RESEARCH STATEMENT IN: SEMICLASSICAL ANALYSIS

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My research interests include the area of **semiclassical analysis**. In collaboration with Professor Alejandro Uribe, we develop semiclassical analysis on certain phase spaces (symplectic manifolds) with boundary, construct a symbolic calculus for semiclassical operators with singular symbols and study propagation. This has applications to spectral theory and mathematical physics. However, I also enjoy working on the area of **numerical analysis**, which has been a parallel project in my doctoral dissertation. In collaboration with Professor Smadar Karni, I have made contributions to non-linear hyperbolic conservation laws, shallow water flows, and flows in porous media. My research statement in that area is available on my webpage:

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1. Introduction

In semiclassical analysis, the main goal is to study the relationships between the mathematics of classical and quantum mechanics. Traditionally one goes from the quantum setting to the classical setting in this field by letting the Planck's constant tend to zero. Classical objects are symplectic manifolds (X,ω) (phase spaces), together with the set of C^∞ real-valued functions in (X,ω) (**classical observables**), and a Poisson bracket $\{\cdot,\cdot\}$. The corresponding quantum objects consists of Hilbert spaces with an inner product $(\mathcal{H},\langle\cdot,\cdot\rangle)$, together with the set of self-adjoint operators (**quantum observables**), and the commutator operation. In general we want to assign to each classical observable a quantum equivalent, respecting the Poisson bracket and the commutator, at least asymptotically. This process, referred to as **quantization**, is important as it is studied in analysis as well as in physics. An example of this is the so-called **Weyl quantization**, where $X = T^*(\mathbb{R}^n)$ and $\mathcal{H} = L^2(\mathbb{R}^n)$. This quantization comes from the representation theory of the Heisenberg group.

This notion of quantization can be extended to manifolds. For any C^∞ manifold M, there is a canonical symplectic structure for $X = T^*M$, where in local coordinates, $\omega = dx \wedge dp$. The symplectic manifold (T^*M,ω) can be quantized to $\big(L^2(M),\langle\cdot,\cdot\rangle\big)$ by **Semiclassical Pseudodifferential operators** (**sc-ΨDO**). This class consists of families of $\hbar$-dependent operators whose kernels, among other properties, have *frequency set* contained in the diagonal $\Delta \subset T^*(M\times M)$ (see [15]). A closely related theory in [18] has a similar class in the non-semiclassical setting. The following chart summarizes the classical and quantum objects in the standard case T^*M:

	Classical Mechanics:	Quantum Mechanics:
Phase Space	$X = (T^*M,\omega)$	$(\mathcal{H},\langle\cdot,\cdot\rangle)$
Observables	$f : T^*M \longrightarrow \mathbb{R}$	$A : \mathcal{H} \longrightarrow \mathcal{H}$ self-adjoint
Equations of Motion	Hamilton Equations: $\frac{dx_j}{dt} = \frac{\partial H}{\partial p_j},\ \frac{dp_j}{dt} = -\frac{\partial H}{\partial x_j}$	Schrödinger Equation: $i\hbar\frac{\partial\psi}{\partial t} = \widehat{H}\psi$
	Poisson bracket: $\{f,g\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$	Quantum Parenthesis: $\{A,B\}_Q = \frac{i}{\hbar}[A,B]$

There has been interest in quantizing symplectic manifolds in a broader setting. For instance, in the physics literature, Bojowald and Strobl [2, 3] considered for the classical setting a symplectic submanifold of T^*M with boundary, which is obtained by a symplectic cut that will be described below. Furthermore, they proposed as a corresponding Hilbert space the image of an associated projector. In the non-semiclassical setting, Guillemin and Lerman in [8] studied operators commuting with such projectors and proposed that as a quantization of an orbifold given by a symplectic cut. Quantization of symplectic manifolds with boundary may have other applications. In particular, they are related to ΨDO with singular symbols, which can be applied to construct parametrices of classical

ΨDO of real principal type on pseudo-convex manifolds (see [16, 10]). In the semiclassical setting, the trace describes how quantum observables corresponding to symplectic manifolds with boundary are concentrated microlocally in the given region with boundary. We use it to give a symbolic proof of a Szegö limit theorem.

