Problem 1 One hundred kilograms of radioactive substance decay to 40 kg in 10 years. How much remains after 20 years?

The amount of the radioactive substance (in kilograms) as a function of time is of the form

\[ Q = Q_0 a^t, \]

where \( Q_0 = 100 \) kg is the initial amount, and \( a \) is the decay factor, and \( t \) is given in years. Since the substance decay to 40 kg in 10 years, then

\[ 100a^{10} = 40. \]

Then \( a^{10} = 0.4 \), which implies

\[ a = (0.4)^{\frac{1}{10}}. \]

Therefore

\[ Q = 100 \text{ kg } (0.4)^{\frac{t}{10}} \]

After 20 years,

\[ Q(20) = 100 \text{ kg } (0.4)^{2} = 16 \text{ kg}. \]

Problem 2 The Bay of Fundy in Canada has the largest tides in the world. The difference between low and high water levels is 15 meters (nearly 50 feet). At a particular point the depth of water, \( y \) meters, is given as a function of time, \( t \), in hours since the midnight by

\[ y = D + A \cos(B(t - C)) \]

(a) What is the physical meaning of \( D \)?

*This is the midline, or the average height between the low and high water level.*

(b) What is the value of \( A \)?

We know

\[ |A| = 7.5 \text{ meters}, \]

So either \( A = 7.5 \) meters, or \( A = -7.5 \) meters, depending on the sign of \( A \).
(c) What is the value of $B$? Assume the time between successive high tides is 12.4 hours.

Since the period is 12.4 hours, then

$$\frac{2\pi}{B} = 12.4,$$

So

$$B = \frac{2\pi}{12.4} \approx 0.506708$$

(d) What is the physical meaning of $C$?

At time $t = C$, we are in a low or high water level, depending on the sign of $A$.

Problem 3

(a) If $f(x) = ax^2 + bx + c$, what can you say about the values of $a, b,$ and $c$ if

(1) $(1, 1)$ is on the graph of $f(x)$?

If $(1, 1)$ is on the graph of $f(x)$, this means that substituting $x = 1$ and $y = 1$ the equation above holds, which gives

$$1 = a + b + c,$$

So the condition for $a, b, c$ is simply

$$a + b + c = 1.$$

(2) $(1, 1)$ is the vertex of the graph of $f(x)$? [Hint: The axis of symmetry is $x = -b/(2a)$]

According to the hint, the axis of symmetry in general is $x = -\frac{b}{2a}$. If $(1, 1)$ is the vertex, then $x = 1$ is the axis of symmetry. As a result we get

$$1 = -\frac{b}{2a}, \text{ or } b = -2a$$

Together with the condition above $(a + b + c = 1)$ we get

$$-a + c = 1$$
$$b = -2a$$

(3) The $y$ intercept of the graph is $(0, 6)$?

The $y$-intercept is simple $c$. So $c = 6$.

(b) Find a quadratic function satisfying all three conditions.

We need

$$-a + c = 1$$
$$b = -2a$$
$$c = 6$$

So $c = 6, a = c - 1 = 5, \text{ and } b = -2 \cdot 5 = -10$. Therefore

$$y = 5 \cdot x^2 - 10 \cdot x + 6$$