity ($\sin^2 t + \cos^2 t = 1$)
  - $\tan t = (\sin t)/(\cos t)$
- Similar Triangles

Formulas That Would Be Provided If Needed (by no means a complete list!)

- Area
  - of quadrilaterals other than rectangles and trapezoids
  - of sector of a circle
  - Hero's formula
- Volume of
  - sphere
  - cone
- Surface area of
  - sphere
  - cylinder
  - cone
- Trigonometry
  - double angle identities
- Arclength in a circle

4.4 Families of Functions and Modeling

In this section we will introduce 3 common family of functions that are used in modeling real life scenarios.

The **Bell curve** $\boxed{y = e^{-(x-a)^2/b}}$. Notice that a, b are numbers. Assume $b > 0$.

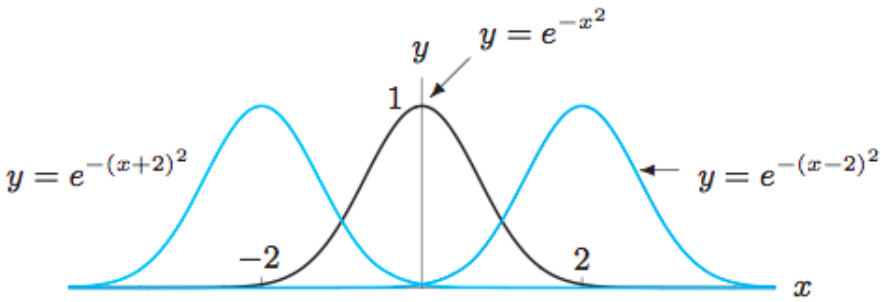


Figure 4.53: The family $y = e^{-(x-a)^2}$

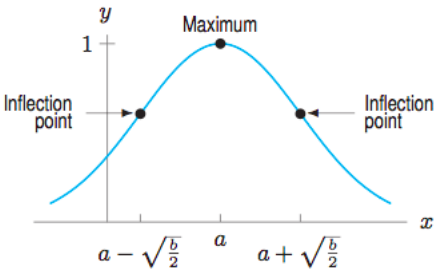


Figure 4.54: Graph of $y = e^{-(x-a)^2/b}$: bell-shaped curve with peak at $x = a$

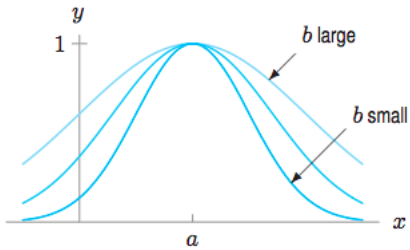


Figure 4.55: Graph of $y = e^{-(x-a)^2/b}$ for fixed a and various b

Observations:

1. $y' = \frac{-2(x-a)}{b} e^{-(x-a)^2/b}$
2. Local and Global max at $x = a$.
3. b gets bigger, bell gets wider.

The **Exponential Model** $\boxed{y = a(1 - e^{-bx})}$. Again, a, b are just numbers. Here $a > 0, b > 0$.

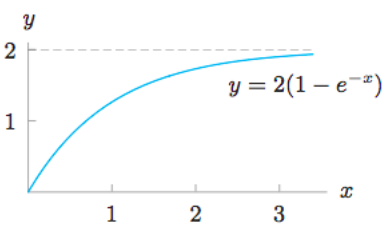


Figure 4.56: One member of the family $y = a(1 - e^{-bx})$, with $a = 2, b = 1$

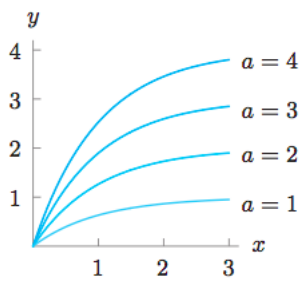


Figure 4.57: Fixing $b = 1$ gives $y = a(1 - e^{-x})$, graphed for various a

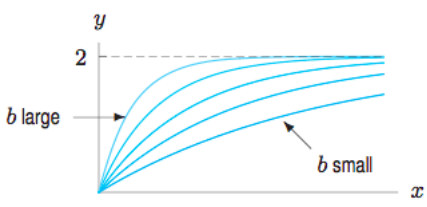


Figure 4.58: Fixing $a = 2$ gives $y = 2(1 - e^{-bx})$, graphed for various b

Observations:

1. Represents a quantity that starts from 0 and approaches to a when x gets bigger.
2. $y(0) = 0, \lim_{x\to\infty} y(x) = a$. This is a horizontal asymptote. Stauration value.
3. $y' = abe^{-bx}$.
4. When b gets bigger, the approach gets "faster".

The **Logistic Model** $\boxed{y = L/(1 + Ae^{-kt}}$ used to model population growth when it is limited by the environment, e.g. limited resource. Assume $L, A, k > 0$.

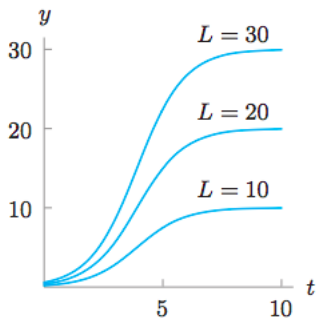


Figure 4.60: Graph of $y = L/(1 + Ae^{-kt})$ varying L with $A = 50$, $k = 1$

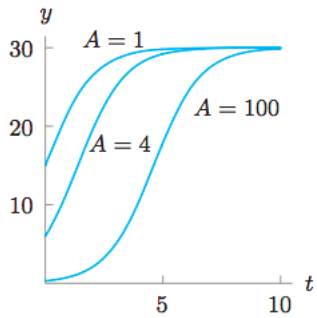


Figure 4.61: Graph of $y = L/(1 + Ae^{-kt})$ varying A with $L = 30$, $k = 1$

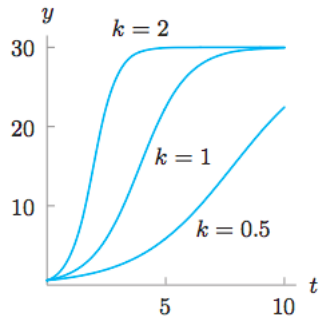


Figure 4.62: Graph of $y = L/(1 + Ae^{-kt})$ varying k, with $L = 30$, $A = 50$

Observations:

1. $y(0) = L/(1 + A)$. $\lim_{x\to\infty} y(x) = L$. This is a horizontal asymptote. Stauration value.
2. $y' = \frac{LAke^{-kx}}{(1+Ae^{-kx})^2}$.
3. A controls the y-intercept.
4. k controls how fast we reach saturation.

Exercise 4.4.1.

3. (2+8 points) The logistic model for population growth is a model that accounts for the fact that population cannot grow indefinitely. The formula for the logistic model is given by $P(t) = \dfrac{L}{1 + Ae^{-kt}}$ where L and A are positive constants.

(a) The carrying capacity is the horizontal asymptote of $P(t)$. What is the carrying capacity? What does this mean in practical terms?

(b) List the steps you would take to find the value of t for which the population is growing the fastest? Give reasons for each step. You do **NOT** have to carry out any of these steps!!!!

Solution.

3. (2+8 points) The logistic model for population growth is a model that accounts for the fact that population cannot grow indefinitely. The formula for the logistic model is given by $P(t) = \dfrac{L}{1 + Ae^{-kt}}$ where L and A are positive constants.

(a) The carrying capacity is the horizontal asymptote of $P(t)$. What is the carrying capacity? What does this mean in practical terms?

The carrying capacity $= \lim_{t\to\infty} P(t) = L$. This gives an upper bound on the population, i.e., the population can approach a value of L but can never quite reach it.

(b) List the steps you would take to find the value of t for which the population is growing the fastest? Give reasons for each step. You do **NOT** have to carry out any of these steps!!!!

1. Find P' as this gives the rate of growth and is the function we are interested in maximizing.

2. Find P'' in order to find the critical points of P'.

3. Find the critical points of P' by determining where P'' is equal to zero or undefined.

4. Test the critical points of P' by either
   - taking the third derivative and testing the critical points,
   - looking at the sign of P'' around the critical points, or
   - giving a graphical argument based on the graph of P or P'.

□

Exercise 4.4.2.

(4.) (12 points) Consider the function:

$$f(x) = e^{\frac{-(ax)^2}{2}}, \qquad \text{for } a \text{ a positive constant.}$$

The graph of $y = f(x)$ is the (in)famous "bell curve," which occurs frequently in statistics, and occasionally in heated political debates as well.

(a) Compute $f''(x)$. Show your work.

(b) For which value of a does the function f have an inflection point at $x = 3$?

Solution.

(4.) (12 points) Consider the function:

$$f(x) = e^{\frac{-(ax)^2}{2}}, \qquad \text{for } a \text{ a positive constant.}$$

The graph of $y = f(x)$ is the (in)famous "bell curve," which occurs frequently in statistics, and occasionally in heated political debates as well.

(a) Compute $f''(x)$. Show your work.

Use the chain rule:

$$f'(x) = e^{\frac{-(ax)^2}{2}} \cdot (-(ax)) \cdot a = (-a^2x) \cdot e^{\frac{-(ax)^2}{2}}$$

Now use the product rule, together with the previous line:

$$f''(x) = -a^2 \cdot e^{\frac{-(ax)^2}{2}} + (-a^2x) \cdot \left[(-a^2x) \cdot e^{\frac{-(ax)^2}{2}}\right]$$

$$f''(x) = a^2 e^{\frac{-(ax)^2}{2}}(a^2x^2 - 1)$$

(b) For which value of a does the function f have an inflection point at $x = 3$?

First let's find out where $f''(x) = 0$, since this is a prerequisite for an inflection point. Since $e^k > 0$ for any k and $a^2 \neq 0$, we need only find out where $a^2x^2 - 1 = 0$. This happens when $x = \pm 1/a$. If $x = 3$ and a is positive, we must have $a = 1/3$. To assure that this is an inflection point of f, we can check the sign of $f''(x)$ to the left and right of $x = 3$. We see that $f''(x)$ is negative to the left of $x = 3$ and positive to the right of $x = 3$. Thus, the function changes from concave down to concave up at $x = 3$ when $a = 1/3$.

□

4.5 Marginality (How to Make Money)

Suppose that we have a great product. We would like to manufacture it by minimizing the cost, and sell it by maximizing the profit. If we can put it in an equation then we can optimize and find global extrema.
Let's say we manufacturing q units of product. The cost involved with manufacturing q product is denoted as $C(q)$. The revenue is denoted as $R(q)$. Usually the cost can be a complicated function while the revenue is simply the price p times the quantity q. So $R(q) = pq$. The profit is denoted as

$$\boxed{\pi(q) = R(q) - C(q)}$$

The Goal is to find global max for the profit. Usually, the domain of q is from 0 to some big number, and the function π is CTS. So we can use EVT to maximize π. In order to do that we need to compare $\pi'(q) = 0$. So we need $R'(q) = C'(q)$.

$C'(q)$ is called the **Marginal Cost** and $R'(q)$ is called the marginal profit. So maximizing π is done by finding the q's were **Marginal Cost = Marginal Profit** (and don't forget to test the end points).
Since q is a whole number, we need to use approximations:

$$\boxed{C'(q) \approx C(q+1) - C(q)}$$

$$\boxed{R'(q) \approx R(q+1) - R(q)}$$

So Marginal cost at q is approximately the cost involved in producing 1 more product. The marginal profit is the profit involved in producing one more product.
Graphical insights:

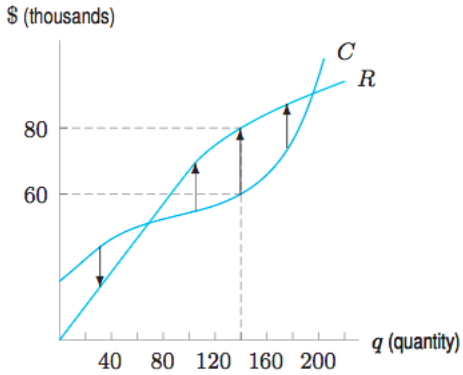


Figure 4.75: Maximum profit at $q = 140$

- $\pi = R - C$ so we wish to maximize the length of the up right arrows.
- $MC = MR$ is a point where the slops are the same, i.e. the tangent lines are parallel.
- Average cost $a(q) = C(q)/q$ is the slop of the line between the origin $(0, 0)$ and the point $(q, C(q))$.

Exercise 4.5.1.

3. [12 points] Oren plans to grow kale on his community garden plot, and he has determined that he can grow up to 160 bunches of kale on his plot. Oren can sell the first 100 bunches at the market and any remaining bunches to wholesalers. The revenue in dollars that Oren will take in from selling b bunches of kale is given by

$$R(b) = \begin{cases} 6b & \text{for } 0 \leq b \leq 100 \\ 4b + 200 & \text{for } 100 < b \leq 160. \end{cases}$$

a. [2 points] Use the formula above to answer each of the following questions.

i. What is the price (in dollars) that Oren will charge for each bunch of kale he sells at the market?

Answer: _____

ii. What is the price (in dollars) that Oren will charge for each bunch of kale he sells to wholesalers?

Answer: _____

For $0 \leq b \leq 160$, it will cost Oren $C(b) = 20 + 3b + 24\sqrt{b}$ dollars to grow b bunches of kale.

b. [1 point] What is the fixed cost (in dollars) of Oren's kale growing operation?

Answer: _____

c. [4 points] At what production level(s) does Oren's marginal revenue equal his marginal cost?

Answer: _____

d. [5 points] Assuming Oren can grow up to 160 bunches of kale, how many bunches of kale should he grow in order to maximize his profit, and what is the maximum possible profit? *You must use calculus to find and justify your answer. Be sure to provide enough evidence to justify your answer fully.*

Answer: bunches of kale: _____ and max profit: _____

Solution.

3. [12 points] Oren plans to grow kale on his community garden plot, and he has determined that he can grow up to 160 bunches of kale on his plot. Oren can sell the first 100 bunches at the market and any remaining bunches to wholesalers. The revenue in dollars that Oren will take in from selling b bunches of kale is given by

$$R(b) = \begin{cases} 6b & \text{for } 0 \le b \le 100 \\ 4b + 200 & \text{for } 100 < b \le 160. \end{cases}$$

a. [2 points] Use the formula above to answer each of the following questions.

i. What is the price (in dollars) that Oren will charge for each bunch of kale he sells at the market?

Answer: $6

ii. What is the price (in dollars) that Oren will charge for each bunch of kale he sells to wholesalers?

Answer: $4

For $0 \le b \le 160$, it will cost Oren $C(b) = 20 + 3b + 24\sqrt{b}$ dollars to grow b bunches of kale.

b. [1 point] What is the fixed cost (in dollars) of Oren's kale growing operation?

Answer: $20

c. [4 points] At what production level(s) does Oren's marginal revenue equal his marginal cost?

Solution: Oren's marginal revenue is $R'(b) = 6$ for $0 < b < 100$ and $R'(b) = 4$ for $100 < b < 160$. His marginal cost is $C'(b) = 3 + 12/\sqrt{b}$.
Thus, $R'(b) = C'(b)$ for $b = 16$ and $b = 144$.

Answer: **at 16 bunches and 144 bunches**

d. [5 points] Assuming Oren can grow up to 160 bunches of kale, how many bunches of kale should he grow in order to maximize his profit, and what is the maximum possible profit? *You must use calculus to find and justify your answer. Be sure to provide enough evidence to justify your answer fully.*

Solution: Since Oren's profit function, $\pi(b) = R(b) - C(b)$, is continuous on $0 \le b \le 160$, it has a global maximum (by the Extreme Value Theorem) and the global maximum occurs at a critical point or an endpoint.
The critical points of $\pi(b)$ occur when $\pi'(b) = 0$ (at $b = 16$ and 144 (when MR=MC)), and when $\pi'(b)$ is undefined (at $b = 100$).
We check the value of $\pi(b)$ at the critical points and end points:
$\pi(0) = -20$, $\pi(16) - -68$, $\pi(100) = 40$, $\pi(144) = 36$, and $\pi(160) \approx 36.42$, and conclude that the maximum occurs at $b = 100$, with a resulting maximum profit of \$40.

Answer: bunches of kale: 100 and max profit: $40

□

Exercise 4.5.2.