My doctoral dissertation contains the following project (details and notation below):

Project: Quantization of Symplectic Manifolds with Boundary. Let M be a C^∞ manifold, and $X_c \subset T^*M$ a symplectic submanifold of T^*M with boundary. The boundary ∂X_c is then foliated by curves tangent to the kernel of the pull-back of the symplectic form. We will assume that the foliation of ∂X_c is fibrating. Given these conditions, we

- Give a definition of a quantization of X_c by assigning to X_c an algebra of pseudodifferential operators with singular symbols that "concentrate microlocally" in X_c .
- Develop a symbolic calculus in the operator algebra. A closely related class has been defined and studied in the non-semiclassical setting in [16] and [10], where general pairs of conic Lagrangian manifolds that intersect cleanly are considered. In [19], a precise calculus for a more general class is provided. More relevant to our work is [1], which contains a non-semiclassical part of the theory.
- Study cases where the operator algebra admits projectors.
- Study the asymptotic behavior of elements in the algebra as $\hbar \to 0$. In addition to the symbol calculus, as part of our own results, we compute the asymptotics of the trace of elements in the class and give a symbolic proof of the Szegö limit theorem as an application.
- Study the propagator given by certain elements in the algebra. We describe the propagation of special elements, and state an Egorov-type theorem.
- Finally, we observe examples of numerical propagation of coherent states where particles approaching to the boundary, ∂X_c, disappear and travel at infinite velocity, appearing again in more than one trajectory (multiples coherent states) at inflow with the same energy (see Figure 1). As future work, we would like to explain this phenomenon theoretically.

Notation. Now I introduce the notation and terminology necessary to speak about the results. The following is the simple case where we define X_c (not in the most general setting). Let $\widehat{P}(\hbar)$ be a sc-ΨDO of order zero on $L^2(M)$ which we will use to "cut" $L^2(M)$, with principal symbol $P : T^*M \to \mathbb{R}$. For simplicity we assume that the spectrum of $\widehat{P}(\hbar)$ is purely discrete and $\operatorname{Spec}(\widehat{P}(\hbar)) \subset \hbar\mathbb{Z}$. It follows that the Hamilton flow of P, $\{\phi_t\}$, is 2π-periodic. Fix two integers, $E_1 < E_2$, and assume that, for $j = 1, 2$, the trajectories on $P^{-1}(E_j)$ satisfy the **Bohr-Sommerfeld** condition. For each $N = 1/\hbar$, let

$$\mathcal{H}_N = \text{span of eigenvectors of } \widehat{P}(\hbar) \text{ with eigenvalues in } [E_1, E_2],$$

and denote by $\Pi_N : L^2(M) \to \mathcal{H}_N$ the orthogonal projector. Let

$$X_c = \left\{ \overline{x} \in T^*M \;\middle|\; E_1 \le P(\overline{x}) \le E_2 \right\},$$

and let $X_{cut} = X_c/S^1$ be the C^0 manifold obtained by collapsing the orbits of the Hamiltonian flow H_P in each boundary $P^{-1}(E_k)$, $k = 1, 2$, into points. The region X_{cut} admits an orbifold structure, which is denoted by (X_{++}, ω_{++}). Operators of the form $\Pi_N \hat{Q}_N \Pi_N$, where $\hat{Q} \in$ scΨDO, generalize the **Töplitz matrices**. In [8], the authors study the projector associated to a classical (non-semiclassical) ΨDO $\widehat{P}$, and show how the operators commuting with the projectors can be thought of as quantizing the algebra of classical observables $C^\infty(X_{++})$. As part of our results we prove the analogous result in the semiclassical setting.

The projector Π_N behaves as an sc-ΨDO with principal symbol the characteristic function of X_c. Specifically, the projector belongs to a larger class of operators with singular symbols that forms an algebra, and whose frequency set is contained in the union of three admissible Lagrangian, the flow-out Λ_k on each energy level E_k by the principal symbol of the cut operator P, $k = 1, 2$, and the diagonal Δ, which intersects each flow-out cleanly.

2. Results

A sketch of the results is as follows. The class sc-ΨDO can be generalized to a bigger class where now the frequency set is contained in an admissible Lagrangian submanifold of $T^*(M \times M)$. This is the class of **semiclassical Fourier Integral Operators (FIO)**, and have been defined in different contexts; see for instance [12, 17] (and [14, 6] for the non-semiclassical setting). When the Lagrangian submanifold is $\Lambda \subset T^*(M \times M)$, they are denoted by sc-$I(M \times M, \Lambda)$. An example of a semiclassical Fourier Integral operator is the propagator $e^{-i\hbar^{-1}t\widehat{P}}$, where $\widehat{P}$ is a zeroth order self-adjoint semiclassical pseudodifferential operator of Schrödinger type. The corresponding Lagrangian submanifold is the graph of the Hamiltonian flow ϕ_t^P, $\left\{\left(\phi_t^P(\overline{x}), \overline{x}\right\}, \overline{x} \in T^*M\right\}$, where P is the principal symbol of $\widehat{P}(\hbar)$.

