The Arithmetic Geometry of Resonant Rossby Wave Triads

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Part One: Rossby Waves and Resonance
Atmospheric **Rossby waves** on Earth are large-scale meanders in high-altitude winds. They are a major influence on the weather.
Rossby Waves

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- with periodic boundary conditions.
- The same PDE and boundary conditions describe drift waves in plasma in a tokamak.
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- Example from quantum mechanics: Bohr’s hydrogen atom. Integral number of modes for the electron.
- On a circle, wavenumber $= 2\pi/(\text{frequency})$ is an integer. On a torus, wavenumbers are a vector in $\mathbb{Z}^2$.
- In the $\beta$-plane model of Rossby waves, there is a zonal (east/west) wavenumber $a$ and a meridional (north/south) wavenumber $b$. 
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We [K] give a new algebraic classification and an even quicker algorithm.
Part Two: Application of Number Theory
For a wavenumber vector \((a, b) \in \mathbb{Z}^2\), the angular frequency is given by the dispersion relation

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A resonant triad consists of wavenumbers \((a, b), (x - a, y - b)\), and \((x, y)\), satisfying the equation

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\frac{a}{a^2 + b^2} + \frac{x - a}{(x - a)^2 + (y - b)^2} = \frac{x}{x^2 + y^2}
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Resonance Equation

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The resonance equation defines a degree five surface $X \subset \mathbb{P}^3$.

Call $C(a, b)$ the curve defined by the resonance equation for fixed $(a, b) \in \mathbb{Z}$. This curve is defined over $\mathbb{Z}$, and $C(a, b)_\mathbb{Q}$ is the fiber of the map

$$X \rightarrow \mathbb{P}^1$$

$$[a : b : x : y] \mapsto [a : b].$$

Resonance Equation

$$x(a^2 + b^2)(a^2 + b^2 - 2ax - 2by) = a(x^2 + y^2)(x^2 + y^2 - 2ax - 2by)$$
Theorem

X is birational to $\mathbb{P}^2$ over $\mathbb{Q}$.

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Theorem

For $(a, b) \in \mathbb{Z}$, $C(a, b)$ is birational to an elliptic curve with torsion $\mathbb{Z}/2\mathbb{Z}$ and rank $\geq 1$, except when $a = 0$ or $b = 0$.

As a corollary, for any $(a, b)$ with $ab \neq 0$, we can find $C(an, bn)$ with arbitrarily many integer points.
Birational Equivalence of $X$ with $\mathbb{P}^2$

A rational parametrization $\mathbb{P}^2 \rightarrow X$, $[s : t : u] \mapsto [a : b : x : y]$

\[
\begin{bmatrix}
a \\ \. \\ b \\ \. \\ x \\ \. \\ y
\end{bmatrix} =
\begin{bmatrix}
s^3 t(s - 2u) \\ \. \\ s(-s^2u(s - 2u) + (t^2 + u^2)(t^2 - 2su + u^2)) \\ \. \\ t(t^2 + u^2)(t^2 - 2su + u^2) \\ \. \\ (t^2 + u^2)(-s^2(s - 2u) + u(t^2 - 2su + u^2))
\end{bmatrix}
\]

Its rational inverse $X \rightarrow \mathbb{P}^2$, $[a : b : x : y] \mapsto [s : t : u]$.

\[
\begin{bmatrix}
s \\ \. \\ t \\ \. \\ u
\end{bmatrix} =
\begin{bmatrix}
a^2 + b^2 \\ \. \\ bx - ay \\ \. \\ ax + by
\end{bmatrix}
\]
The real points of $C(a, b)$ form a smooth closed loop, so there are finitely many integer points.
A Fibration by Elliptic Curves

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- $C(a, b)$ is genus one, unless $[a : b] = [0 : 1], [\pm i : 1], [\pm 2i : 1]$, in which cases it is genus zero.
- $C(a, b)$ has two (non-real) singularities. We normalize the curve to obtain a smooth model, and convert to Weierstrass form.

Weierstrass form of smooth model of $C(a, b)$

$$W^2 = Z^3 + (a^2 - 2b^2)Z^2 + (a^2 + b^2)^2 Z.$$
There are four “trivial” integer points on $C(a, b)$, giving rise to zonal resonances and unstable single-wave “resonances.”

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- There are four “trivial” integer points on $C(a, b)$, giving rise to zonal resonances and unstable single-wave “resonances.”
- The point $(0, 0)$ is taken to the identity of this elliptic curve. $T = (a, b)$ has order 2, $P = (0, 2b)$ has infinite order, and $(a, -b) = P + T$.

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Computational Power

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- Plug triples of integers into our parametrization and clear common factors to enumerate all triads.
- While we don’t yet have good bounds on how long it will take to enumerate all triads up to a given wavenumber bound, in practice we have enumerated all $463 \times 24$ triads—more than found in the literature—up to wavenumber 5000 in 80 minutes on a MacBook Pro.
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We’ve also enumerated the first few resonant wavevectors with zonal group velocity zero: $(n, \pm n)$ for integers $n$ in the sequence 13, 229, 3277, 504613, 155870857, 34589637433, 58803854910601, . . . . (This computation took a trivial amount of time.)
Thank you to the organizers!

Definition of the Wavenumber Set

Wavenumber Set

\[ \Lambda := \{(a, b) \in \mathbb{Z}^2 : C(a, b) \text{ has a non-trivial integer point}\} . \]
\[ \Lambda_{\text{new}} := \{(a, b) \in \mathbb{Z}^2 : C(a, b) \text{ has a new non-trivial integer point}\} . \]
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- non-trivial \((x, y) \in C(a, b)\): \(x \neq 0\) and \(x - a \neq 0\).
- new: \(\gcd(a, b, x, y) = 1\).
Plot of the Wavenumber Set $\Lambda_{\text{new}}$