6. [10 points] The Green Bag Company (GBC) makes hand bags out of recycled materials. A table of the company's marginal cost, MC, and marginal revenue, MR, at various production levels q is given below. The variable q is the number of hand bags produced, and marginal cost and marginal revenue are measured in dollars per bag.

q	1000	2000	3000	4000	5000	6000
MC	100	81	75	96	112	123
MR	125	123	114	110	107	106

Assume for this problem that GBC's cost and revenue functions are twice differentiable and that MC and MR are either increasing or decreasing on each interval shown in the table.

a. [3 points] At which production level from the table is GBC's profit increasing the fastest? Explain your answer.

b. [3 points] The CEO of the company thinks profit is maximized at 3000 bags, but the CFO of the company thinks that profit will be maximized at 4500 bags. Who could be correct, and why? [Note: The terms "CEO" and "CFO" refer to officers in the company.]

c. [4 points] Assuming GBC has no fixed costs, use a right-hand sum to estimate the cost to produce the first 3000 bags. Be sure to show your work.

Solution.

6. [10 points] The Green Bag Company (GBC) makes hand bags out of recycled materials. A table of the company's marginal cost, MC, and marginal revenue, MR, at various production levels q is given below. The variable q is the number of hand bags produced, and marginal cost and marginal revenue are measured in dollars per bag.

q	1000	2000	3000	4000	5000	6000
MC	100	81	75	96	112	123
MR	125	123	114	110	107	106

Assume for this problem that GBC's cost and revenue functions are twice differentiable and that MC and MR are either increasing or decreasing on each interval shown in the table.

a. [3 points] At which production level from the table is GBC's profit increasing the fastest? Explain your answer.

Solution: The derivative of the profit is $MR - MC$. We are looking for the place where this derivative is largest, so by inspecting the table, we see that $MR - MC$ is largest (with a value of 42) when $q = 2000$ bags.

b. [3 points] The CEO of the company thinks profit is maximized at 3000 bags, but the CFO of the company thinks that profit will be maximized at 4500 bags. Who could be correct, and why? [Note: The terms "CEO" and "CFO" refer to officers in the company.]

Solution: The CFO could be correct because the profit will be maximized when $MC = MR$ and this *could* occur at $q = 4500$ since $MR - MC$ changes sign somewhere in the interval $4000 < q < 5000$. It definitively does not occur at $q = 3000$ because $MR > MC$ at this point which means producing more bags will result in higher profits for the company.

c. [4 points] Assuming GBC has no fixed costs, use a right-hand sum to estimate the cost to produce the first 3000 bags. Be sure to show your work.

Solution: Doing a right hand sum, we have

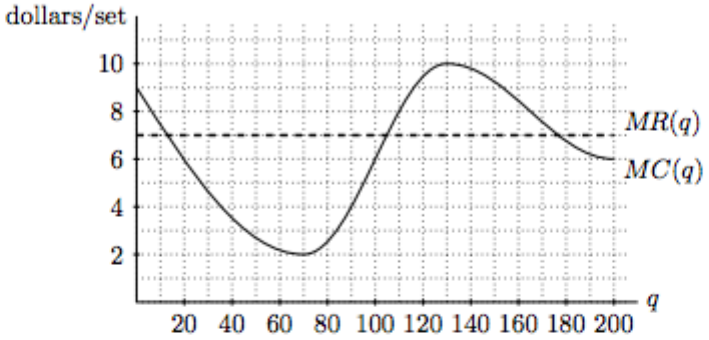
$$C(3000) = \int_0^{3000} MC(q)dq \approx 1000MC(1000)+1000MC(2000)+1000MC(3000) = 256000.$$

So the approximation predicts the cost to produce the first 3000 bags will be \$256,000.

□

Exercise 4.5.3.

2. [12 points] Link has started a business selling winter clothes for cats. Among his most successful products are his new kitten mittens. He is currently selling his mittens for \$7 per set. Below is a graph of Link's marginal cost $MC(q)$ and marginal revenue $MR(q)$, in dollars per set of mittens, if he makes q sets of mittens this winter. Due to a shortage of yarn, Link can make a maximum of 200 sets of mittens this winter. In order to start making mittens, Link must spend \$40 on knitting supplies (in other words, it costs \$40 to make 0 sets of mittens).



You do not need to show any work for this problem.

a. [3 points] Approximately how many sets of mittens should Link make this winter in order to maximize his profit?

Answer: Link should make about _____ sets of mittens.

b. [2 points] If the price per set is raised to \$9, approximately how many sets of mittens should Link make in order to maximize his profit?

Answer: Link should make about _____ sets of mittens.

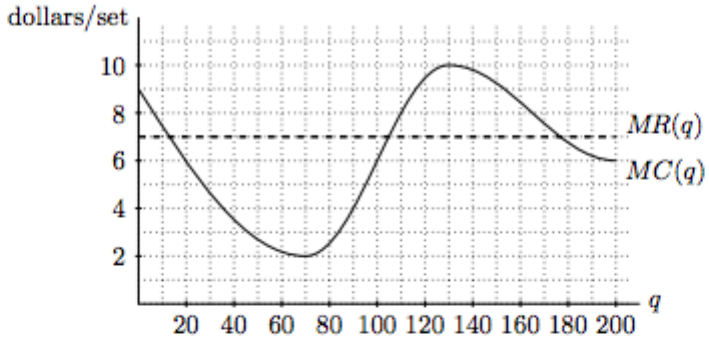
c. [3 points] Write an expression involving integrals which equals Link's total profit if Link makes 150 sets of mittens. Your expression may involve the functions $MR(q)$ and $MC(q)$.

d. [4 points] Link makes a deal with a store that would like to buy his cat hats. If the store buys up to 50 hats, then each one will cost \$10. If the store buys more than 50 hats, then Link will reduce the price of the entire order by \$0.05 per hat for every additional hat over 50. (For example, if the store buys 52 hats, they will pay \$9.90 per hat.) Write a formula for a function $L(q)$ which gives Link's revenue if he sells q hats to the store.

$$L(q) = \begin{cases} \underline{\qquad\qquad\qquad\qquad} & \text{if } 0 \le q \le 50 \\ \underline{\qquad\qquad\qquad\qquad} & \text{if } q > 50 \end{cases}$$

Solution.

2. [12 points] Link has started a business selling winter clothes for cats. Among his most successful products are his new kitten mittens. He is currently selling his mittens for \$7 per set. Below is a graph of Link's marginal cost $MC(q)$ and marginal revenue $MR(q)$, in dollars per set of mittens, if he makes q sets of mittens this winter. Due to a shortage of yarn, Link can make a maximum of 200 sets of mittens this winter. In order to start making mittens, Link must spend \$40 on knitting supplies (in other words, it costs \$40 to make 0 sets of mittens).



You do not need to show any work for this problem.

a. [3 points] Approximately how many sets of mittens should Link make this winter in order to maximize his profit?

Answer: Link should make about **104** sets of mittens.

b. [2 points] If the price per set is raised to \$9, approximately how many sets of mittens should Link make in order to maximize his profit?

Answer: Link should make about **200** sets of mittens.

c. [3 points] Write an expression involving integrals which equals Link's total profit if Link makes 150 sets of mittens. Your expression may involve the functions $MR(q)$ and $MC(q)$.

Solution:

$$\int_0^{150} (MR(q) - MC(q))\, dq - 40$$

d. [4 points] Link makes a deal with a store that would like to buy his cat hats. If the store buys up to 50 hats, then each one will cost \$10. If the store buys more than 50 hats, then Link will reduce the price of the entire order by \$0.05 per hat for every additional hat over 50. (For example, if the store buys 52 hats, they will pay \$9.90 per hat.) Write a formula for a function $L(q)$ which gives Link's revenue if he sells q hats to the store.

$$L(q) = \begin{cases} 10q & \text{if } 0 \le q \le 50 \\ (10 - 0.05(q-50))q & \text{if } q > 50 \end{cases}$$

□

4.6 Rates and Related Rate

Alternatively, this section should be called "modeling and the chain rule". In this section we need to take a derivative of a function like $f(t) = r(t)^2$ where $r(t)$ is a function of t itself. So we apply the chain rule, take the derivative with respect to r and then multiply by the derivative of r:

$$f'(t) = 2r(t) \cdot r'(t)$$

One way student like to remember the chain rule by is to write:

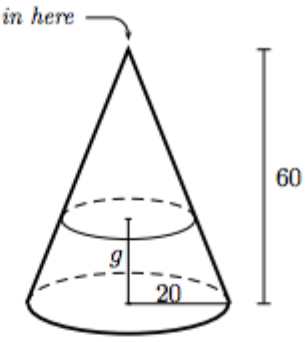
$$\frac{dy}{dt} = \frac{dy}{dr}\frac{dr}{dt}$$

Another way to approach the problem is to apply implicit differentiation on f and taking into account that r is a function. We get the same result.

Exercise 4.6.1.

4. [12 points]

Having taken care of Sebastian and sent Erin into the hands of the *Illumisqati*, King Roderick is pleased that his plan is proceeding well. Our wicked villain decides to relax with a handmade chocolate before he heads to his farmhouse. The process of making the chocolate involves pouring molten chocolate into a mould. The mould is a cone with height 60 mm and base radius 20 mm. Roderick places the mould on the ground and begins pouring the chocolate through the apex of the cone. A diagram of the situation is shown on the right.



In case they are helpful, recall the following formulas for a cone of radius r and height h:

$$\text{Volume} = \tfrac{1}{3}\pi r^2 h \quad \text{and} \quad \text{Surface Area} = \pi r(r + \sqrt{h^2 + r^2}).$$

a. [6 points] Let g be the depth of the chocolate, in mm, as shown in the diagram above. What is the value of g when Roderick has poured a total of 20,000 mm^3 of chocolate into the mould? *Show your work carefully, and make sure your answer is accurate to at least two decimal places.*

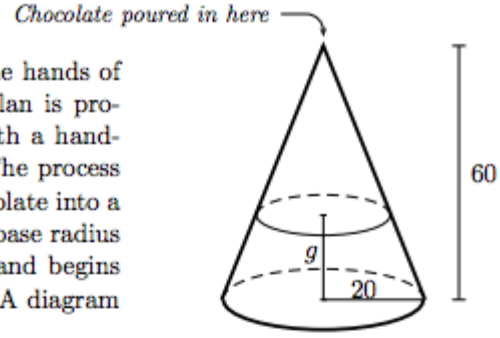
Answer: $g \approx$ _____

b. [6 points] How fast is the depth of the chocolate in the mould (g in the diagram above) changing when Roderick has already poured 20,000 mm^3 of chocolate into the mould if he is pouring at a rate of 5,000 mm^3 per second at this time? *Show your work carefully and make sure your answer is accurate to at least two decimal places. Be sure to include units.*

Answer: _____

Solution.

4. [12 points]
Having taken care of Sebastian and sent Erin into the hands of the *Illumisqati*, King Roderick is pleased that his plan is proceeding well. Our wicked villain decides to relax with a handmade chocolate before he heads to his farmhouse. The process of making the chocolate involves pouring molten chocolate into a mould. The mould is a cone with height 60 mm and base radius 20 mm. Roderick places the mould on the ground and begins pouring the chocolate through the apex of the cone. A diagram of the situation is shown on the right.



In case they are helpful, recall the following formulas for a cone of radius r and height h:

$$\text{Volume} = \tfrac{1}{3}\pi r^2 h \quad \text{and} \quad \text{Surface Area} = \pi r(r + \sqrt{h^2 + r^2}).$$

a. [6 points] Let g be the depth of the chocolate, in mm, as shown in the diagram above. What is the value of g when Roderick has poured a total of 20,000 mm^3 of chocolate into the mould? *Show your work carefully, and make sure your answer is accurate to at least two decimal places.*

Solution: The volume of the solid is given by $V = \frac{1}{3}\pi(20)^2 60 - \frac{1}{3}\pi r^2(60-g)$ where r is the radius of the cross-section at height g. We want to rewrite r in terms of g. Using similar triangles we find the equation

$$\frac{r}{20} = \frac{60-g}{60},$$

which implies $r = \frac{60-g}{3}$. Therefore, $V = 8000\pi - \frac{\pi}{27}(60-g)^3$. So, to find the appropriate g we need to solve $8000\pi - \frac{\pi}{27}(60-g)^3 = 20{,}000$. Solving, we get

$$(60-g)^3 = \frac{27}{\pi}(8000\pi - 20{,}000), \tag{2}$$

which implies $g = 60 - \sqrt[3]{\frac{27}{\pi}(8000\pi - 20{,}000)} \approx 24.67$. The chocolate is approximately 24.67 mm deep when he has poured a total of 20,000 mm^3 of chocolate into the mould.

Answer: $g \approx$ **24.67**

b. [6 points] How fast is the depth of the chocolate in the mould (g in the diagram above) changing when Roderick has already poured 20,000 mm^3 of chocolate into the mould if he is pouring at a rate of 5,000 mm^3 per second at this time? *Show your work carefully and make sure your answer is accurate to at least two decimal places. Be sure to include units.*

Solution: We want to find $\frac{dg}{dt}$ when $g = 60 - \sqrt[3]{\frac{27}{\pi}(8000\pi - 20{,}000)}$ (from part (a)) if $\frac{dV}{dt} = 5000$ at this time. Differentiating our formula (2) from part (a) with respect to t, we have

$$\frac{dV}{dt} = \frac{\pi}{9}(60-g)^2\frac{dg}{dt}.$$

Substituting $g = 60 - \sqrt[3]{\frac{27}{\pi}(8000\pi - 20{,}000)}$ and $\frac{dV}{dt} = 5000$ into this equation, we find

$$5000 = \frac{\pi}{9}\left(60 - \left[60 - \sqrt[3]{\frac{27}{\pi}(8000\pi - 20{,}000)}\right]\right)^2 \frac{dg}{dt} \quad \text{so}$$

$$\frac{dg}{dt} = \frac{5000 \cdot 9}{\pi}\left(\frac{27}{\pi}(8000\pi - 20{,}000)\right)^{-2/3} \approx 11.47.$$

(Using our approximation $g \approx 24.67$ instead gives us $\frac{dg}{dt} \approx 11.48$.)
So the depth of the chocolate is increasing at an instantaneous rate of about 11.47 mm/sec at that moment.

Answer: **11.47 mm/ sec.**

□

Exercise 4.6.2.

5. [12 points] A certain type of spherical melon has weight proportional to its volume as it grows. When the melon weighs 0.2 pounds, it has a volume of 36 cm^3 and its weight is increasing at a rate of 0.1 pounds per day. [Note: The volume of a sphere is $V = \frac{4}{3}\pi r^3$.]

a. [3 points] Find $\frac{dV}{dt}$ when the melon weighs 0.2 pounds (t measured in days).

b. [5 points] Find the rate at which the radius of the melon is increasing when it weighs 0.2 pounds.

c. [4 points] Use a local linearization to approximate the volume of the melon 36 hours after it weighs 0.2 pounds.

Solution.

5. [12 points] A certain type of spherical melon has weight proportional to its volume as it grows. When the melon weighs 0.2 pounds, it has a volume of 36 cm^3 and its weight is increasing at a rate of 0.1 pounds per day. [Note: The volume of a sphere is $V = \frac{4}{3}\pi r^3$.]

a. [3 points] Find $\dfrac{dV}{dt}$ when the melon weighs 0.2 pounds (t measured in days).

Solution: When the melon weighs 0.2 pounds, it has a volume of 36 cm^3, so if $V = kw$ where V is volume, w is weight and k is a proportionality constant, then $36 = 0.2k$ giving $k = 180$. Now $\frac{dV}{dt} = k\frac{dw}{dt} = 180(0.1) = 18$ cm^3 per day.

b. [5 points] Find the rate at which the radius of the melon is increasing when it weighs 0.2 pounds.

Solution: We are looking for $\frac{dr}{dt}$ when $w = 0.2$. Differentiating the equation for a sphere gives

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

We know $36 = \frac{4}{3}\pi r^3$, so $r = \frac{3}{\pi^{1/3}}$. We also know that $\frac{dV}{dt} = 18$. Putting this together we have

$$18 = 4\pi \frac{9}{\pi^{2/3}} \frac{dr}{dt}.$$

which means $\frac{dr}{dt} = \frac{1}{2\pi^{1/3}}$ cm per day.

c. [4 points] Use a local linearization to approximate the volume of the melon 36 hours after it weighs 0.2 pounds.

Solution: Since $\frac{dV}{dt} = 18$ cm^3 per day and 36 hours is 1.5 days we know the melon will increase by approximately 18(1.5)=27 cm^3 in the 36 hours after it weighs 0.2 pounds. Since it is 36 cm^3 at the time in question, the volume will be about 63 cm^3 36 hours later.