The class of operators we will describe below has frequency set contained in the union of two Lagrangians that intersect cleanly. They are microlocally FIO on each Lagrangian away from the intersection. We begin by discussing the microlocal model case: $M = \mathbb{R}^n$ and $\widehat{P} = \hbar D_{x_1}$. We will use this case to define operators $J^{\ell,m}(\mathbb{R}^n; \Delta, \Lambda)$ where Δ is the diagonal and Λ is the flow-out of $\{p_1 = 0\}$. This case models our states microlocally near any of the intersections $\Delta \cap \Lambda_j$, $j = 1, 2$. Roughly speaking, operators in $J^{\ell,m}(\mathbb{R}^n; \Delta, \Lambda)$ are those whose Schwartz kernels are of the form

$$A(x, y, \hbar) = \frac{1}{(2\pi\hbar)^n} \int e^{\frac{i}{\hbar}\left[(x_1 - y_1 - s)p_1 + (x' - y')p'\right] + is\sigma} \; a(s, x, y, p, \sigma, \hbar)\; ds\, dp\, d\sigma,$$

where $a(s, x, y, p, \sigma, \hbar)$ is an amplitude with an asymptotic expansion

$$a(s, x, y, p, \sigma, \hbar) \sim \sum_{j=-\ell}^{\infty} \hbar^j a_j(s, x, y, p, \sigma),$$

where, for each j, $a_j(s, x, y, p, \sigma)$ is a polyhomogeneous classical symbol in σ of degree m. The hybrid nature of the amplitude gives states that have frequency sets contained in the union of the two Lagrangians:

$$FS\,(A) \subset \Delta \cup \Lambda,$$

and in fact, away from the intersection $\Sigma = \Delta \cap \Lambda$, A is microlocally in the spaces of semiclassical Lagrangian states sc$-I^{\ell+m}(\Delta \setminus \Sigma)$ and sc$-I^{\ell}(\Lambda \setminus \Sigma)$, respectively. One has two symbol maps:

$$\begin{tikzcd} & {|\wedge|^{1/2}(\Delta \setminus \Sigma)} \\ J^{\ell,m}(\mathbb{R}^n; \Delta, \Lambda) \arrow[ur, "\sigma_0"] \arrow[dr, "\sigma_1"'] & \\ & {|\wedge|^{1/2}(\Lambda \setminus \Sigma).} \end{tikzcd}$$

It is easy to see that, for A as above, they are given by the following formulae:

$$\sigma_0(A) := 2\pi \; a_{-\ell', m'}(s, x, y, p, \sigma)\sqrt{dxdp}\,\Big|_{y=x, s=0, p_1=\sigma} \qquad \text{and}$$

$$\sigma_1(A) := \sqrt{2\pi} \; \int a_{-\ell'}(s, x, y, p, \sigma) e^{is\sigma} d\sigma \sqrt{dx dy_1 dp'}\,\Big|_{y'=x', p_1=0, s=x_1-y_1} .$$

In general, we can define the class above for any manifold M and more general regions X_c. Let $X_c \subset T^*M$ be a symplectic manifold with boundary. The boundary ∂X_c is then **foliated** by curves tangent to the kernel of the pull-back of the symplectic form. We will assume that the foliation of ∂X_c is **fibrating**, i. e., there exists a C^∞ Hausdorff manifold S and a smooth fiber map $\rho : \partial X_c \to S$ whose fibers are the connected leaves of the foliation defined above. In addition, we assume that the **Bohr-Sommerfeld** conditions are satisfied, i.e., for each closed trajectory γ in ∂X_c, $\int_\gamma \alpha \in 2\pi\mathbb{Z}$, where α is the tautological one-form in T^*M. Each connected component of ∂X_c defines a flow-out Λ. Furthermore, locally each connected component of ∂X_c can be described by an equation $\{P = 0\}$ for some P that defines 2π periodic trajectories coinciding with the fibers of ρ. Under these conditions, we can define $J^{\ell,m}(M \times M; \Delta, \partial X_c)$, where ∂X_c indicates one has as many flow-outs (only intersecting the diagonal) as the number of connected components.

