Representation Theory for the Heisenberg Group
Student Analysis Seminar
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Heisenberg Lie Algebra and its relation with Quantum Mechanics.

Canonical Commutation Relations
\((aa^* - a^*a = 1)\)

Realizations of the CCR.

Stone-von Neumann uniqueness theorem.

Classification of unitary irreducible representations of the Heisenberg group.
Heisenberg Lie algebra

The Heisenberg Lie Algebra $\mathfrak{h}$ is a 3-dimensional real vector space with basic vectors $X, Y, Z$ satisfying the following commutation relations:

$$[X, Y] = Z, [X, Z] = [Y, Z] = 0$$

Matrix realization:

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Generalization: $\mathfrak{h}_n$ denote the $(2n+1)$-dimensional algebra of block-triangular matrices of the form:

$$\begin{pmatrix} 0 & \vec{x}^t & \vec{y}^t & 2z \\ 0 & 0 & 0 & \vec{y} \\ 0 & 0 & 0 & -\vec{x} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
The **Generalized Heisenberg Lie group** $H_n$ is the simply connected Lie group with $\text{Lie}(H_n) = \mathfrak{h}_n$. Realizations:

$$h(\vec{x}, \vec{y}, z) = \exp(\sum_{i=1}^{n}(x_i X_i + y_i Y_i) + z) = \begin{pmatrix} 1 & \vec{x}^t & \vec{y}^t & 2z \\ 0 & 1 & 0 & \vec{y} \\ 0 & 0 & 1 & -\vec{x} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

**Phase Space**

<table>
<thead>
<tr>
<th>Classical Mechanics:</th>
<th>Quantum Mechanics:</th>
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</thead>
<tbody>
<tr>
<td>$X = (T^* M, \omega)$</td>
<td>$(\mathcal{P}\mathcal{H}, \langle, \rangle)$</td>
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<tr>
<td>$f : T^* M \to \mathbb{R}$</td>
<td>$A : \mathcal{H} \to \mathcal{H}$ self-adjoint</td>
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<tr>
<td>Hamilton’s Equations</td>
<td>Schrödinger equation</td>
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<tr>
<td>$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}$, $\frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}$</td>
<td>$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$</td>
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<td>Poisson parenthesis :</td>
<td>Quantum parenthesis :</td>
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<tr>
<td>${f, g} = \sum_{i=1}^{n}(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i})$</td>
<td>${A, B}_Q = \frac{i}{\hbar} [A, B]$</td>
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Realizations of $\hbar$

Weyl quantization:

$$a(P, Q)\Psi(x) = aW\Psi(x) = \frac{1}{(2\pi)^{n/2}\hbar^n} \int \exp(-\frac{i}{\hbar}(y - x) \cdot p) a(p, \frac{y + x}{\hbar}) \Psi(y) dp dy$$

Considering $M = \mathbb{R}$. Let be $q$ the position and $p$ the momentum variables. Then $q^W = m_q0 : Q$ (multiplication by $q$) is the position operator and $p^W = -i\hbar \frac{\partial}{\partial q} =: P$ is the momentum operator.

**Canonical commutation relations:**

$$pq - qp = -i\hbar$$

$P, Q$ is the **standard realization** of CCR.
Another quantization:

The **Bargman space**

\[ \mathcal{B}_n = \{ f : \mathbb{C}^n \to \mathbb{C} \text{ holomorphic s.t. } \int |f(z)|^2 e^{\frac{-|z|^2}{\hbar}} d\zeta d\overline{\zeta} < \infty \} \]

with the inner product:

\[ \langle f, g \rangle = \frac{1}{(\pi \hbar)^n} \int \overline{f}(z) g(z) e^{\frac{-|z|^2}{\hbar}} d\zeta d\overline{\zeta} \]

forms a Hilbert space and there is a Bargman quantization as well. Define:

\[ a = \frac{1}{\sqrt{2}} (q + ip), \quad a^* = \frac{1}{\sqrt{2}} (q - ip), \quad aa^* - a^*a = \hbar \]

In the Bargman quantization:

\[ (a^*)^B = mz, \quad a^B = \frac{\partial}{\partial z} \]
Reformulation of CCR

CCR:

\[ aa^* - a^*a = 1 \]

Two realizations:

**Standard realization:**
\[ a = P \text{ and } a^* = Q \text{ acting on } L^2(\mathbb{R}, dt) \]

**Fock realization:**
\[ a = \frac{\partial}{\partial z}, \quad a^* = mz \]

Reformulation of CCR:

\[ u(s) := e^{isp}, \quad v(t) := e^{itq} \]

CCR is equivalent to:
\[ u(s)v(t) = e^{ist\hbar}v(t)u(s) \]
Theorem. (Stone-von Neumann uniqueness theorem). Let $p$ and $q$ be self adjoint operators in a Hilbert space $H$ satisfying the canonical commutation relation in the Heisenberg form. Then $H$ is isomorphic to a direct sum of several copies of $L^2(\mathbb{R}, dt)$ where the operators $p$ and $q$ act by the standard realization.

In particular, any irreducible realization of CCR is equivalent to the standard one.

Proposition.

a) In any realization of CCR there exists a vacuum vector.

b) For any irreducible realization the vacuum vector is unique (up to a scalar factor).

c) All irreducible realizations of CCR are equivalent to each other. There exists an orthonormal basis $\{v_k\}_{k \geq 0}$ such that the operators $a, N = a^* a, a^*$ are given by:

$$av_k = \sqrt{k} \cdot v_{k-1}, \quad Nv_k = k \cdot v_k, \quad a^* \cdot v_k = \sqrt{k + 1} \cdot v_{k+1}$$
Classification of unitary irreducible representations of $H$

**Theorem.** Any unitary irreducible representation of the Heisenberg group $H$ is equivalent to exactly one of the following:

**a)** the 1-parametric family of equivalence classes of infinite-dimensional unireps $\pi_\lambda$, $\lambda \neq 0$ given by

$$
\pi_\lambda(g_{a,b,c}) = \pi_\lambda(exp(cZ)exp(bY)exp(aX)) = e^{2\pi i \lambda c} \nu(\lambda b) u\left(\frac{a}{\hbar}\right), \text{ and}
$$

**b)** the 2-parametric family of 1-dimensional representation $\pi_{\mu,\nu}$ given by

$$
\pi_{\mu,\nu} = e^{2\pi i (a\mu + c\nu)}
$$