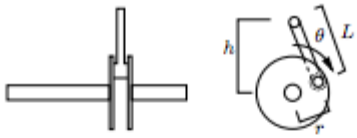
□

Exercise 4.6.3.

3. [10 points] For each of the following determine the indicated quantity.

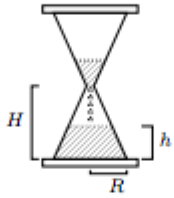
a. [4 points] In an internal combustion engine, pistons are pushed up and down by a crank shaft similar to the diagram shown to the right. As the shaft rotates the height of the piston, h, is related to the rotational angle θ of the shaft by $h = r\cos\theta + \sqrt{L^2 - r^2\sin^2\theta}$, where r and L are constant lengths. If $r = 10$ cm, $L = 15$ cm, and h is decreasing at a rate of 5000 cm/s when $\theta = 3\pi/4$, how fast is θ changing then?

Crank shaft diagram (part a)



side view end view

Hourglass diagram (part b)

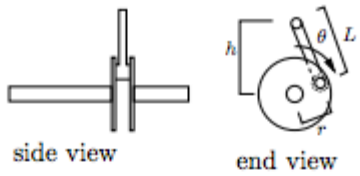


b. [6 points] The lower chamber of an hourglass is shaped like a cone with height H in and base radius R in, as shown in the figure to the right, above. Sand falls into this cone. Write an expression for the volume of the sand in the lower chamber when the height of the sand there is h in *(Hint: A cone with base radius r and height y has volume $V = \frac{1}{3}\pi r^2 y$, and it may be helpful to think of a difference between two conical volumes.)*. Then, if $R = 0.9$ in, $H = 2.7$ in, and sand is falling into the lower chamber at $2\,\text{in}^3/\text{min}$, how fast is the height of the sand in the lower chamber changing when $h = 1$ in?

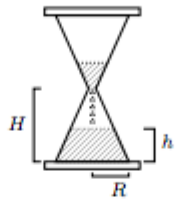
Solution.

3. [10 points] For each of the following determine the indicated quantity.

Crank shaft diagram (part a)



Hourglass diagram (part b)



a. [4 points] In an internal combustion engine, pistons are pushed up and down by a crank shaft similar to the diagram shown to the right. As the shaft rotates the height of the piston, h, is related to the rotational angle θ of the shaft by $h = r\cos\theta + \sqrt{L^2 - r^2\sin^2\theta}$, where r and L are constant lengths. If $r = 10$ cm, $L = 15$ cm, and h is decreasing at a rate of 5000 cm/s when $\theta = 3\pi/4$, how fast is θ changing then?

Solution: Using the chain rule, we know that $h'(t) = \left(-r\sin\theta - \frac{r^2\sin\theta\cos\theta}{\sqrt{L^2-r^2\sin^2\theta}}\right)\cdot\frac{d\theta}{dt}$. Thus if $\theta = 3\pi/4$, $h' = -5000$, $r = 10$ and $L = 15$, we have $-5000 = \left(-\frac{10}{\sqrt{2}} - \frac{50}{\sqrt{225-50}}\right)\cdot\frac{d\theta}{dt} \approx -3.29\cdot\frac{d\theta}{dt}$. Thus $\frac{d\theta}{dt} \approx 1500$ radians/sec

b. [6 points] The lower chamber of an hourglass is shaped like a cone with height H in and base radius R in, as shown in the figure to the right, above. Sand falls into this cone. Write an expression for the volume of the sand in the lower chamber when the height of the sand there is h in *(Hint: A cone with base radius r and height y has volume $V = \frac{1}{3}\pi r^2 y$, and it may be helpful to think of a difference between two conical volumes.)*. Then, if $R = 0.9$ in, $H = 2.7$ in, and sand is falling into the lower chamber at 2 in^3/min, how fast is the height of the sand in the lower chamber changing when $h = 1$ in?

Solution: The whole volume of the lower chamber is $V_{tot} = \frac{1}{3}\pi R^2 H$. The volume of the empty space above the sand is similarly $V_{emp} = \frac{1}{3}\pi r^2 (H-h)$, where r is the radius at the height h. By comparing the similar triangles delimiting the full lower chamber and the empty top section, we see that $r = \frac{R}{H}(H-h)$. Thus $V_{emp} = \frac{1}{3}\pi\frac{R^2}{H^2}(H-h)^3$. The volume of the lower, sand-filled region is therefore

$$V = \frac{1}{3}\pi\frac{R^2}{H^2}\left(H^3 - (H-h)^3\right).$$

Then, differentiating, we have

$$\frac{dV}{dt} = \pi\frac{R^2}{H^2}(H-h)^2\frac{dh}{dt}.$$

Thus, when $\frac{dV}{dt} = 2$, $R = 0.9$, $H = 2.7$, and $h = 1$,

$$2 = \pi\frac{1}{3^2}(1.7)^2\frac{dh}{dt},$$

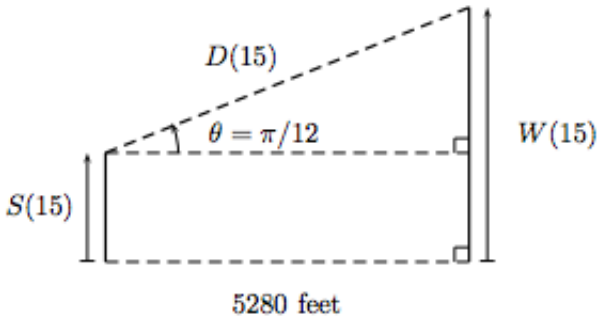
so that $\frac{dh}{dt} = \frac{18}{\pi(1.7)^2} \approx 1.98$ in/min.

□

Exercise 4.6.4.

9. [10 points] You are sitting on a ship traveling at a constant speed of 6 ft/sec, and you spot a white object due east moving parallel to the ship. After tracking the object through your telescope for 15 seconds, you identify it as the white whale. Let $W(t)$ denote the distance of the whale from its starting point in feet, and $S(t)$ denote the distance of the ship from its starting point in feet, with t the time in seconds since you saw the whale. You also note that at the end of the 15 seconds you were tracking the whale, you have turned your head $\pi/12$ radians north to keep it in your sights.

a. [1 point] If initially the creature is 5280 ft (1 mile) from the ship due east, use the angle you have turned your head to find the distance $D(t)$ in feet between the ship and the creature at 15 seconds. The figure below should help you visualize the situation. Recall that, for right triangles, $\cos(\theta)$ is the ratio of the adjacent side to the hypotenuse.



b. [2 points] Let $\theta(t)$ give the angle you've turned your head after t seconds of tracking the whale. Write an equation $D(t)$ for the distance between the ship and the whale at time t (Hint: your answer may involve $\theta(t)$).

c. [3 points] If at precisely 15 seconds, you are turning your head at a rate of .01 radians per second, what is the instantaneous rate of change of the distance between the ship and the whale?

d. [4 points] What is the speed of the whale at $t = 15$ seconds? Hint: Use the Pythagorean theorem.

Solution.

5280 feet

Solution: Since $\cos(\pi/12) = \frac{5280}{D(15)}$, we find that $D(15) = \frac{5280}{\cos(\pi/12)} \approx 5466.258$ft.

b. [2 points] Let $\theta(t)$ give the angle you've turned your head after t seconds of tracking the whale. Write an equation $D(t)$ for the distance between the ship and the whale at time t (Hint: your answer may involve $\theta(t)$).

Solution: From the previous part, we know that the distance between the ship and the whale is 5280 divided by the cosine of the angle you've turned your head. Since $\theta(t)$ gives how far you've turned your head, we can find the distance at any time t using the function $D(t) = \frac{5280}{\cos(\theta(t))}$.

c. [3 points] If at precisely 15 seconds, you are turning your head at a rate of .01 radians per second, what is the instantaneous rate of change of the distance between the ship and the whale?

Solution: Since $D(t) = \frac{5280}{\cos(\theta(t))}$, we take the derivative with respect to t on both sides to get

$$\frac{dD}{dt} = \frac{5280\sin(\theta(t))}{\cos^2(\theta(t))}\theta'(t).$$

Since $\theta(15) = \pi/12$ and we are given $\theta'(15) = .01$, we get

$$\left.\frac{dD}{dt}\right|_{t=15} = \frac{5280\sin(\pi/12)}{\cos^2(\pi/12)}(.01) \approx 14.6468\text{ft/sec}.$$

d. [4 points] What is the speed of the whale at $t = 15$ seconds? Hint: Use the Pythagorean theorem.

an Department of Mathematics Fall, 2011 Math 115 Exam 3 Problem 9 (white whale) So

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Solution: The right triangle in the figure above has hypotenuse $D(t)$ and sides with length $D(t)$ and $W(t) - S(t)$, so the Pythagorean theorem states

$$D(t)^2 = 5280^2 + (W(t) - S(t))^2.$$

If we take the t derivative of both sides, we get

$$2D(t)\frac{dD}{dt} = 2(W(t) - S(t))\left(\frac{dW}{dt} - \frac{dS}{dt}\right).$$

To find $\left.\frac{dW}{dt}\right|_{t=15}$, we will need $D(15) = 5466.258$ and $\left.\frac{dD}{dt}\right|_{t=15} = 14.6468$. We also need to find $W(15) - S(15)$, but since this is one side of the right triangle, we can use tangent to find this distance: $W(15) - S(15) = 5280\tan(\pi/12) \approx 1414.7717$. Finally, we also need to know $\frac{dS}{dt}$, but in the description of the problem it says that ship is traveling at a constant speed of 6 ft/sec. Plugging all of this information into our equation we have

$$2(5466.258)(14.6468) = 2(1414.7717)\left(\frac{dW}{dt} - 6\right) \Rightarrow 56.5909 = \left.\frac{dW}{dt}\right|_{t=15} - 6.$$

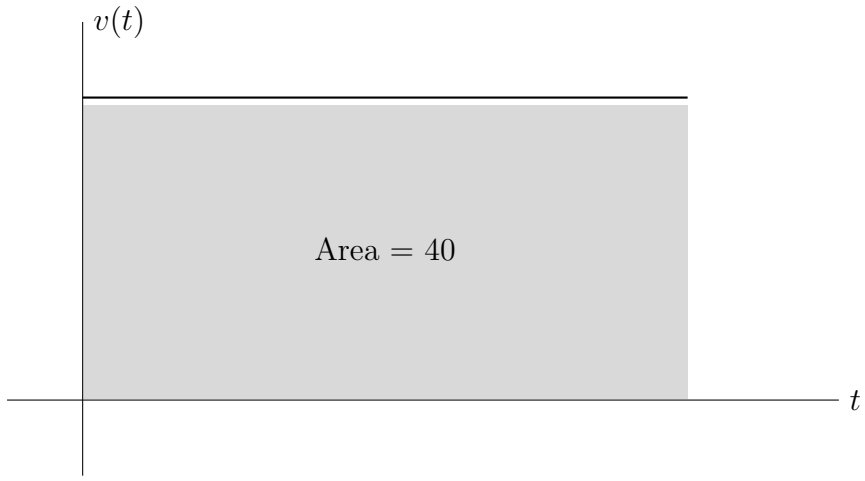
Therefore, $\left.\frac{dW}{dt}\right|_{t=15} \approx 62.5909$ ft/sec

□

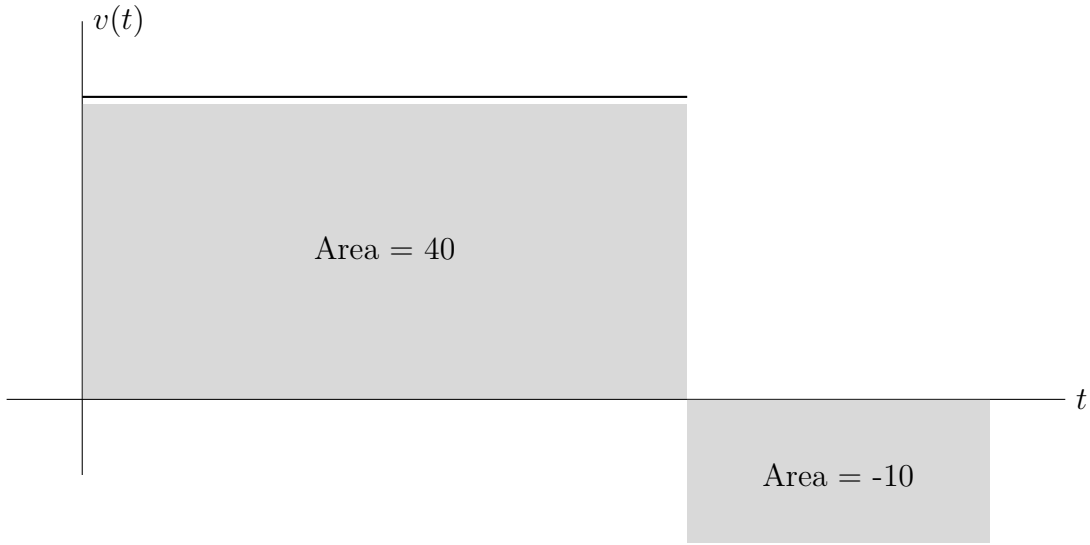
5 Definite Integral

5.1 Distance

Given a function $v(t)$, how to measure the distance traveled? If $v(t)$ is constant, say 20 miles per hour, and the total time is 2 hours. What is the total distance traveled? $20 \cdot 2 = 40 miles$. Graphically, 40 is the area under the graph:

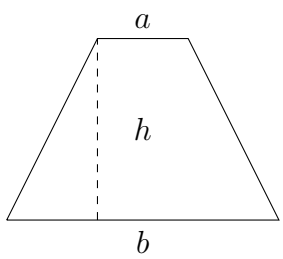


This idea generalizes to any function $v(t)$ and the distance is the area under the graph with the convention of taking area under the x axis as negative. So if I traveled at 20 mph for 2 hours and then traveled at 10 mph in the opposite direction for 1 hour, the total distance is $40 - 10 = 30$ miles.