For the next proposition, see [16] and [13] for more details.

Proposition 1. *Elements in* $J^{\ell,m}(M \times M; \Delta, \partial X_c)$ *have* ***pairs of symbols*** *that are smooth on each Lagrangian away from the intersection* $\Sigma = \Delta \cap \Lambda$, $\Delta \setminus \Sigma$ *and* $\Lambda \setminus \Sigma$, *and* ***blow-up*** *as we approach* Σ. *The symbol in* $\Lambda \setminus \Sigma$ *can be extended to* Λ *as a* ***conormal distribution*** *to* Σ. *Furthermore, the symbol in* Λ *can be identified with a family of classical* ΨDO *of order* $m' = m + 1/2$ *acting on functions defined on each fiber in the flow-out.*

Our results emerge more naturally if expressed using the identification of Proposition 1. We will henceforth use that identification to show the results:

Proposition 2. ***Symbolic Compatibility Condition.*** *For each connected component of* ∂X_c *and each flow-out* Λ *and* P *as above, the symbol in* Λ *is a classical* ΨDO *of order* $m' = m + 1/2$, *acting on functions on the fibers of* $\rho : \partial X_c \to S$, *and satisfies*

$$\left[\frac{\sigma(\sigma_1(A)_s)(\tau)}{\tau^{m'}}\right]\Big|_{\Sigma} = \left[\frac{\sigma_0(A)}{P^{m'}}\right]\Big|_{\Sigma},$$

where F_s *is a fiber in* ∂X_c, τ *is the momentum variables in* T^*F_s, *and* $\sigma_1(A)_s$ *is the classical* ΨDO *corresponding to that fiber.*

The class J is closed under composition, as described in our first main result:

Theorem 3. ***Semiclassical Algebras.*** *The class* $J\,(M \times M; \Delta, \partial X_c)$ *is closed under composition and under the adjoint operation. For each* $u \in J^{\ell,m}$ *and* $v \in J^{\ell',m'}$, $u \circ v \in J^{\ell+\tilde{\ell}+1/2, m+\tilde{m}-1/2}\,(M \times M; \Delta, \partial X_c)$. *In particular,* $J^{-1/2,1/2}$ *is an algebra. The symbols are given as follows:*

$$\sigma_0(u \circ v)(\bar{x}, \bar{x}) = \sigma_0(u)(\bar{x}, \bar{x})\sigma_0(v)(\bar{x}, \bar{x}), \text{ and}$$

$$\sigma_k(u \circ v)\Big|_{F_s} = \sigma_k(u)\Big|_{F_s} \circ \sigma_k(v)\Big|_{F_s}, \ \sigma_k(u^*)\Big|_{F_s} = \left(\sigma_k(u)\Big|_{F_s}\right)^* \text{ for } k = 1, 2.$$

Now we are in a position to propose a quantization for the symplectic manifold with boundary $X_c \subset T^*M$.

Definition 4. Quantization. The manifold X_c has an associated operator algebra, $\mathcal{A}_{X_c}$, which consists of elements in the algebra $J\,(M \times M; \Delta, \partial X_c)$ which are microlocally of order $\mathrm{O}(\hbar^\infty)$ in the complement $T^*M \setminus X_c$.

It is expected that the trace of elements in $J^{\ell,m}$, if it exists, has contributions from the two symbols in the leading coefficient. Our second main result describes the leading contribution, which comes from either Lagrangian, depending on the orders ℓ and m:

Theorem 5. ***Trace.*** *Let* A *be an operator in the class* $J^{\ell,m}$, *and assume for simplicity that* A *has compact microsupport contained in* X_c. *Then, if* $m \geq 1/2$,

$$\mathrm{Tr}(A) = (2\pi)^{-n}\hbar^{-\ell-m-n}\int_{X_c}\sigma_0(x, x, p, -p)dxdp + O(\hbar^{-\ell-m-n}\ \hbar\ \log(1/\hbar)).$$

where σ_0 *is the symbol of* A *in the diagonal. Furthermore, for* $m \leq -7/2$,

$$\mathrm{Tr}(A) = (2\pi)^{-n+1/2}\hbar^{-n-\ell+1/2}\int_S Tr(\sigma_1(A)_s\Big|_{F_s})ds + O(\hbar^{-n-\ell+3/2}).$$

Now we give the results concerning the projector Π_N. In [8], Guillemin and Lerman study the projector in the non-semiclassical setting.

