Good luck!

Problem 1

For each problem below, find a value if the constant $k$ such that the limit exists. Show your reasoning.

• $\lim_{x \to 4} \frac{x^2 - k^2}{x - 4}$

Since $x - 4$ approaches zero as $x \to 4$, then the only chance for the limit to exist is if the limit of the numerator is also zero as $x \to 4$, otherwise the fractional function has a vertical asymptote at $x = 4$. So we need

$$0 = \lim_{x \to 4} (x^2 - k^2) = 4^2 - k^2,$$

So $k = 4$ satisfies the required condition. In fact, if $k = 4$,

$$\lim_{x \to 4} \frac{x^2 - k^2}{x - 4} = \lim_{x \to 4} \frac{x^2 - 4^2}{x - 4} = \lim_{x \to 4} \frac{(x - 4)(x + 4)}{x - 4} = \lim_{x \to 4} (x + 4) = 4 + 4 = 8$$

• $\lim_{x \to 1} \frac{x^2 - kx + 4}{x - 1}$

Analogously, we need the limit of the numerator to be zero as $x \to 1$. So, we need

$$0 = \lim_{x \to 1} (x^2 - kx + 4) = 1 - k + 4 = 0$$

So, $k = 5$ works. In fact, if $k = 5$,

$$\lim_{x \to 1} \frac{x^2 - 5x + 4}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x - 4)}{x - 1} = \lim_{x \to 1} (x - 4) = 1 - 4 = -3$$

• $\lim_{x \to \infty} \frac{x^2 + 3x + 5}{4x + 1 + x^2}$

Here, we need to check the long-run behavior of the fractional function. The leading term of the numerator is $x^2$. If $k$ is a non-negative integer, the denominator is a polynomial. If $k = 0$ or $k = 1$, the order of the polynomial in the numerator would be greater than the order of the polynomial of the denominator, and the limit would be $\infty$. On the other hand, if $k = 2$, then the leading term of the polynomial in the denominator is $x^2$. So $k = 2$ works, and in fact

$$\lim_{x \to \infty} \frac{x^2 + 3x + 5}{4x + 1 + x^2} = \lim_{x \to \infty} \frac{x^2}{x^2} = 1$$
**Problem 2** In a time of $t$ in seconds, a particle moves a distance of $s$ meters from its starting point, where $s = 4t^2 + 3$. **Include units.**

(a) Find the average velocity between $t = 1$ and $t = 1 + h$ if

(i) $h = 0.1$

The average velocity is

$$\frac{4(1+h)^2 + 3 - 4 - 3}{h} = \frac{4(1+h)^2 - 4}{h} \approx 8.4 \text{m/s if } h = 0.1$$

(ii) $h = 0.01$

The average velocity here is

$$\frac{4(1+h)^2 - 4}{h} \approx 8.04 \text{m/s if } h = 0.01$$

(iii) $h = 0.001$

Finally, the average velocity here is

$$\frac{4(1+h)^2 - 4}{h} \approx 8.004 \text{m/s if } h = 0.001$$

(b) Use your answer to part (a) to estimate the instantaneous velocity of the particle at time $t = 1$.

As $h \to 0$, the average velocity seems to be converging to $8 \text{m/s}$

**Problem 3**

(a) Sketch the graph of a continuous function $f$ with all of the following properties:

(i) $f(0) = 2$

(ii) $f(x)$ is decreasing for $0 \leq x \leq 3$

(iii) $f(x)$ is increasing for $3 < x \leq 5$

(iv) $f(x)$ is decreasing for $x > 5$

(v) $f(x) \to 9$ as $x \to \infty$

(b) Is it possible that the graph of $f$ is concave down for all $x > 6$? Explain

No. Since the function has to be decreasing for $x \geq 5$, it needs to approach the horizontal asymptote $y = 9$ from above, as can be seen from the graph. As a result, it needs to be concave up for $x >> 0$ big enough. If if was concave down, it would cross the horizontal asymptote and will keep decreasing.