Ways to approximate area:

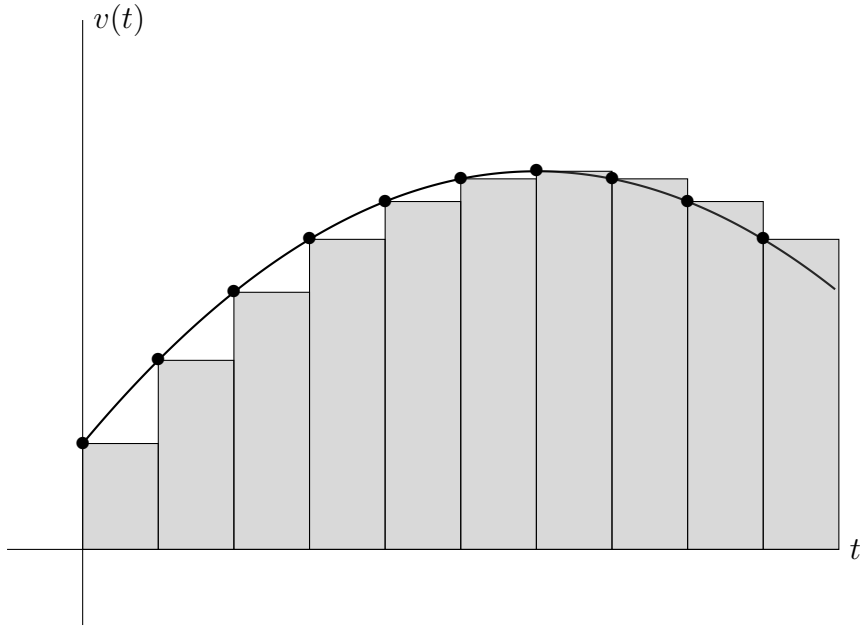
- Using area of shapes like triangles, rectangles and trapezoids. For example:



Area $= \frac{a+b}{2}h$

- Using the approximations: **Left Sum and Right Sum**:

Left sum:



- Take the interval $[a, b]$ and divide to n equal subintervals of size Δt. $(b-a)/n = \Delta t$.
- The intervals are $[t_0, t_1], [t_1, t_2], ..., [t_{n-1}, t_n]$ with:

$$\begin{aligned}
&t_0 = a \\
&t_1 = t_0 + \Delta t = a + \Delta t \\
&t_2 = t_1 + \Delta t = a + 2\Delta t \\
&t_3 = t_2 + \Delta t = a + 3\Delta t \\
&\vdots \\
&t_n = t_{n-1} + \Delta t = a + n\Delta t = b
\end{aligned}$$

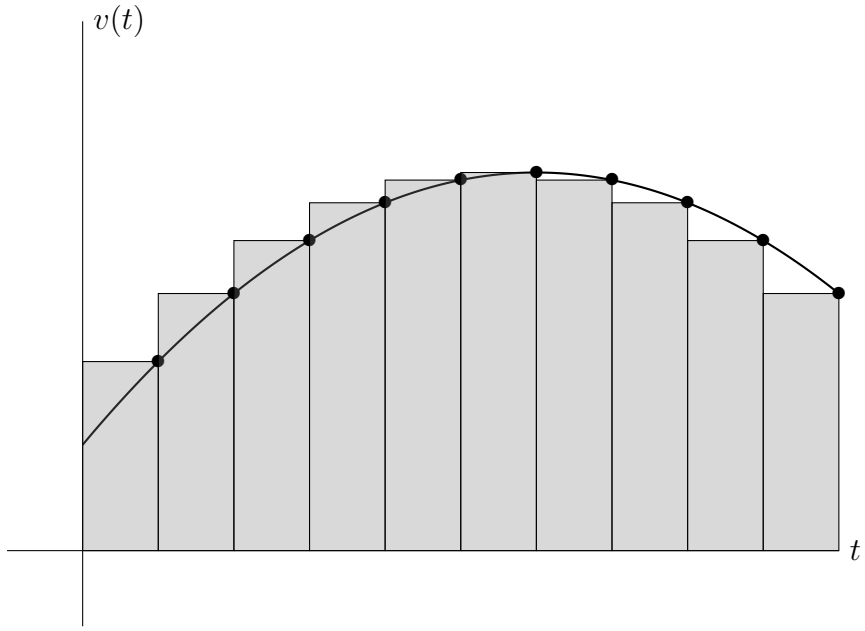
- For each interval we pick the left value of the function $v(t)$:

$$\begin{aligned}
&[t_0, t_1] \text{ pick } v(t_0) = v(a) \\
&[t_1, t_2] \text{ pick } v(t_1) \\
&[t_2, t_3] \text{ pick } v(t_2) \\
&\vdots \\
&[t_{n-1}, t_n] \text{ pick } v(t_{n-1})
\end{aligned}$$

- The approximation using the left sum is $v(t_0)\Delta t + v(t_1)\Delta t + ... + v(t_{n-1})\Delta t =$

$$LS = \Delta t(v(t_0) + v(t_1).... + v(t_{n-1}))$$

Right sum:



- Take the interval $[a, b]$ and divide to n equal subintervals of size Δt. $(b-a)/n = \Delta t$.
- The intervals are $[t_0, t_1], [t_1, t_2], ..., [t_{n-1}, t_n]$ with:

$$\begin{aligned}
&t_0 = a \\
&t_1 = t_0 + \Delta t = a + \Delta t \\
&t_2 = t_1 + \Delta t = a + 2\Delta t \\
&t_3 = t_2 + \Delta t = a + 3\Delta t \\
&\vdots \\
&t_n = t_{n-1} + \Delta t = a + n\Delta t = b
\end{aligned}$$

- For each interval we pick the left value of the function $v(t)$:

$$\begin{aligned}
&[t_0, t_1] \text{ pick } v(t_1) \\
&[t_1, t_2] \text{ pick } v(t_2) \\
&[t_2, t_3] \text{ pick } v(t_3) \\
&\vdots \\
&[t_{n-1}, t_n] \text{ pick } v(t_n) = v(b)
\end{aligned}$$

- The approximation using the left sum is $v(t_1)\Delta t + v(t_2)\Delta t + ... + v(t_n)\Delta t =$

$$RS = \Delta t(v(t_1) + v(t_2).... + v(t_n))$$

Remark 5.1.1. Observe:

- $|RS - LS| = \Delta t|v(b) - v(a)|$
- When $v(t)$ is increasing, LS is an underestimate and RS is an overestimate.
- When $v(t)$ is decreasing, LS is an overestimate and RS is an underestimate.
- When we take more intervals, n gets bigger and the approximation gets better.

Exercise 5.1.2.

1. Figure 5.11 shows the velocity of a car for $0 \leq t \leq 12$ and the rectangles used to estimate of the distance traveled.

(a) Do the rectangles represent a left or a right sum?
(b) Do the rectangles lead to an upper or a lower estimate?
(c) What is the value of n?
(d) What is the value of Δt?
(e) Give an approximate value for the estimate.

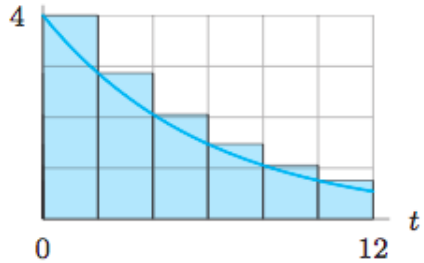


Figure 5.11

Solution. (a) For each interval, we are picking the left point — Left sum.

(b) Graphically — an upper bound.

(c) n is the number of rectangles so $n = 6$.

(d) Δt is the width of each rectangle so $\Delta t = 2$. Also $(b - a)/n = \Delta t$ and we have $(12 - 0)/6 = 2$

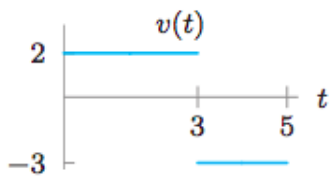
(e) LS=2(4+3+2+1.5+1 +0.75) = 2(12.25)=24.5

□

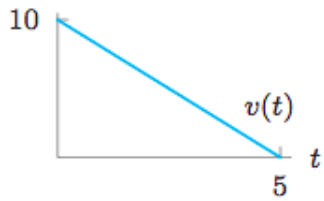
Exercise 5.1.3.

Exercises 9–12 show the velocity, in cm/sec, of a particle moving along a number line. (Positive velocities represent movement to the right; negative velocities to the left.) Compute the change in position between times $t = 0$ and $t = 5$ seconds.

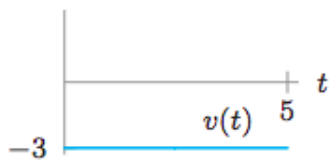
9.



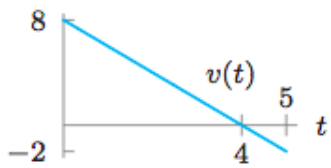
10.



11.



12.



Solution. 9. Distance = 6 - 6=0
10. Distance $= 10 \cdot 5/2 = 25$ cm
11. Distance $= -3 \cdot 5 = -15$ cm
12. Distance $= 8 \cdot 4/2 - 2 \cdot 1/2 = 32/2 - 1 = 15$. □

Exercise 5.1.4.

13. Use the expressions for left and right sums on page 276 and Table 5.5.

(a) If $n = 4$, what is Δt? What are t_0, t_1, t_2, t_3, t_4? What are $f(t_0), f(t_1), f(t_2), f(t_3), f(t_4)$?

(b) Find the left and right sums using $n = 4$.

(c) If $n = 2$, what is Δt? What are t_0, t_1, t_2? What are $f(t_0), f(t_1), f(t_2)$?

(d) Find the left and right sums using $n = 2$.

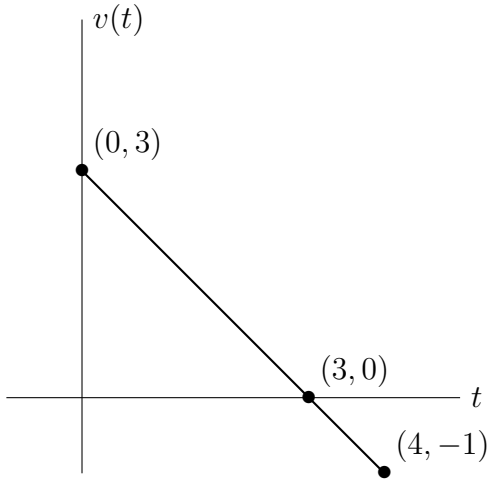
Table 5.5

t	15	17	19	21	23
$f(t)$	10	13	18	20	30

Solution. (a) $a = 15, b = 23, b - a = 8, \Delta t = 8/4 = 2$. The t_i's are 15,17,19,21,23. The $f(t_i)$'s are given in the table.
(b) LS = 2(10+13+18+20) = 2(61)=122. RS=2(13+18+20+30)=2(81)=162
(c) $\Delta t = 4$, t's are 14,19,23.
(d) LS = 4(10+18)=112. RS = 4(18+30)=192.

□

Exercise 5.1.5.
Let $v(t)$ be in meters per second, represented by the following graph. Find LS, RS using 4 intervals and the exact distance traveled for $v(t)$ between $t = 0$ and $t = 4$ seconds. Is the LS an upper bound or a lower bound?



Solution. The exact distance is the area under the graph. Using the area for two triangles we get $3 \cdot 3/2 - 1 \cdot 1/2 = 4.5 - 0.5 = 4$ meters.
$n = 4$ so $\Delta t = 1$.
The t's are $0, 1, 2, 3, 4$.
LS = 1(3+2+1+0)=6
RS = 1(2+1+0-1)=2
As expected, LS is an upper bound since the function is decreasing. $\square$

5.2 Definite Integral

We repeat the the discussion from 5.1 for a general function $f(x)$. Given some interval $[a, b]$ we would like to measure the area under the graph. We have two approximations so far, the Left sum and the right sum. In the context of general functions, those are called left/right Riemann sums:

$$LS = \sum_{i=0}^{n-1} f(t_i)\Delta t, \quad RS = \sum_{i=1}^{n} f(t_i)\Delta t$$

As n gets bigger the approximation gets better. We define:

$$\int_a^b f(x)dx = \lim_{n\to\infty} RS = \lim_{n\to\infty} LS$$

If both exists.
We have a **Theorem** that guarantees that when f is **CTS** then $\int_a^b f(x)dx$ is the area under the graph, it exists.
A **General Riemann Sum**:

- Take the interval $[a, b]$ and divide to n equal subintervals of size Δt. $(b - a)/n = \Delta t$.
- The intervals are $[t_0, t_1], [t_1, t_2], ..., [t_{n-1}, t_n]$ with:

$$\begin{aligned}
&t_0 = a \\
&t_1 = t_0 + \Delta t = a + \Delta t \\
&t_2 = t_1 + \Delta t = a + 2\Delta t \\
&t_3 = t_2 + \Delta t = a + 3\Delta t \\
&\vdots \\
&t_n = t_{n-1} + \Delta t = a + n\Delta t = b
\end{aligned}$$

- For each interval we pick the left value of the function $v(t)$:

$$\begin{aligned}
&[t_0, t_1] \text{ pick } v(c_1) \text{ for some } t_0 \le c_1 \le t_1 \\
&[t_1, t_2] \text{ pick } v(c_2) \text{ for some } t_1 \le c_2 \le t_2 \\
&[t_2, t_3] \text{ pick } v(c_3) \text{ for some } t_2 \le c_3 \le t_3 \\
&\vdots \\
&[t_{n-1}, t_n] \text{ pick } v(c_n) \text{ for some } t_{n-1} \le c_n \le t_n
\end{aligned}$$

- The approximation using the left sum is $v(c_1)\Delta t + v(c_2)\Delta t + ... + v(c_n)\Delta t =$

$$\text{General Riemann Sum} = \Delta t(v(c_1) + v(c_2).... + v(c_n))$$

The average values of $f(x)$ on $[a, b]$ is:

$$\boxed{\frac{1}{b-a}\int_a^b f(x)dx}$$

Exercise 5.2.1.

8. [11 points] Suppose k and p are positive constants. Consider the function

$$R(x) = p - \ln(x^2 + k)\,.$$

a. [5 points] Use the limit definition of the derivative to write down an explicit expression for $R'(3)$.
Your answer should not include the letter R.
Do not attempt to evaluate or simplify the limit.

Answer: $R'(3) =$

b. [4 points] Write out all the terms for the right-hand Riemann sum with three subdivisions of equal length which approximates the integral

$$\int_1^{13} R(x)\,dx\,.$$

Your answer should not include the letter R but may involve k and/or p.

c. [2 points] Is the right-hand Riemann sum with three subdivisions of equal length from part (b) an overestimate or an underestimate of $\int_1^{13} R(x)\,dx$, or is there not enough information to make this determination? Briefly explain your reasoning.

Answer: (Circle one choice.)

Overestimate Underestimate Not enough info

Reasoning:

Solution.

8. [11 points] Suppose k and p are positive constants. Consider the function

$$R(x) = p - \ln(x^2 + k)\,.$$

a. [5 points] Use the limit definition of the derivative to write down an explicit expression for $R'(3)$.
Your answer should not include the letter R.
Do not attempt to evaluate or simplify the limit.

Answer: $R'(3) = \lim_{h\to 0} \dfrac{(p - \ln((3+h)^2 + k)) - (p - \ln(3^2 + k))}{h}$

b. [4 points] Write out all the terms for the right-hand Riemann sum with three subdivisions of equal length which approximates the integral

$$\int_1^{13} R(x)\,dx\,.$$

Your answer should not include the letter R but may involve k and/or p.

Solution: This right hand sum is given by

$$4R(5) + 4R(9) + 4R(13) = 4(p - \ln(25 + k)) + 4(p - \ln(81 + k)) + 4(p - \ln(169 + k))\,.$$

c. [2 points] Is the right-hand Riemann sum with three subdivisions of equal length from part (b) an overestimate or an underestimate of $\int_1^{13} R(x)\,dx$, or is there not enough information to make this determination? Briefly explain your reasoning.

Answer: (Circle one choice.)

Overestimate $\boxed{\text{Underestimate}}$ Not enough info

Reasoning:

Solution: Since $R'(x) = -\frac{2x}{x^2+k}$ is negative for $x > 0$, $R(x)$ is decreasing on the interval $1 \le x \le 13$. Thus, the above right-hand Riemann sum is an underestimate for the integral.

□

Exercise 5.2.2.

3. [14 points] Let g be a differentiable function defined for all real numbers. A table of some values of g is given below.

w	-1	1	3	5
$g(w)$	-2	3	5	6

Assume that g is always strictly increasing on the interval $[-1, 5]$ and that g' is always strictly decreasing on the interval $[-1, 5]$.

a. [2 points] Estimate $g'(5)$.

Answer: $g'(5) \approx$ _____

b. [4 points] Rank the following quantities in order from least to greatest by filling in the blanks below with the options I-V.

I. 0 II. $g'(1)$ III. $g(1) - g(-1)$ IV. $g'(3)$ V. $\dfrac{g(3) - g(1)}{2}$

_____ < _____ < _____ < _____ < _____

c. [4 points] Find the best possible estimate of $\int_{-1}^{5} (g(w) + 1)\, dw$ using a right hand sum and the data provided. Be sure to write all of the terms in the sum.

d. [1 point] Is your estimate from part (c) an overestimate or underestimate of $\int_{-1}^{5} (g(w) + 1)\, dw$? *You do not need to explain your answer.*

Underestimate Overestimate Impossible to determine

e. [3 points] Find the average value of $g'(w)$ on the interval $[-1, 5]$.

Answer: _____

Solution.

3. [14 points] Let g be a differentiable function defined for all real numbers. A table of some values of g is given below.

w	-1	1	3	5
$g(w)$	-2	3	5	6

Assume that g is always strictly increasing on the interval $[-1, 5]$ and that g' is always strictly decreasing on the interval $[-1, 5]$.

a. [2 points] Estimate $g'(5)$.

Solution: $g'(5) \approx \frac{g(5)-g(3)}{5-3} = \frac{1}{2}$.

Answer: $g'(5) \approx$ $\frac{1}{2}$

b. [4 points] Rank the following quantities in order from least to greatest by filling in the blanks below with the options I-V.

I. 0 II. $g'(1)$ III. $g(1) - g(-1)$ IV. $g'(3)$ V. $\dfrac{g(3) - g(1)}{2}$

$0 < g'(3) < \dfrac{g(3) - g(1)}{2} < g'(1) < g(1) - g(-1)$

c. [4 points] Find the best possible estimate of $\int_{-1}^{5} (g(w) + 1)\, dw$ using a right hand sum and the data provided. Be sure to write all of the terms in the sum.