Theorem 6. *Let $\widehat{Q}(\hbar) \in sc - \Psi DO$ be of order zero and compact microsupport. Then for $m \in \mathbb{Z}$, and any Schwartz function f, $\left(\Pi_N \widehat{Q}_N \Pi_N\right)^m$ and $f(\Pi_N \widehat{Q}_N \Pi_N)$ are both in the algebra $J^{-1/2,1/2}$, with principal symbols:*

$$\sigma_0 \left((\Pi_N \widehat{Q}_N \Pi_N)^m \right) (\overline{x}, \overline{x}) = \chi_{X_c}(\overline{x}) Q(\overline{x})^m, \sigma_0 \left(f(\Pi_N \widehat{Q}_N \Pi_N) \right) (\overline{x}, \overline{x}) = \chi_{X_c}(\overline{x}) f(Q(\overline{x})), \ \textit{and}$$

$$\sigma_k \left(f(\Pi_N \widehat{Q}_N \Pi_N) \right)_{\big|_{F_s}} = f \left(\Pi_{F_s} M_{|_{Q_{F_s}}} \Pi_{F_s} \right),$$

where $k = 1, 2$, F_s is an orbit of $P^{-1}(E_k)$, $Q_{|_{F_s}}$ is the restriction of Q to F_s, $M_{Q_{|_{F_s}}}$ is the operator "multiplication by $Q_{|_{F_s}}$", and Π_{F_s} is the Szegö projector in the orbit F_s.

The Szegö limit theorem, a generalization of the theorem in [7], comes as a corollary

Corollary 7. ***Szegö***. *Assume that X_c is compact. Then for any Schwartz function f,*

$$\operatorname{Tr} \left(f(\Pi_N \widehat{Q}_N \Pi_N) \right) = (2\pi)^{-n} \ N^n \int_{X_c} f \circ Q \ \frac{\omega^n}{n!} + O(N^{n-1} \log(N)).$$

The propagator $e^{-it\hbar^{-1}\widehat{Q}}$ is well known to be a FIO. When $\widehat{Q}$ is replaced by $\Pi_N \widehat{Q}_N \Pi_N$, the corresponding propagator hasn't been studied symbolically. Suppose that the Hamiltonian flow of the principal symbol of $\widehat{Q}$ preserves the region X_c, i.e., for each $\sigma \in \partial X_c$, $\Xi_Q(\sigma) \in T_\sigma \partial X_c$. This is equivalent to the condition that Q is constant along the orbits of P in the boundary ∂X_c. Our third main result describes the propagator in this case:

Theorem 8. ***Propagator***. *Suppose $\widehat{Q} \in sc - \Psi DO$ is of order zero, with principal symbol Q being constant along the orbits in $P^{-1}E_k$, $k = 1, 2$ generated by H_P. Assume sub$\widehat{Q}(\hbar) = 0$. Then for $\hbar = \frac{1}{N}$, $\Pi_N e^{-itN\Pi_N \widehat{Q}_N \Pi_N} \Pi_N \in J^{-1/2,1/2}\left(M \times M; \Delta(t), \Lambda_1(t), \Lambda_2(t)\right)$, where*

$$\Delta(t) = \left\{ (\overline{x}, \overline{y}) \left| \overline{x}, \overline{y} \in T^*(M \times M), \overline{x} = \phi_t^Q(\overline{y}) \right. \right\}$$

$$\Lambda_{k,t} = \left\{ (\overline{x}, \overline{y}) \left| \overline{x}, \overline{y} \in P^{-1}(E_k) \exists s \in \mathbb{R} \textit{ such that } \overline{x} = \phi_s^P \phi_t^Q(\overline{y}) \right. \right\}, \ k = 1, 2, \ \textit{with principal symbols}$$

$$\sigma_0(\phi_t^Q \overline{x}, \overline{x}) = \chi_{X_c}(\overline{x}) \sigma_{exp}(\phi_t^Q \overline{x}, \overline{x}), \qquad \textit{for } (\phi_t^Q \overline{x}, \overline{x}) \in \Delta(t)$$

$$\sigma_1(\phi_t^Q \phi_S^P \overline{x}, \overline{x}) = \tfrac{1}{\sqrt{2\pi}} \tfrac{1}{1 - e^{is}} \sigma_{exp}(\phi_t^Q \overline{x}, \overline{x}), \quad \textit{for } (\phi_t^Q \phi_s^P \overline{x}, \overline{x}) \in \Lambda_t,$$

where σ_{exp} is the principal symbol of the semiclassical Fourier integral operator $e^{-itN\hat{Q}_N}$ on $\Delta(t)$, and χ_{X_c} is the characteristic function on X_c.

