Math 115 — Practice for Exam 3

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Name: SOLUTIONS

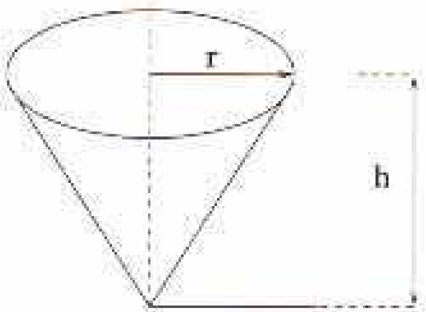
Instructor: ______________________________ Section Number: ____________

1. This exam has 3 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2002	2	12	cone	9	
Winter 2006	3	7	Mackinac Bridge	12	
Winter 2009	2	6	spherical balloon	6	
Total				27	

Recommended time (based on points): 28 minutes

12. (9 points) Fluid flows out of the bottom of a cone-shaped vessel at the rate of 3 cubic cm per second (see figure below). If the radius of the cone is one-third of its height, how fast is the height of the fluid changing when the fluid is 6 cm deep in the center of the cone. Be sure to show your work and give the correct units in your answer. (Remember that the volume of a cone is $\frac{1}{3}\pi r^2 h$.)



Given:

$$V = \frac{1}{3}\pi r^2 h$$

and $r = \frac{1}{3} h$

$$\frac{dV}{dt} = -3 \frac{cm^3}{sec}$$

We can write

$$V(h) = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 \cdot h = \frac{\pi}{3}\left(\frac{h^3}{9}\right)$$

So

$$\frac{dV}{dt} = \frac{1}{3}\pi \left(\frac{1}{3} h^2\right)\frac{dh}{dt} = \frac{\pi}{9} h^2 \frac{dh}{dt}$$

Want

$\frac{dh}{dt}$ when $h = 6$ cm, using $\frac{dV}{dt} = -3$

$$-3 = \frac{1}{9}\pi (36) \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{-3}{4\pi} \text{ cm/sec}$$

7. (12 points) You are heading due North across the Mackinac Bridge, and you sight your favorite fudge factory, located 1/4 mile due East of the end of the bridge. You might be fooling yourself, but the minute you sight the factory you are sure you can smell the chocolate fudge. You have to put the car on cruise control (at 55 mph) to resist the temptation to speed the rest of the way across the bridge. Show all work on parts (a) and (b) below.

   (a) If you are still 1.25 miles from the end of the bridge when you spot the factory, what is the distance (across the water) between your car and the factory?

   - $D = \sqrt{(1.25)^2 + (0.25)^2} \simeq 1.275$ miles.

   So, the distance across the water between my car and the factory is about 1.275 miles.

   (b) If θ is the angle formed by a line between the factory and end of the bridge and the line from the factory to your car, how fast is θ changing at the time you spot the factory?

   - $\tan\theta = \dfrac{x}{1/4} = 4x$, where
   - x is the distance between your car and the end of the bridge, and $\dfrac{dx}{dt} = -55$ mph, so
   - $\theta = \arctan(4x)$.

   Therefore,

   - $\dfrac{d\theta}{dt} = \dfrac{4}{1+(4x)^2}\dfrac{dx}{dt} = \dfrac{4(-55)}{1+16x^2}$, so
   - when $x = 1.25$, we have $\dfrac{d\theta}{dt} = \dfrac{-220}{26} = -8.46$ radians/hr.

   Thus, the angle θ is decreasing by about 8.46 radians per hour when you first spot the fudge factory!

5. Your friend starts a small company which sells awesome t-shirts for \$10 apiece. The table below shows the cost of making different numbers of shirts:

q (number of shirts made)	5	10	15	20	25	30	35	40	45	50
$C(q)$ (cost, in \$)	100	130	150	168	184	196	206	218	236	256

(a) (2 points) Write an expression for the revenue function $R(q)$.

$R(q) = 10q$, measured in dollars.

(b) (4 points) How many shirts should your friend aim to sell, if her goal is to maximize profit? Explain.

The profit function $\pi(q) = R(q) - C(q)$. The critical points of this function are those q which make $\pi'(q) = 0$, as well as the endpoints. We check these.

From above, we know that $\pi'(q) = R'(q) - C'(q) = 10 - C'(q)$. In the table above, the largest difference between consecutive values of $C(q)$ is 30 (which is $C(10) - C(5)$), which means the largest value $C'(q)$ takes on the interval is 6. Therefore, as far as we can glean from the information given in the table, we should expect that $\pi'(q)$ is positive everywhere in the interval $0 \leq q \leq 50$. So, the maximum should be at one of the endpoints.

When $q = 5$ (i.e. 5 shirts are sold), your friend loses money, since $\pi(5) = -50$. At the other extreme, if your friend sells 50 awesome t-shirts, she will make a tidy profit of $\pi(50) = 244$ dollars. Thus, she should aim to sell 50 shirts (and probably more, if that's a possibility).

6. (6 points) The radius of a spherical balloon is increasing by 3 cm per second. At what rate is air being blown into the balloon at the moment when the radius is 9 cm? Make sure you include units! [*Hint: the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.*]

If $V(t)$ denotes the volume of the balloon at some time t, the rate at which air is being blown into it is $\frac{dV}{dt}$. By chain rule, since the volume of the balloon is $V = \frac{4}{3}\pi r^3$, we have

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dr} \cdot \frac{dr}{dt} \\ &= (4\pi r^2) \cdot \frac{dr}{dt} \end{aligned}$$

We are given in the problem statement that $\frac{dr}{dt} = 3$, whence

$$\frac{dV}{dt} = 12\pi r^2.$$

Thus when $r = 9$,

$$\frac{dV}{dt} = 972\pi \approx 3053.6 \text{ cm}^3/\text{s}.$$