Solution:

$$\begin{aligned}\int_{-1}^{5} (g(w) + 1)\, dw &\approx \Delta w((g(1) + 1) + (g(3) + 1) + (g(5) + 1)) \\ &= 2(4 + 6 + 7) \\ &= 34.\end{aligned}$$

d. [1 point] Is your estimate from part (c) an overestimate or underestimate of $\int_{-1}^{5} (g(w) + 1)\, dw$?

You do not need to explain your answer.

Underestimate $\boxed{\text{Overestimate}}$ Impossible to determine

Solution: The function $g(w) + 1$ is always increasing (since it is a vertical shift of $g(w)$, which is always increasing) so the right hand sum gives an overestimate.

e. [3 points] Find the average value of $g'(w)$ on the interval $[-1, 5]$.

Solution: By definition, the average value of $g'(w)$ on $[-1, 5]$ is

$$\begin{aligned}g'(w) &= \frac{1}{6}\int_{-1}^{5} (g'(w))\, dw \\ &= \frac{1}{6}[g(5) - g(-1)] \\ &= \frac{8}{6} = \frac{4}{3}.\end{aligned}$$

(The average value of g' on the interval is the average rate of change of g over the interval.)

Answer: $\frac{4}{3}$

□

Exercise 5.2.3.

6. [12 points] The rate $q(t)$ at which cars passed through the intersection of Main Street and Huron after a football game is presented in the table below.

t (in minutes after the game ended)	0	20	40	60	80	100	120
$q(t)$ (in cars per minute)	10	15	19	21	20	17	13

a. [4 points] What is the meaning of $\int_0^{120} q(t)\,dt$? Using a left Riemann sum and $n = 6$, estimate $\int_0^{120} q(t)\,dt$. (Write out the terms of your sum.)

b. [2 points] Write an expression for the average rate at which cars passed through the intersection for the first two hours after the game ended.

c. [3 points] Estimate $q'(30)$.

d. [3 points] If $Q(t)$ denotes the total number of cars that have passed through the intersection t minutes after the game ended, find and interpret $Q'(60)$.

Solution.

6. [12 points] The rate $q(t)$ at which cars passed through the intersection of Main Street and Huron after a football game is presented in the table below.

t (in minutes after the game ended)	0	20	40	60	80	100	120
$q(t)$ (in cars per minute)	10	15	19	21	20	17	13

a. [4 points] What is the meaning of $\int_0^{120} q(t)\,dt$? Using a left Riemann sum and $n = 6$, estimate $\int_0^{120} q(t)\,dt$. (Write out the terms of your sum.)

Solution: The expression $\int_0^{120} q(t)\,dt$ gives the total number of cars that passed through the intersection in the first two hours after the game. A left-hand approximation with 6 subdivisions is given by

$$(20)(10 + 15 + 19 + 21 + 20 + 17) = 2040 \text{ cars}.$$

b. [2 points] Write an expression for the average rate at which cars passed through the intersection for the first two hours after the game ended.

Solution: The average rate at which cars passed through the intersection during this time period is given by $\frac{1}{120}\int_0^{120} q(t)\,dt$.

c. [3 points] Estimate $q'(30)$.

Solution: The best estimate we can get from the table is

$$q'(30) \approx \frac{19 - 15}{40 - 20} = 0.2 \text{ cars per minute per minute.}$$

d. [3 points] If $Q(t)$ denotes the total number of cars that have passed through the intersection t minutes after the game ended, find and interpret $Q'(60)$.

Solution: We can read $Q'(60)$ from the table. We have $Q'(60) = 21$ and indicates that one hour after the game, approximately 21 additional cars would pass through the intersection in the next minute.

□

Exercise 5.2.4.

11. [8 points] You are not required to show your work on this page.

a. [2 points] A function $f(x)$ is differentiable. Some values of f and f' are shown in the table below.

x	0	1	2	3	4
$f(x)$	3	4	1	-1	-2
$f'(x)$	2	-2	-3	0	3

Let $g(x) = \cos(\frac{\pi}{2}f(x))$. Which of the following values of x <u>must</u> be a critical point of $g(x)$? Circle <u>all</u> such values.

0 1 2 3 4 NONE OF THESE

b. [2 points] Which of the following expressions gives the linear approximation for $\arctan(x)$ near $x = 1$? Circle <u>all</u> such expressions.

i. $\frac{\pi}{4} + \frac{1}{2}(x-1)$

ii. $\frac{1}{2} + \frac{\pi}{4}(x-1)$

iii. $\frac{1}{1+x^2} + \frac{\pi}{4}(x-1)$

iv. $\arctan(x) + \frac{1}{2}(x-1)$

v. NONE OF THESE

c. [2 points] Which of the following functions are antiderivatives of $f(x) = \frac{1}{x}$? Circle <u>all</u> such functions.

i. $\ln(|x+1|)$

ii. $\ln(|x|)$

iii. $\ln(|x|) + 2$

iv. $\ln(4|x|)$

v. $4\ln(|x|)$

vi. NONE OF THESE

d. [2 points] Suppose n is a positive integer, f is a decreasing, continuous function on the interval $[2, 6]$, the value of the left Riemann sum with n equal subdivisions for $\int_2^6 f(x)dx$ is A, and $f(2) = f(6) + 8$. Circle <u>all</u> the statements that must be true.

i. A is an overestimate for $\int_2^6 f(x)dx$.

ii. $\int_2^6 f(x)\,dx = 8$.

iii. $\int_1^5 f(x+1)\,dx = \int_2^6 f(x)\,dx$.

iv. The left Riemann sum for $\int_2^6 (f(x))^2\,dx$ with n equal subdivisions is equal to A^2.

v. NONE OF THESE

Solution.

11. [8 points] You are not required to show your work on this page.

a. [2 points] A function $f(x)$ is differentiable. Some values of f and f' are shown in the table below.

x	0	1	2	3	4
$f(x)$	3	4	1	-1	-2
$f'(x)$	2	-2	-3	0	3

Let $g(x) = \cos(\frac{\pi}{2}f(x))$. Which of the following values of x must be a critical point of $g(x)$? Circle all such values.

0 $\boxed{1}$ 2 $\boxed{3}$ $\boxed{4}$ NONE OF THESE

b. [2 points] Which of the following expressions gives the linear approximation for $\arctan(x)$ near $x = 1$? Circle all such expressions.

i. $\boxed{\frac{\pi}{4} + \frac{1}{2}(x-1)}$

ii. $\frac{1}{2} + \frac{\pi}{4}(x-1)$

iii. $\frac{1}{1+x^2} + \frac{\pi}{4}(x-1)$

iv. $\arctan(x) + \frac{1}{2}(x-1)$

v. NONE OF THESE

c. [2 points] Which of the following functions are antiderivatives of $f(x) = \frac{1}{x}$? Circle all such functions.

i. $\ln(|x+1|)$

ii. $\boxed{\ln(|x|)}$

iii. $\boxed{\ln(|x|) + 2}$

iv. $\boxed{\ln(4|x|)}$

v. $4\ln(|x|)$

vi. NONE OF THESE

d. [2 points] Suppose n is a positive integer, f is a decreasing, continuous function on the interval $[2, 6]$, the value of the left Riemann sum with n equal subdivisions for $\int_2^6 f(x)dx$ is A, and $f(2) = f(6) + 8$. Circle all the statements that must be true.

i. $\boxed{A \text{ is an overestimate for } \int_2^6 f(x)dx.}$

ii. $\int_2^6 f(x)\,dx = 8.$

iii. $\boxed{\int_1^5 f(x+1)\,dx = \int_2^6 f(x)\,dx.}$

iv. The left Riemann sum for $\int_2^6 (f(x))^2\,dx$ with n equal subdivisions is equal to A^2.

v. NONE OF THESE

□

Quiz #8. Please write you name: _____ and email: _____

Let $f(x) = p - \ln(x^2)$. Suppose p is a positive constant.

1. Write out all the terms of the right-hand Riemann sum using 3 subdivisions of equal length which approximate the integral

$$\int_1^{13} f(x)dx$$

   Your answer should not include the letter f but may involve p.

2. Is the approximation of the previous part an overestimate or an underestimate? Circle one choice and explain your reasoning:
   Overestimate Underestimate Not enough information

Solution. $n = 3, b = 13, a = 1, \Delta t = (13 - 1)/3 = 4$, so we need to write a sum using the values at 5,9,13:

$$RS = 4(p - \ln(25) + p - \ln(81) + p - \ln(169))$$

Notice $f'(x) = \frac{-2x}{x^2} = \frac{-2}{x}$. In the interval [1,13], the derivative is negative, so the function is decreasing. Ergo, the right-sum is an underestimate.

$\square$

5.3 The fundamental Theorem and interpretations

When $f(x)$ is CTS, we have a way to compute integrals using the "antiderivative" of the function.

Theorem 5.1: The Fundamental Theorem of Calculus[2]

If f is continuous on the interval $[a, b]$ and $f(t) = F'(t)$, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Another way to see it: if $F(x)$ is CTS in [a,b] and differentiable on (a,b):

$$F(b) - F(a) = \int_a^b F'(x)dx$$

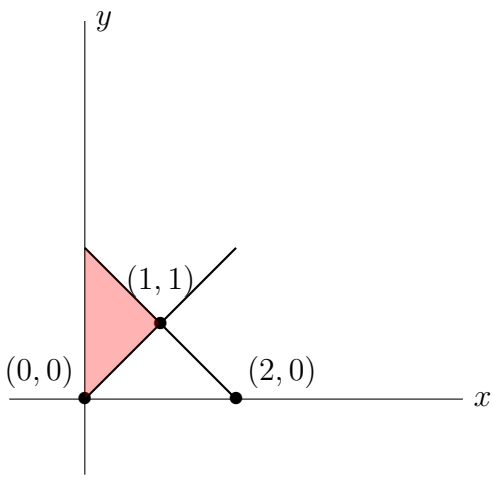
This gives us a way to compute limits of Riemann sums and areas under the graphs is an exact way.

5.4 Theorems about Integrals

Let $f(x), g(x)$ be CTS, a, b, c some numbers. Then:

- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$
- $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx.$
- The Area between $f(x)$ and $g(x)$ is $\int_a^b (f(x) - g(x))dx$

Example 5.4.1. Find the area between $f(x) = 2 - x$ and $g(x) = x$ as illustrated.



We need the area from 0 to 1 of $f(x) - g(x)$:

$$\int_0^1 2 - x - xdx = \int_0^1 2 - 2xdx$$

Now we need to estimate using a Riemann sum or wait for the next chapter or use basic geometry.

Using symmetry:

If f is even, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$ If g is odd, then $\displaystyle\int_{-a}^{a} g(x)\,dx = 0.$

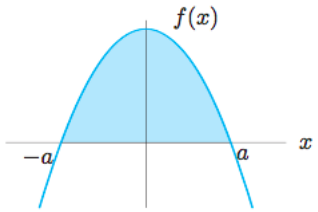


Figure 5.58: For an even function, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

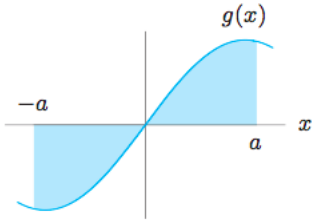


Figure 5.59: For an odd function, $\int_{-a}^{a} g(x)\,dx = 0$

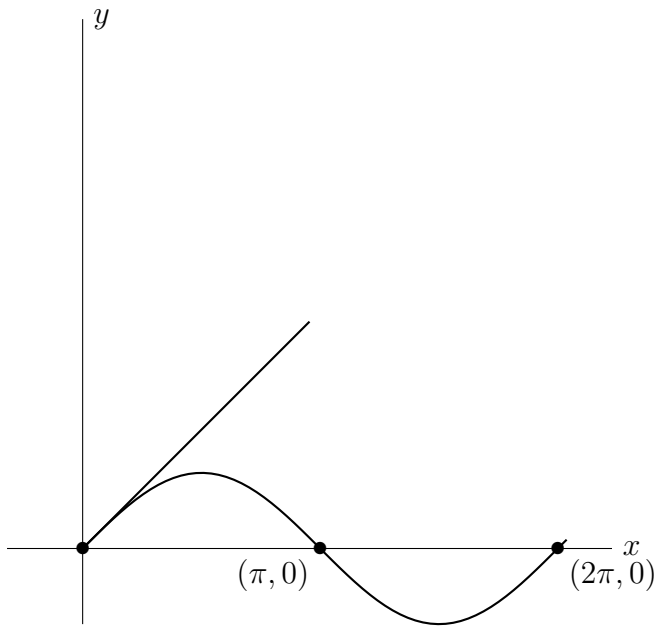
Comparison of integrals:

Theorem 5.4: Comparison of Definite Integrals

Let f and g be continuous functions.

1. If $m \le f(x) \le M$ for $a \le x \le b$, then $\displaystyle m(b-a) \le \int_{a}^{b} f(x)\,dx \le M(b-a).$
2. If $f(x) \le g(x)$ for $a \le x \le b$, then $\displaystyle\int_{a}^{b} f(x)\,dx \le \int_{a}^{b} g(x)\,dx.$

Exercise 5.4.2. an upper bound to the area of $\sin(x)$ in the following intervals: $[0, 2\pi], [0, \pi], [0, \pi/2]$. Use symmetry when possible. Here is a graph of $\sin(x)$ together with $g(x) = x$.



Solution. Due to symmetry, the area is exactly 0 on $[0, 2\pi]$.
On $[0, \pi/2]$ we can bound the area using the area of $g(x) = x$ from 0 to $\pi/2$. Using a formula of an area of a triangle we get a bound of:

$$\frac{1}{2}(\pi/2)^2 \approx 1.24$$

Due to symmetry, the area on $[0, \pi]$ is twice the area on $[0, \pi/2]$ so we get a bound of 1.48.
In the next chapter we shall see that the area on $[0, \pi/2]$ is 1 and on $[0, \pi]$ is 2.
BTW, if one bounds $\sin(x)$ using the max value 1, we get a bound on $[0, \pi/2]$ of $\pi/2 \approx 1.57$ which is not as good.

