Problem 1. Differentiate the following functions. If you need more space, use the last page for your computations.

(a) \( y = \sqrt{z} e^{-z} \)

\[
\frac{dy}{dx} = \frac{1}{2\sqrt{z}} e^{-z} + \sqrt{z}(-e^{-z}) \\
= e^{-z} \left( \frac{1 - 2z}{2\sqrt{z}} \right)
\]

(b) \( y = \left( \frac{x^2 + 2}{\ln(x)} \right)^2 \)

\[
\frac{dy}{dx} = 2 \left( \frac{x^2 + 2}{\ln(x)} \right) \cdot \frac{\left( 2x \ln(x) - (x^2 + 2) \frac{1}{x} \right)}{\ln(x)^2} \\
= \frac{2(x^2 + 2) \cdot (2x \ln(x) - x^2 - 2)}{x \ln(x)^3}
\]

(c) \( f(x) = 2x \tan(cos(x)) \)

Using the chain rule twice, and the product rule we get

\[
f'(x) = 2 \tan(cos(x)) + 2x \frac{1}{\cos^2(cos(x))}(-\sin(x)) \\
= 2 \tan(cos(x)) - \frac{2x \sin(x)}{\cos^2(cos(x))}
\]

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(d) \( r(\theta) = \arctan(\theta) \sqrt{\cos(3\theta)} \)

Using the product rule, and chain rule twice, we get
\[
r'(\theta) = \frac{1}{1 + \theta^2} \sqrt{\cos(3\theta)} + \arctan(\theta) \frac{1}{2 \sqrt{\cos(3\theta)}} 3 \cdot (-\sin(3\theta))
\]
\[
= \frac{\sqrt{\cos(3\theta)}}{1 + \theta^2} - \frac{3 \arctan(\theta) \sin(3\theta)}{2 \sqrt{\cos(3\theta)}}.
\]

(e) \( f(x) = e^{-2x} \sin(x) \)

Here we use the product rule and the derivative of exponential functions to obtain
\[
f'(x) = -2e^{-2x} \sin(x) + e^{-2x} \cos(x)
\]
\[
= e^{-2x} (-2 \sin(x) + \cos(x))
\]

(f) \( G(x) = \frac{\sin^2(x) - 1}{\cos(x) + 1} \)

Using the quotient rule, and derivatives of trigonometric functions, we find
\[
G'(x) = \frac{2 \sin(x) \cos(x)(\cos^2(x) + 1) - (\sin^2(x) - 1)(2 \cos(x))(-\sin(x))}{(\cos^2(x) + 1)^2}
\]
\[
= \frac{\sin(x) \cos(x) (2 \cos^2(x) + 2 + 2 \sin^2(x) - 2)}{(\cos^2(x) + 1)^2}
\]
\[
= \frac{2 \sin(x) \cos(x)}{(\cos^2(x) + 1)^2}
\]
(g) \( g(t) = \cos(ln(t)) \)

Chain rule applied several times gives:

\[
g'(t) = -\sin(ln(t)) \cdot \frac{1}{t} = -\frac{\sin(ln(t))}{t}.
\]

(h) \( T(u) = \arctan \left( \frac{u}{1+u} \right) \)

Chain rule plus quotient rules gives:

\[
T'(u) = \frac{1}{1 + \left( \frac{u}{1+u} \right)^2} \left( \frac{1 \cdot (1 + u) - 1 \cdot u}{(1 + u)^2} \right) = \frac{1}{1 + \left( \frac{u}{1+u} \right)^2} \cdot \frac{1}{(1 + u)^2}
\]

\[
= \frac{1}{(1 + u)^2 + u^2}
\]

Problem 2.

- For \( x > 0 \), find and simplify the derivative of \( f(x) = \arctan(x) + \arctan(1/x) \)

Using the derivative of \( \arctan \) and the chain rule, we get for \( x > 0 \):

\[
f'(x) = \frac{1}{1 + x^2} + \frac{1}{1 + \left( \frac{1}{x} \right)^2} \left( -x^{-2} \right)
\]

\[
= \frac{1}{1 + x^2} - \frac{1}{1 + \left( \frac{1}{x} \right)^2} \cdot \frac{1}{x^2} = \frac{1}{1 + x^2} - \frac{1}{1 + x^2} = 0
\]

- What does the result tell you about \( f \)?

Since the derivative is zero, then the function is constant for \( x > 0 \). In fact, using properties of trigonometric functions, one can show that

\[
\arctan(x) + \arctan \left( \frac{1}{x} \right) = \frac{\pi}{2}
\]

for \( x > 0 \).