Corollary 9. ***Egorov-type theorem*** *Under the conditions above, and for any $\widehat{A}_N \in \ sc - \Psi DO$ of order zero, we have*

$e^{itN\Pi_N \widehat{Q}_N \Pi_N} \Pi_N \widehat{A}_N \Pi_N e^{-itN\Pi_N \widehat{Q}_N \Pi_N} \in J^{-1/2,1/2}\left(M \times M, \Delta, \Lambda_1, \Lambda_2\right)$, *with the following principal symbols:*

$$\sigma_0(\overline{x}, \overline{x}) = \chi_{X_c}(\overline{x}) a(\Phi_t^Q \overline{x}) \ , \ \sigma_k \left(e^{itN\Pi_N \widehat{Q}_N \Pi_N} \Pi_N \widehat{A}_N \Pi_N e^{-itN\Pi_N \widehat{Q}_N \Pi_N} \right) = \Pi_{F_s} M_{(a \circ \phi_t^Q)_{|_{F_s}}} \Pi_{F_s}.$$

3. FUTURE WORK

We would like to expand Theorem 8 to the case where the principal symbol of $\widehat{Q}$ doesn't preserve the region X_c. In the interior of X_c, we believe the propagator has to behave as the Fourier integral operator $e^{-it\hbar^{-1}\widehat{Q}}$, as Π_N behaves as the identity there. If we apply the propagator to a coherent state that concentrates in the interior of X_c, the center will propagate in the energy level, until it hits the boundary. We don't know what happens after that critical time. Done in Bargmann space, numerical computations in Figure 1 shows the results of the propagation of a projected coherent state by $\Pi_N e^{-itN\Pi_N \widehat{Q}_N \Pi_N}\Pi_N$ where $Q = q^2 - p^2$. The coherent state is initially concentrated at $w = -0.6 - 0.25i$, and propagated at $t = 0.2$, where the coherent state hits the boundary, and at time $t = 0.4$, after the collision time. We observe that the coherent state follows the trajectories of Q, hits the boundary, and returns to the cut region X_c with the same energy, but concentrated in two points. We are interested in finding a mathematical explanation for this phenomenon.

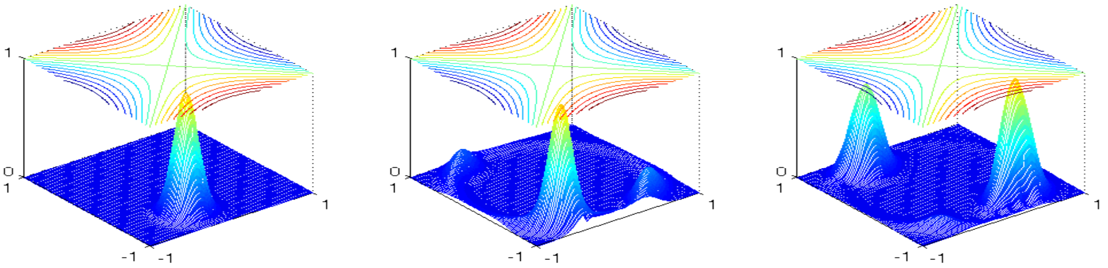

Figure 1. A coherent state concentrated at $w = -0.6 - 0.25i$ is propagated by $e^{-it\hbar^{-1}\Pi_N \hat{Q} \Pi_N}$ ($Q = q^2 - p^2$) at time $t = 0$ (left), $t = .2$ (middle) and $t = 0.4$ (right).

The trace formula has shown to be a good tool that relates the spectrum and the principal symbol of operators (see for example [17], [4], and [5]). In the future we would like to get a trace formula for elements of the form $\Pi_N \hat{Q}_N \Pi_N$ when the Hamiltonian flow of Q preserves ∂X_c. We expect to get different formulas for the contributions depending on the orders ℓ, m.

Finally, we would like to extend the construction of the operator algebra for abstract compact symplectic manifolds with boundary, using the techniques of geometric quantization.

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