$\square$

6 Antiderivatives

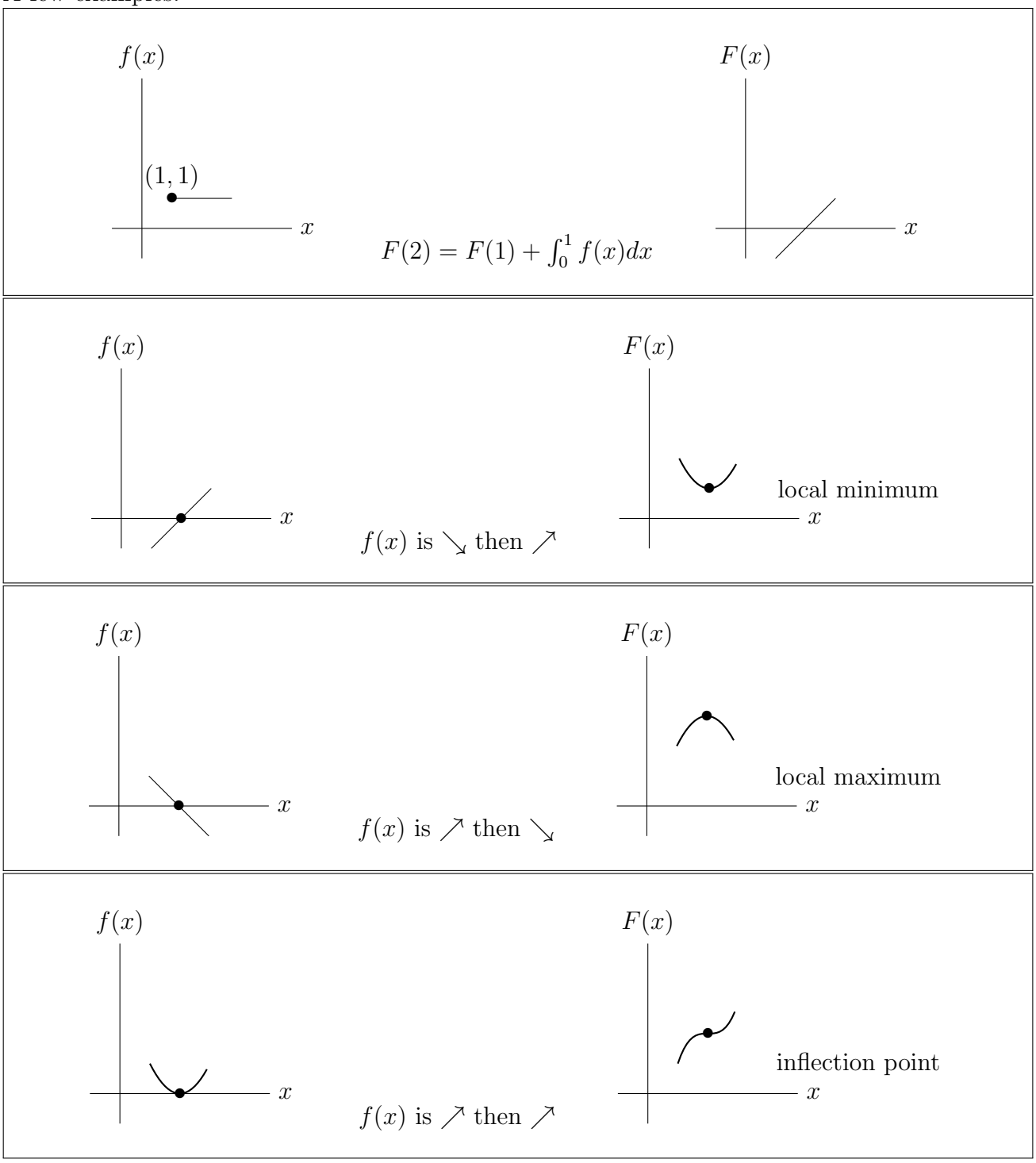
6.1 Graphically and Numerically

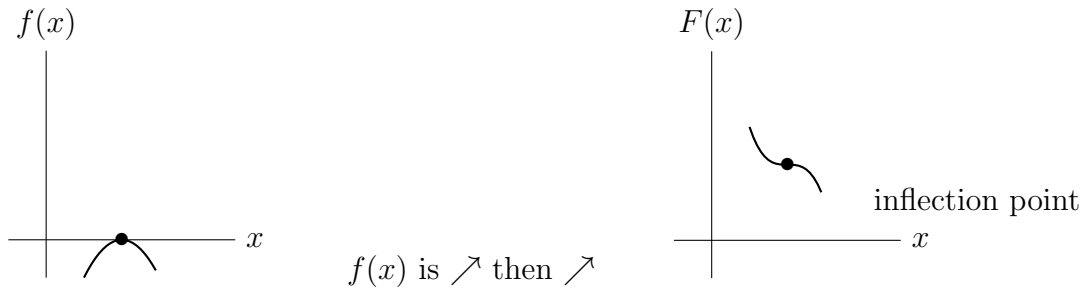
Remark 6.1.1. Let $f(x)$ be a funciton and $F(x)$ its antiderivative, $F'(x) = f(x)$. Then for any constant c, $F(x) + c$ is also an antiderivative for $f(x)$. The antiderivative is always set up to a constant.

Recall the relation:

$$F(b) = F(a) + \int_a^b f(x)dx$$

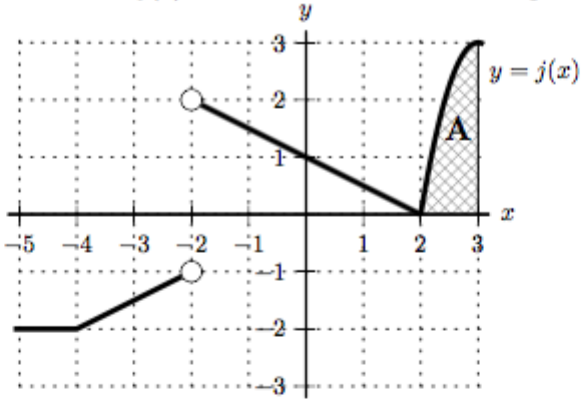
So if we know the value of $F(a)$, and the area under the graph of $f(x)$ between a and b, then we can compute $F(b)$.
A few examples:




$f(x)$
x
$f(x)$ is $\nearrow$ then $\nearrow$
$F(x)$
inflection point
x

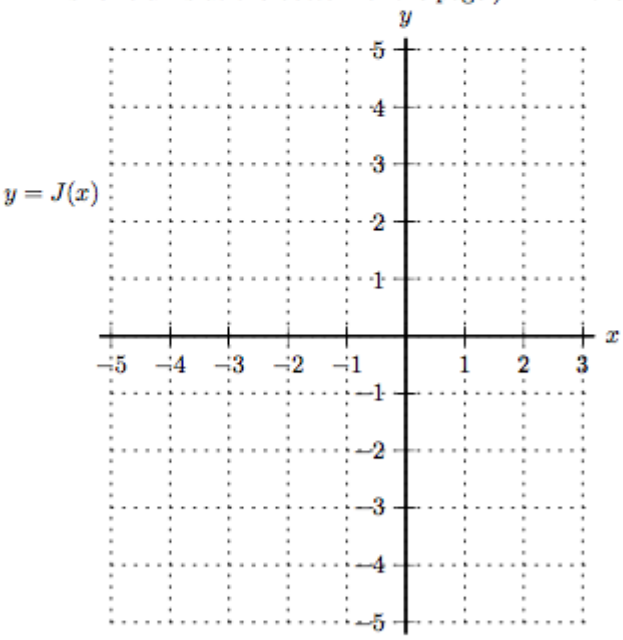
Exercise 6.1.2.

7. [10 points] The graph of a function $j(x)$ is shown below. The shaded region A has area 2.



On the axes provided below, sketch a well-labeled graph of an antiderivative of $J(x)$ of $j(x)$ that is defined and continuous on the interval $-5 \leq x \leq 3$ and that satisfies $J(0) = 1$.
Be sure that you pay close attention to each of the following:

- the value of $J(x)$ at each of its critical points and inflection points (Be sure to also write this data in the answer blanks at the bottom of the page.)
- where J is/is not differentiable
- where J is increasing/decreasing/constant
- the concavity of the graph of $y = J(x)$



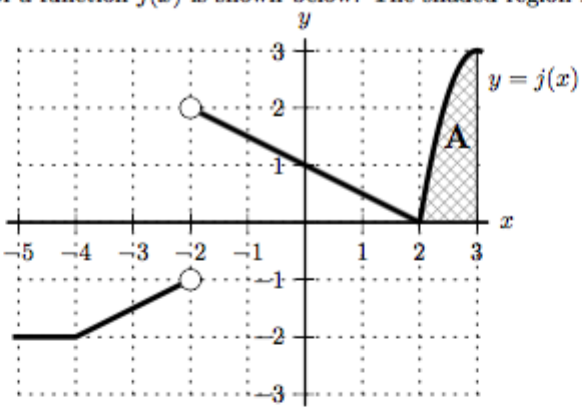
On the answer blanks below, write both the x– **and** y–coordinates of all critical points and all inflection points of $J(x)$. Write NONE if $J(x)$ has no such points.

Both coordinates of all critical points: _____

Both coordinates of all inflection points: _____

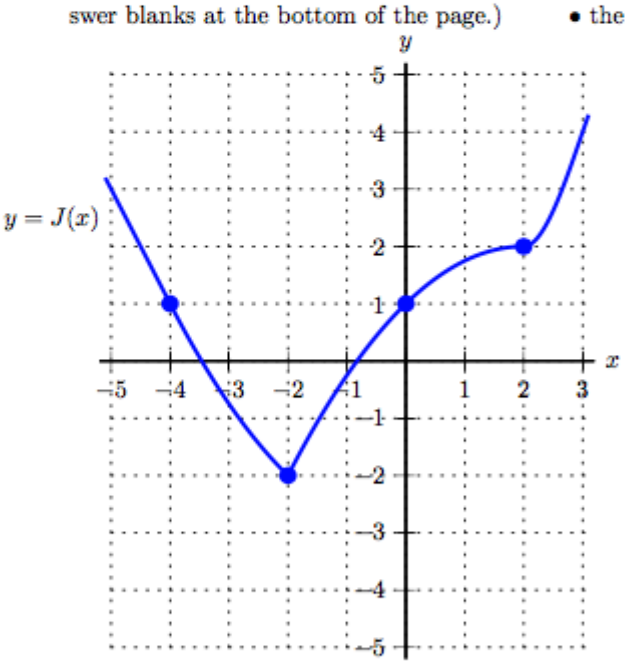
Solution.

7. [10 points] The graph of a function $j(x)$ is shown below. The shaded region A has area 2.



On the axes provided below, sketch a well-labeled graph of an antiderivative of $J(x)$ of $j(x)$ that is defined and continuous on the interval $-5 \leq x \leq 3$ and that satisfies $J(0) = 1$.
Be sure that you pay close attention to each of the following:

- the value of $J(x)$ at each of its critical points and inflection points (Be sure to also write this data in the answer blanks at the bottom of the page.)
- where J is/is not differentiable
- where J is increasing/decreasing/constant
- the concavity of the graph of $y = J(x)$



On the answer blanks below, write both the x– **and** y–coordinates of all critical points and all inflection points of $J(x)$. Write NONE if $J(x)$ has no such points.

Both coordinates of all critical points: $(-2, -2)$, $(2, 2)$

Both coordinates of all inflection points: $(-2, -2)$, $(2, 2)$

□

6.2 Analytically

- $\int_a^b 0dx = C|_a^b = C - C = 0$
- $\int_a^b kdx = kx + C|_a^b = kb - ka$
- $\int_a^b x^n dx = \frac{x^{n+1}}{n+1} + C\Big|_a^b = \frac{b^{n+1}-a^{n+1}}{n+1}$, for $n \neq -1$
- $\int_a^b x^{-1}dx = \ln|x| + C|_a^b = \ln|b| - \ln|a|$
- $\int_a^b e^x dx = e^x + C|_a^b = e^b - e^a$
- $\int_a^b \cos(x) = \sin(x) + C|_a^b = \sin(b) - \sin(a)$
- $\int_a^b \sin(x) = -\cos(x) + C|_a^b = \cos(a) - \cos(b)$
- $\int_a^b f + gdx = F(x) + G(x)|_a^b = F(b) + G(b) - F(a) - G(a)$
- $\int_a^b kfdx = kF(x)|_a^b\Big|_a^b = k(F(b) - F(a))$

Exercise 6.2.1.

1. [10 points] The table below gives several values of a function $f(x)$ and its derivative. Assume that both $f(x)$ and $f'(x)$ are defined and differentiable for all x.

x	0	1	2	3	4	5	6
$f(x)$	0	3	4	2	-1	-3	5
$f'(x)$	4	2	-1	-5	-2	7	9
$f''(x)$	-1	-3	-5	0	4	3	1

Compute each of the following. Do not give approximations. If it is not possible to find the value exactly, write NOT POSSIBLE

a. [2 points] Find $\int_0^4 f''(x)\,dx$.

Answer: $\int_0^4 f''(x)\,dx =$ _____

b. [2 points] Find $\int_2^5 (3f(x)+1)\,dx$.

Answer: $\int_2^5 (3f(x)+1)\,dx =$ _____

c. [3 points] Find the average value of $4f'(x)+x$ on the interval $[1,6]$.

Answer: _____

d. [3 points] Assuming that $f(x)$ is an odd function, find $\int_{-3}^3 f(x)\,dx$ and $\int_{-3}^3 f'(x)\,dx$.

Answer: $\int_{-3}^3 f(x)\,dx =$ _____ and $\int_{-3}^3 f'(x)\,dx =$ _____

Solution.

1. [11 points] The table below gives several values of a function $f(x)$ and its derivative. Assume that both $f(x)$ and $f'(x)$ are defined and differentiable for all x.

x	0	1	2	3	4	5	6
$f(x)$	0	3	4	2	-1	-3	5
$f'(x)$	4	2	-1	-5	-2	7	9
$f''(x)$	-1	-3	-5	0	4	3	1

Compute each of the following. Do not give approximations. If it is not possible to find the value exactly, write NOT POSSIBLE

a. [2 points] Find $\int_0^4 f''(x)\,dx$.

Solution: By the Fundamental Theorem of Calculus,

$$\int_0^4 f''(x)\,dx = f'(4) - f'(0) = -2 - 4 = -6.$$

Answer: $\int_0^4 f''(x)\,dx =$ -6

b. [2 points] Find $\int_2^5 (3f(x)+1)\,dx$.

Solution: In order to evaluate this exactly, we would need to know an antiderivative of $f(x)$. Since we don't know one, this is not possible to evaluate exactly.

Answer: $\int_2^5 (3f(x)+1)\,dx =$ NOT POSSIBLE

c. [3 points] Find the average value of $4f'(x)+x$ on the interval $[1,6]$.

Solution: The average value can be computed as an integral. Since an antiderivative of $4f'(x)+x$ is $4f(x)+\frac{1}{2}x^2$, we can compute the exact value of this integral with the Fundamental Theorem of Calculus:

$$\frac{1}{6-1}\int_1^6 \left(4f'(x)+x\right)\,dx = \frac{1}{5}\left(\left(4f(6)+\frac{1}{2}6^2\right) - \left(4f(1)+\frac{1}{2}1^2\right)\right) = 5.1$$

Answer: 5.1

d. [4 points] Assuming that $f(x)$ is an odd function, find $\int_{-3}^3 f(x)\,dx$ and $\int_{-3}^3 f'(x)\,dx$.

Solution: Note that

$$\int_{-3}^3 f(x)\,dx = \int_{-3}^0 f(x)\,dx + \int_0^3 f(x)\,dx,$$

and since $f(x)$ is an odd function, the two integrals on the right cancel out, leaving us with 0.
Also, since $f(x)$ is odd, we have $f(-3) = -f(3)$, and hence by the Fundamental Theorem of Calculus,

$$\int_{-3}^3 f'(x)\,dx = f(3) - f(-3) = f(3) - (-f(3)) = 2f(3) = 4.$$

Answer: $\int_{-3}^3 f(x)\,dx =$ 0 and $\int_{-3}^3 f'(x)\,dx =$ 4

□

Exercise 6.2.2.

8. [11 points] Public opinion has swung against the King since his arrest. Elphaba has been travelling the Sovereign lands collecting donations of acorns to help launch an attack against the King. Let $P(x)$ be the total mass (in kg) of acorns that Elphaba has collected after she has travelled a total of x km. Let $Q(t)$ be Elphaba's velocity (in km/day) when she has been travelling for t days. You may assume that $Q(t)$ is continuous and always positive and that $P(x)$ is an increasing, differentiable function.
For each of questions (a) through (d) below, circle the <u>one</u> best answer. No points will be given for ambiguous or multiple answers.

a. [2 points] Circle the one equation below that best supports the following statement:
When Elphaba has travelled 100 km, she has collected approximately 3 kg less acorns than she will have collected when she has travelled 100.5 km.

i. $P'(100) = 6$

ii. $P'(100) = -3$

iii. $P'(100) = 1.5$

iv. $P'(100.5) = -6$

v. $P'(100.5) = 3$

vi. $P'(100.5) = -1.5$

b. [2 points] Which one of the following expressions is equal to the amount (in kg) by which Elphaba's collection of acorns increases over the course of the 50th km of her travels.

i. $P(50)$

ii. $P'(49)$

iii. $\int_{49}^{50} P(t)\,dt$

iv. $\int_{49}^{50} P'(x)\,dx$

c. [2 points] Which one of the following expressions is equal to the mass (in kg) of acorns that Elphaba collected during the 4th day of her travels?

i. $P'(4)$

ii. $P\left(\int_0^4 Q(t)\,dt\right) - P\left(\int_0^3 Q(t)\,dt\right)$

iii. $P(4) - P(3)$

iv. $P\left(\int_3^4 Q(t)\,dt\right)$

d. [2 points] Let m be a positive constant and let $R(t)$ be the antiderivative of $Q(t)$ such that $R(0) = 0$. Assuming that both $P(t)$ and $R(t)$ are invertible, which one of the following expressions is equal to the time (in days) it takes Elphaba to collect m kg of acorns?

i. $R(P(m))$

ii. $R^{-1}(P^{-1}(m))$

iii. $R(P^{-1}(m))$

iv. $P(R(m))$

v. $P^{-1}(R^{-1}(m))$

vi. $P(R^{-1}(m))$

e. [3 points] Write an equation that expresses the following statement:
After Elphaba has been travelling for a total of 5 days, she has collected a total of 200 kg of acorns.

Answer: _____

Solution.

8. [11 points] Public opinion has swung against the King since his arrest. Elphaba has been travelling the Sovereign lands collecting donations of acorns to help launch an attack against the King. Let $P(x)$ be the total mass (in kg) of acorns that Elphaba has collected after she has travelled a total of x km. Let $Q(t)$ be Elphaba's velocity (in km/day) when she has been travelling for t days. You may assume that $Q(t)$ is continuous and always positive and that $P(x)$ is an increasing, differentiable function.
For each of questions (a) through (d) below, circle the one best answer. No points will be given for ambiguous or multiple answers.

a. [2 points] Circle the one equation below that best supports the following statement:
When Elphaba has travelled 100 km, she has collected approximately 3 kg less acorns than she will have collected when she has travelled 100.5 km.

i. $\boxed{P'(100) = 6}$

ii. $P'(100) = -3$

iii. $P'(100) = 1.5$

iv. $P'(100.5) = -6$

v. $P'(100.5) = 3$

vi. $P'(100.5) = -1.5$

b. [2 points] Which one of the following expressions is equal to the amount (in kg) by which Elphaba's collection of acorns increases over the course of the 50th km of her travels?

i. $P(50)$

ii. $P'(49)$

iii. $\int_{49}^{50} P(t)\,dt$

iv. $\boxed{\int_{49}^{50} P'(x)\,dx}$

c. [2 points] Which one of the following expressions is equal to the mass (in kg) of acorns that Elphaba collected during the 4th day of her travels?

i. $P'(4)$

ii. $\boxed{P\left(\int_0^4 Q(t)\,dt\right) - P\left(\int_0^3 Q(t)\,dt\right)}$

iii. $P(4) - P(3)$

iv. $P\left(\int_3^4 Q(t)\,dt\right)$

d. [2 points] Let m be a positive constant and let $R(t)$ be the antiderivative of $Q(t)$ such that $R(0) = 0$. Assuming that both $P(t)$ and $R(t)$ are invertible, which one of the following expressions is equal to the time (in days) it takes Elphaba to collect m kg of acorns?

i. $R(P(m))$

ii. $\boxed{R^{-1}(P^{-1}(m))}$

iii. $R(P^{-1}(m))$

iv. $P(R(m))$

v. $P^{-1}(R^{-1}(m))$

vi. $P(R^{-1}(m))$

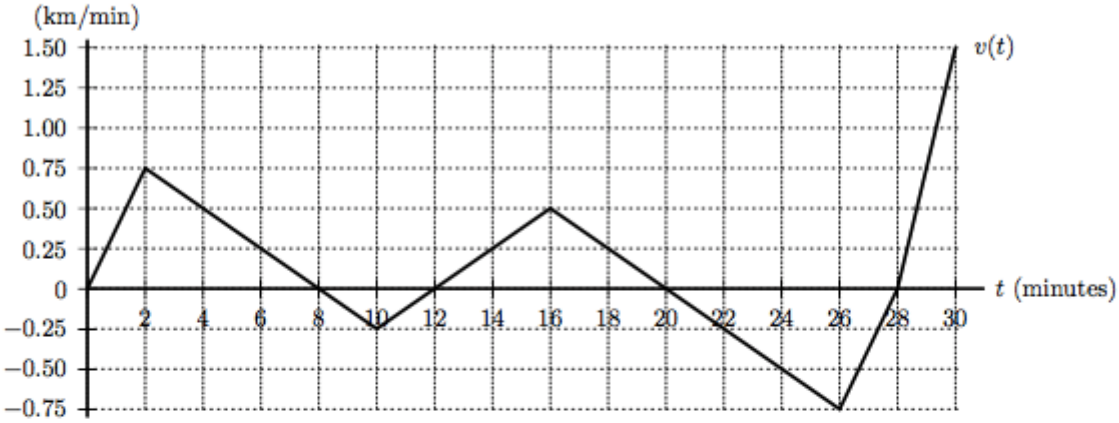
e. [3 points] Write an equation that expresses the following statement:
After Elphaba has been travelling for a total of 5 days, she has collected a total of 200 kg of acorns.

Answer: $P\left(\int_0^5 Q(t)\,dt\right) = 200$ **or** $\int_0^5 Q(t)\,dt = P^{-1}(200)$

□

Exercise 6.2.3.

1. [10 points] Unfortunately, Sebastian left the King's castle but never made it to Adam's manor because the brakes on his car were sabotaged. Sebastian was driving on a straight road between the King's castle and Adam's manor when he found himself unable to brake and racing down a hill. Let $v(t)$ be Sebastian's velocity (in kilometers per minute) t minutes after he left the King's castle. Note that $v(t)$ is positive when Sebastian is traveling towards Adam's manor. Sebastian suspected he was being followed so he occasionally backtracked.
Sebastian crashed 30 minutes into his journey. A graph of $v(t)$ is given below.



a. [3 points] How far from the King's castle was Sebastian 12 minutes into his journey? *Include units.*

Answer: _____

b. [2 points] What was Sebastian's average velocity during the first 12 minutes of his journey?

Answer: _____

c. [2 points] Of the four times below, circle the one at which Sebastian's acceleration was the greatest (i.e. most positive).

$t = 6$ $t = 13$ $t = 20$ $t = 27$

d. [3 points] In the interval $0 \leq t \leq 30$ when was Sebastian the closest to the King's castle? When was he the furthest from the King's castle?

Answer: Sebastian was the closest to the King's castle at $t =$ _____.

Sebastian was the furthest from the King's castle at $t =$ _____.

Solution.

1. [10 points] Unfortunately, Sebastian left the King's castle but never made it to Adam's manor because the brakes on his car were sabotaged. Sebastian was driving on a straight road between the King's castle and Adam's manor when he found himself unable to brake and racing down a hill. Let $v(t)$ be Sebastian's velocity (in kilometers per minute) t minutes after he left the King's castle. Note that $v(t)$ is positive when Sebastian is traveling towards Adam's manor. Sebastian suspected he was being followed so he occasionally backtracked.
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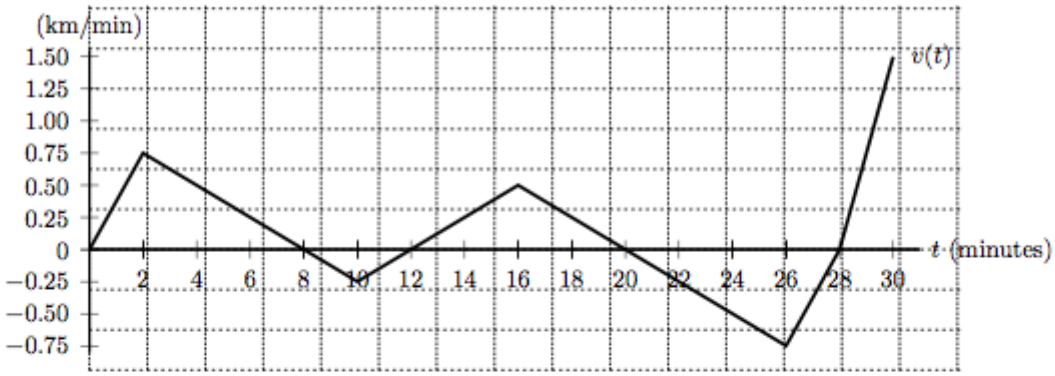


a. [3 points] How far from the King's castle was Sebastian 12 minutes into his journey? *Include units.*

Solution: Since Sebastian initially started at the King's castle, his distance from it after 12 minutes is given by $\int_0^{12} v(t)\,dt$. To calculate this we need to calculate the signed area between the graph of $v(t)$ and the t-axis. Therefore,

$$\int_0^{12} v(t)\,dt = (0.5)(8)(0.75) - (0.5)(4)(0.25) = 2.5 \text{ km}.$$

(Note that 0.5 is the area of each box in the graph.)

Answer: **2.5 km**

b. [2 points] What was Sebastian's average velocity during the first 12 minutes of his journey?

Solution: Sebastian's average velocity during the first 12 minutes is given by the equation

$$frac112 \int_0^{12} v(t)\,dt = \frac{2.5}{12}.$$

Answer: $\frac{2.5}{12}$ **km/min**

c. [2 points] Of the four times below, circle the one at which Sebastian's acceleration was the greatest (i.e. most positive).

$t = 6$ $\quad$ $t = 13$ $\quad$ $t = 20$ $\quad$ $\boxed{t = 27}$

d. [3 points] In the interval $0 \le t \le 30$ when was Sebastian the closest to the King's castle? When was he the furthest from the King's castle?

Answer: Sebastian was the closest to the King's castle at $t =$ **0**.

Sebastian was the furthest from the King's castle at $t =$ **20**.

□

Exercise 6.2.4.

9. [8 points]
Suppose that the standard price of a round-trip plane ticket from Detroit to Paris, purchased t days after April 30, is $P(t)$ dollars. Assume that P is an invertible function (even though this is not always the case in real life).

In the context of this problem, give a practical interpretation for each of the following:

a. [2 points] $P'(2) = 55$

b. [2 points] $P^{-1}(690)$

c. [2 points] $\int_5^{10} P'(t)dt$

d. [2 points] $\frac{1}{5}\int_5^{10} P(t)dt$

Solution.

9. [8 points]
Suppose that the standard price of a round-trip plane ticket from Detroit to Paris, purchased t days after April 30, is $P(t)$ dollars. Assume that P is an invertible function (even though this is not always the case in real life).

In the context of this problem, give a practical interpretation for each of the following:

a. [2 points] $P'(2) = 55$

Solution: The standard price of a round-trip ticket from Detroit to Paris is approximately \$55 more if the ticket is purchased on May 3 than if it is purchased on May 2.

b. [2 points] $P^{-1}(690)$

Solution: The standard price of a round-trip ticket from Detroit to Paris is \$690 if it is purchased $P^{-1}(690)$ days after April 30.

c. [2 points] $\int_5^{10} P'(t)dt$

Solution: The standard price of a round-trip ticket from Detroit to Paris changes by $\int_5^{10} P'(t)dt$ dollars between May 5 and May 10. (If the integral is positive, it will be a price increase. If the integral is negative, it will be a price decrease.)

d. [2 points] $\frac{1}{5}\int_5^{10} P(t)dt$

Solution: This is the average standard price (in dollars) of a round-trip ticket from Detroit to Paris purchased between May 5 and May 10.

□

Exercise 6.2.5.

8. [12 points] Below is a table of values for the function $t(y)$ which gives the number of tweets per day, in millions, on the social media website Twitter, y years after January 1, 2007. For this problem assume $t(y)$ is an increasing function.

year y	0	1	2	3	4
millions of tweets per day $t(y)$	0.005	0.3	2.5	35	50

a. [4 points] Using the table, estimate the expression

$$365 \int_1^4 t(y)dy$$

using a left-hand Riemann sum. Please write all of the terms in the sum for full credit.

b. [4 points] Give a practical interpretation of the expression $365 \int_1^4 t(y)dy$.

c. [4 points] Suppose $T(y)$ is the total number of tweets, in millions, y years after January 1, 2007. If $T(3) = 9797$, estimate the total number of tweets between January 1, 2007 and January 1, 2011. Indicate what method you use to obtain your estimate and be sure to show your work.

Solution.

8. [12 points] Below is a table of values for the function $t(y)$ which gives the number of tweets per day, in millions, on the social media website Twitter, y years after January 1, 2007. For this problem assume $t(y)$ is an increasing function.

year y	0	1	2	3	4
millions of tweets per day $t(y)$	0.005	0.3	2.5	35	50

a. [4 points] Using the table, estimate the expression

$$365 \int_1^4 t(y)dy$$

using a left-hand Riemann sum. Please write all of the terms in the sum for full credit.

Solution: To estimate this integral, we will use a left hand sum with 3 subdivisions. Since $n = 3$, $\Delta y = (4-1)/3 = 1$. Therefore, the left hand sum is

$$365 \int_1^4 t(y)dy \approx 365(t(1) \cdot 1 + t(2) \cdot 1 + t(3) \cdot 1) = 365(37.8) = 13,797 \text{ million tweets}$$

b. [4 points] Give a practical interpretation of the expression $365 \int_1^4 t(y)dy$.

Solution: Since $t(y)$ gives tweets per day when you input a year, the units on the definite integral are (millions of tweets per day)(year). When we multiply by 365 days per year, we have the units of $365 \int_1^4 t(y)dy$ are millions of tweets. So the definite integral represents the total number tweets in millions that appeared on Twitter between January 1, 2008 and January 1, 2011.

c. [4 points] Suppose $T(y)$ is the total number of tweets, in millions, y years after January 1, 2007. If $T(3) = 9797$, estimate the total number of tweets between January 1, 2007 and January 1, 2011. Indicate what method you use to obtain your estimate and be sure to show your work.

Solution: By the fundamental theorem of calculus, we know that

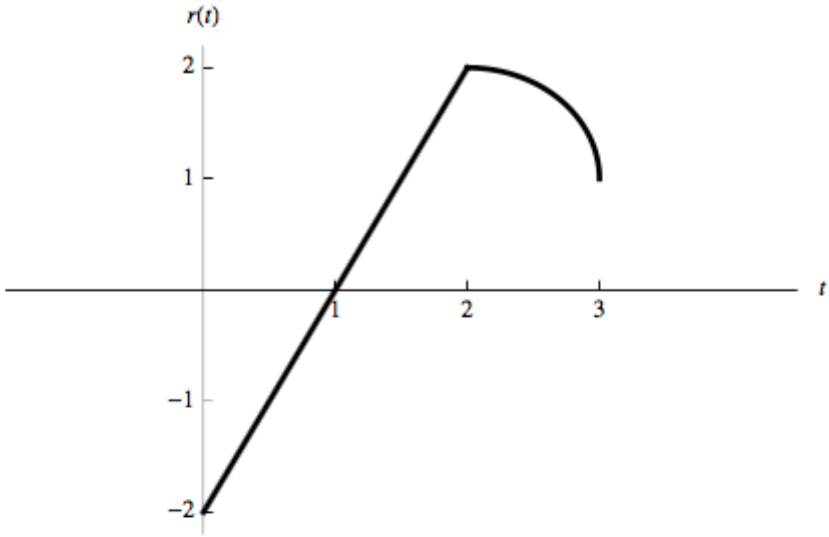
$$T(y) - T(3) = 365 \int_3^y t(w)dw.$$

In order to find $T(4)$, we need to compute the definite integral $\int_3^4 t(w)dw$. From the table above, we can use a left hand sum to compute $365 \int_3^4 t(w)dw = 365(35 \cdot 1) = 12775$. From our formula, we get that $T(4) = 12775 + T(3) = 22572$ million tweets. If we use a right hand sum to compute the value of the integral, we get $365 \int_3^4 t(w)dw = 365(50 \cdot 1) = 18250$ for a total of $T(4) = 28047$ million tweets.

□

Exercise 6.2.6.

3. [12 points] Shown below is a graph of a function $r(t)$. The graph consists of a straight line between $t = 0$ and $t = 2$ and a quarter circle between $t = 2$ and $t = 3$.



Calculate the following using the graph and the properties of integrals.

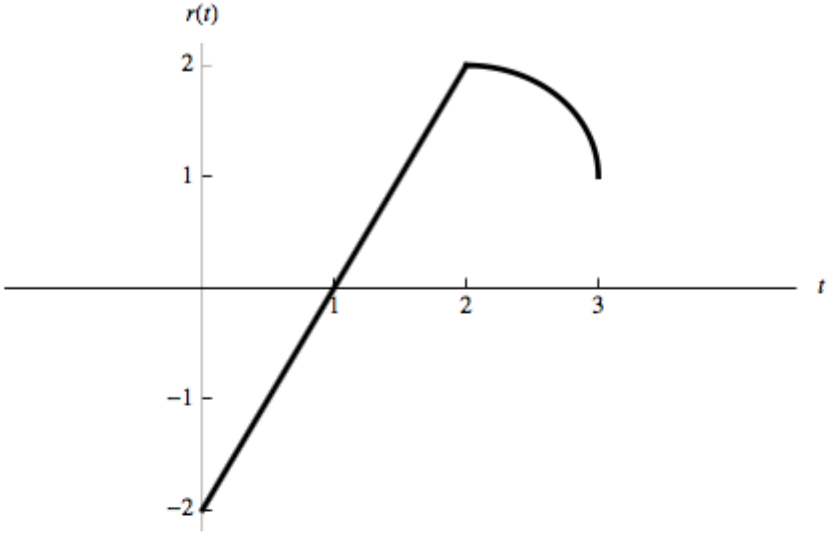
a. [4 points] $-3\int_0^3 (2 + r(t))dt$.

b. [4 points] $\int_{1/2}^{3/2} r'(t)dt$.

c. [4 points] The average value of r on the interval $[1, 3]$.

Solution.

3. [12 points] Shown below is a graph of a function $r(t)$. The graph consists of a straight line between $t = 0$ and $t = 2$ and a quarter circle between $t = 2$ and $t = 3$.



Calculate the following using the graph and the properties of integrals.

a. [4 points] $-3\int_0^3 (2 + r(t))dt$.

Solution: We compute

$$-3\int_0^3 (2 + r(t))dt = -6\int_0^3 1dt \; - \; 3\int_0^3 r(t)dt = -18 - 3(1 + \pi/4) = -21 - 3\pi/4.$$

b. [4 points] $\int_{1/2}^{3/2} r'(t)dt$.

Solution: By the fundamental theorem of calculus,

$$\int_{1/2}^{3/2} r'(t)dt = r(3/2) - r(1/2) = 1 - (-1) = 2.$$

c. [4 points] The average value of r on the interval $[1, 3]$.

Solution: The average value of r on the interval $[1, 3]$ is

$$\frac{1}{3-1}\int_1^3 r(t)dt = \frac{1}{2}(1 + 1 + \pi/4) = 1 + \pi/8.$$

□

Exercise 6.2.7.

6. [10 points] The table below gives the expected growth rate, $g(t)$, in ounces per week, of the weight of a baby in its first 54 weeks of life (which is slightly more than a year).[1] Assume for this problem that $g(t)$ is a decreasing function.

week t	0	9	18	27	36	45	54
growth rate $g(t)$	6	6	4.5	3	3	3	2

a. [6 points] Using six subdivisions, find an overestimate and underestimate for the total weight gained by a baby over its first 54 weeks of life.

b. [4 points] How frequently over the 54 week period would you need the data for $g(t)$ to be measured to find overestimates and underestimates for the total weight gain over this time period that differ by 0.5 lb (8 oz)?

Solution.

6. [10 points] The table below gives the expected growth rate, $g(t)$, in ounces per week, of the weight of a baby in its first 54 weeks of life (which is slightly more than a year).[1] Assume for this problem that $g(t)$ is a decreasing function.

week t	0	9	18	27	36	45	54
growth rate $g(t)$	6	6	4.5	3	3	3	2

a. [6 points] Using six subdivisions, find an overestimate and underestimate for the total weight gained by a baby over its first 54 weeks of life.

Solution: The gain rate is a decreasing function, so a left-hand sum will be an overestimate and a right-hand sum an underestimate. The left-hand sum is

$$\int_0^{54} g(t)\,dt \approx (6+6+4.5+3+3+3)(9) = 229.5 \quad \text{oz},$$

and the right-hand sum

$$\int_0^{54} g(t)\,dt \approx (6+4.5+3+3+3+2)(9) = 193.5 \quad \text{oz}.$$

That is, we expect the weight gain to be between 12 and 14 lb!

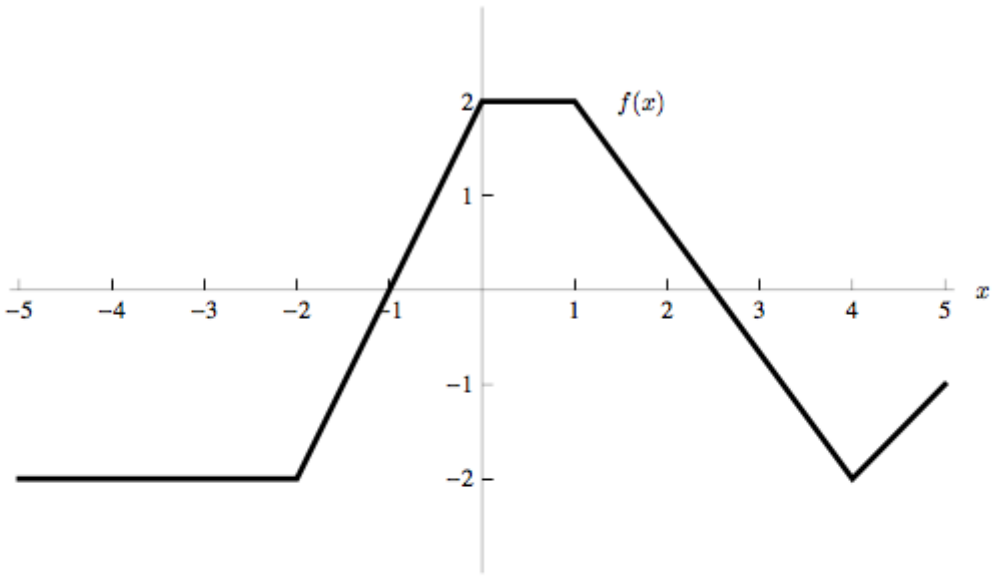
b. [4 points] How frequently over the 54 week period would you need the data for $g(t)$ to be measured to find overestimates and underestimates for the total weight gain over this time period that differ by 0.5 lb (8 oz)?

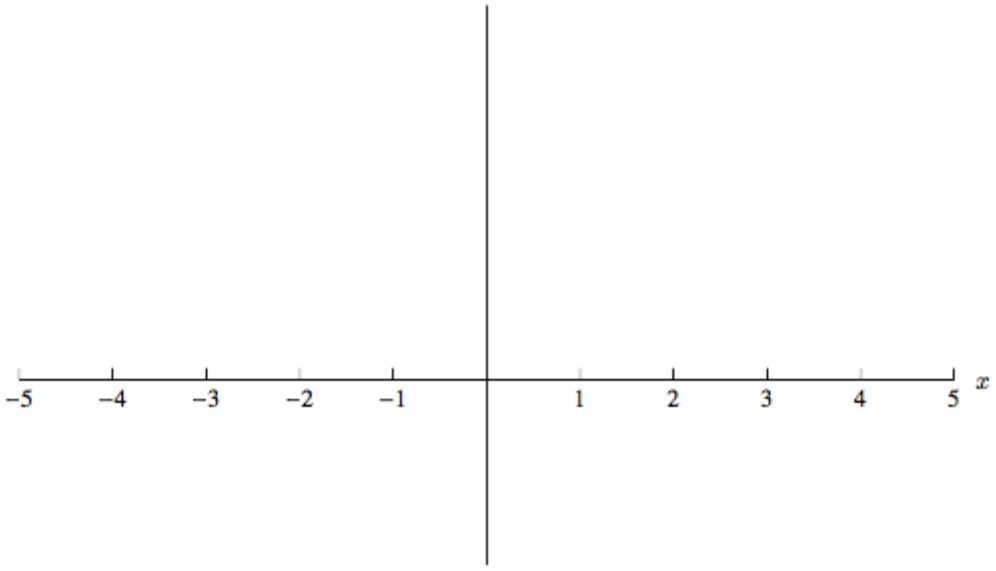
Solution: We know that the difference between the over- and underestimates is over − under $= |g(54) - g(0)|\Delta t$. Thus we need $\Delta t \leq 8/(6-2) = 2$ weeks. So we would need data for $g(t)$ every two weeks.

□

Exercise 6.2.8.

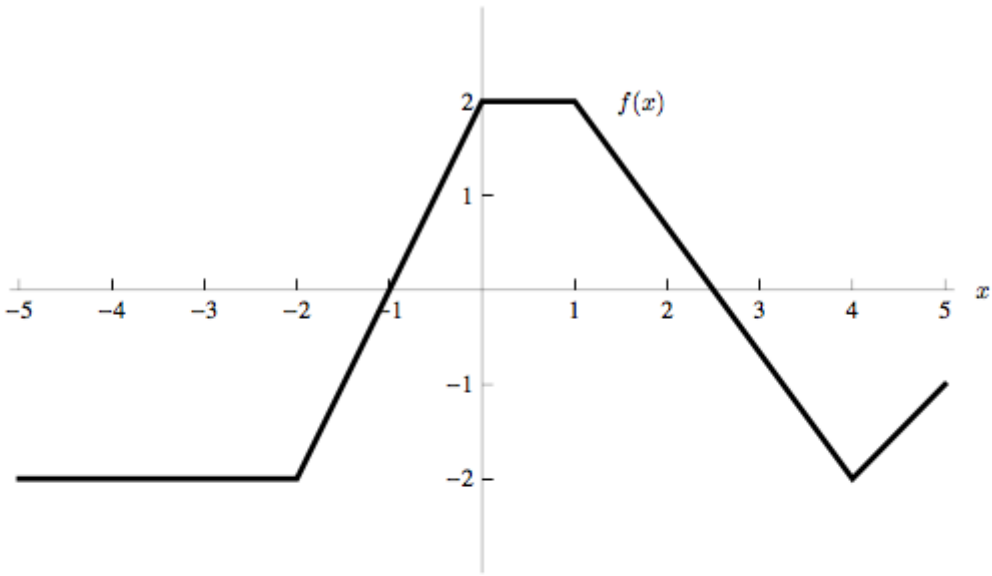
5. [10 points] The graph of a piecewise linear function $f(x)$ is shown below. On the axes provided, sketch a well-labeled graph of an antiderivative $F(x)$ of $f(x)$ satisfying $F(0) = -1$. Be sure to make the concavity of F clear and to label the y-coordinates of the local minima and maxima of F and the y-coordinates of F at $x = 5$ and $x = -5$.

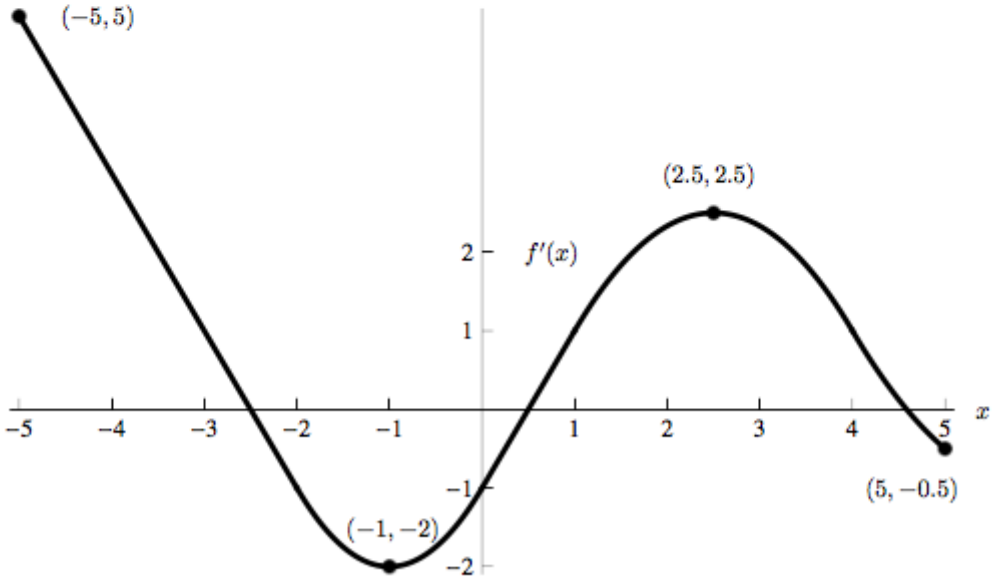




Solution.

5. [10 points] The graph of a piecewise linear function $f(x)$ is shown below. On the axes provided, sketch a well-labeled graph of an antiderivative $F(x)$ of $f(x)$ satisfying $F(0) = -1$. Be sure to make the concavity of F clear and to label the y-coordinates of the local minima and maxima of F and the y-coordinates of F at $x = 5$ and $x = -5$.





□

Exercise 6.2.9.

5. [5 points] After hearing of the *Illumisqati* activities from Erin and Elphaba, the Police storm the King's farmhouse and find ample evidence to convict him of kidnapping. However, since he is the King, charges can only be brought against him if the Police can show proficiency in mathematics. Help them by doing the following problem.

For c a constant, consider the function

$$B(u) = \arctan(u^c + 7).$$

Use the limit definition of the derivative to write an explicit expression for $B'(3)$.

Your answer should not involve the letter B. Do not attempt to evaluate or simplify the limit.

Answer: $B'(3) =$

6. [6 points] Recall the following definitions:

- A function f is *even* if $f(-x) = f(x)$ for all x in the domain of f.
- A function f is *odd* if $f(-x) = -f(x)$ for all x in the domain of f.

Compute each of the integrals below. If not enough information is provided to answer the question, write NOT ENOUGH INFORMATION.

a. [2 points] Suppose g is a differentiable function on $(-\infty, \infty)$ and g' (the **derivative** of g) is a continuous odd function with $g(3) = 2$ and $g(7) = 9$. Find $\int_{-3}^{7} g'(x)\,dx$.

Answer: $\int_{-3}^{7} g'(x)\,dx =$ _____

b. [2 points] Suppose that q is a continuous and even function on $(-\infty, \infty)$ and that $\int_{0}^{5} q(x)\,dx = -4$. Find $\int_{-5}^{5} (3q(x) + 7)\,dx$.

Answer: $\int_{-5}^{5} (3q(x) + 7)\,dx =$ _____

c. [2 points] Let $h(x) = \ln x$ and suppose p is a differentiable function on $(-\infty, \infty)$ with $p(4) = 7$. Find $\int_{4}^{1} (h(x)p'(x) + h'(x)p(x))\,dx$.

Answer: $\int_{4}^{1} (h(x)p'(x) + h'(x)p(x))\,dx =$ _____

Solution.

5. [5 points] After hearing of the *Illumisqati* activities from Erin and Elphaba, the Police storm the King's farmhouse and find ample evidence to convict him of kidnapping. However, since he is the King, charges can only be brought against him if the Police can show proficiency in mathematics. Help them by doing the following problem.

For c a constant, consider the function $B(u) = \arctan(u^c + 7)$.

Use the limit definition of the derivative to write an explicit expression for $B'(3)$.

Your answer should not involve the letter B. Do not attempt to evaluate or simplify the limit.

Answer: $B'(3) = \boxed{\lim_{h\to 0} \frac{\arctan((3+h)^c + 7) - \arctan(3^c + 7)}{h}}$

6. [6 points] Recall the following definitions:

- A function f is *even* if $f(-x) = f(x)$ for all x in the domain of f.
- A function f is *odd* if $f(-x) = -f(x)$ for all x in the domain of f.

Compute each of the integrals below. If not enough information is provided to answer the question, write NOT ENOUGH INFORMATION.

a. [2 points] Suppose g is a differentiable function on $(-\infty, \infty)$ and g' (the **derivative** of g) is a continuous odd function with $g(3) = 2$ and $g(7) = 9$. Find $\int_{-3}^{7} g'(x)\,dx$.

Solution: Since $g'(x)$ is odd, $\int_{-3}^{3} g'(x)\,dx = 0$ so we have

$$\int_{-3}^{7} g'(x)\,dx = \int_{3}^{7} g'(x)\,dx = g(7) - g(3) = 9 - 2 = 7.$$

Answer: $\int_{-3}^{7} g'(x)\,dx =$ 7

b. [2 points] Suppose that q is a continuous and even function on $(-\infty, \infty)$ and that $\int_{0}^{5} q(x)\,dx = -4$. Find $\int_{-5}^{5} (3q(x) + 7)\,dx$.

Solution: By the linearity properties of definite integrals,

$$\int_{-5}^{5} (3q(x) + 7)\,dx = 3\left(\int_{-5}^{5} q(x)\,dx\right) + \int_{-5}^{5} 7\,dx = 3\left(\int_{-5}^{5} q(x)\,dx\right) + 70.$$

Since $q(x)$ is even, $\int_{-5}^{5} q(x)\,dx = 2\int_{0}^{5} q(x)\,dx = -8$.

Therefore, $\int_{-5}^{5} (3q(x) + 7)\,dx = 3(-8) + 70 = 46$.

Answer: $\int_{-5}^{5} (3q(x) + 7)\,dx =$ 46

c. [2 points] Let $h(x) = \ln x$ and suppose p is a differentiable function on $(-\infty, \infty)$ with $p(4) = 7$. Find $\int_{4}^{1} (h(x)p'(x) + h'(x)p(x))\,dx$.

Solution: Note that $h(x)p(x)$ is an antiderivative of $h(x)p'(x) + h'(x)p(x)$ so by the Fundamental Theorem of Calculus,

$$\int_{4}^{1} (h(x)p'(x) + h'(x)p(x))\,dx = h(1)p(1) - h(4)p(4) = \ln(1)p(1) - \ln(4)p(4) = -7\ln(4).$$

Answer: $\int_{4}^{1} (h(x)p'(x) + h'(x)p(x))\,dx =$ $-7\ln(4)$

□