Representation Theory for the Heisenberg Group

Student Analysis Seminar

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Presentation

- Heisenberg Lie Algebra and its relation with Quantum Mechanics.
- Canonical Commutation Relations ($aa^* - a^*a = 1$)
- Realizations of the CCR.
- Stone-von Neumann uniqueness theorem.
- Classification of unitary irreducible representations of the Heisenberg group.

Heisenberg Lie algebra

The **Heisenberg Lie Algebra** $\mathfrak{h}$ is a 3-dimensional real vector space with basic vectors X, Y, Z satisfying the following commutation relations:

$$[X, Y] = Z, [X, Z] = [Y, Z] = 0$$

Matrix realization:

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Generalization: $\mathfrak{h}_n$ denote the (2n+1)-dimensional algebra of block-triangular matrices of the form:

$$\begin{pmatrix} 0 & \overrightarrow{x}^t & \overrightarrow{y}^t & 2z \\ 0 & 0 & 0 & \overrightarrow{y} \\ 0 & 0 & 0 & -\overrightarrow{x} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Heisenberg Lie Group and Quantum Mechanics

The **Generalized Heisenberg Lie group** H_n is the simply connected Lie group with $Lie(H_n) = \mathfrak{h}_n$.Realizations:

$$h(\overrightarrow{x}, \overrightarrow{y}, z) = exp(\sum_{i=1}^{n}(x_i X_i + y_i Y_i) + z) = \begin{pmatrix} 1 & \overrightarrow{x}^t & \overrightarrow{y}^t & 2z \\ 0 & 1 & 0 & \overrightarrow{y} \\ 0 & 0 & 1 & -\overrightarrow{x} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

	Classical Mechanics:	**Quantum Mechanics**:
Phase Space	$X = (T^*M, \omega)$	$(\mathbb{P}\mathcal{H}, \langle , \rangle)$
Observables	$f : T^*M \longrightarrow \mathbb{R}$	$A : \mathcal{H} \longrightarrow \mathcal{H}$ self-adjoint
Equations of Motion	Hamilton's Equations $\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \ \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}$	Schrödinger equation $i\hbar\frac{\partial \psi}{\partial t} = \widehat{H}\psi$
	Poisson parenthesis : $\{f, g\} = \sum_{i=1}^{n}(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i})$	Quantum parenthesis : $\{A, B\}_Q = \frac{i}{\hbar}[A, B]$

Realizations of $\mathfrak{h}$

Weyl quantization:

$$a(\mathbf{P},\mathbf{Q})\Psi(x) = aW\Psi(x) = \frac{1}{(2\pi)^{n/2}\hbar^n}\int exp(-\frac{i}{\hbar}(\mathbf{y}-\mathbf{x})\cdot\mathbf{p})a(\mathbf{p},\frac{\mathbf{y}+\mathbf{x}}{\hbar})\Psi(y)d\mathbf{p}d\mathbf{y}$$

Considering $M = \mathbb{R}$. Let be q the position and p the momentum variables. Then $q^W = m_q 0 : Q$ (multiplication by q) is the position operator and $p^W = -i\hbar\frac{\partial}{\partial q} =: P$ is the momentum operator.

Canonical commutation relations:

$$pq - qp = -i\hbar$$

P,Q is the **standard realization** of CCR.

Bargman space

Another quantization:
The **Bargman space**

$$\mathcal{B}_n = \{f : \mathbb{C}^n \longrightarrow \mathbb{C} \textit{ holomorphic } \text{ s.t. } \int |f(z)|^2 exp(-\frac{|z|^2}{\hbar}) dz d\overline{z} < \infty\}$$

with the inner product:

$$\langle f, g \rangle = \frac{1}{(\pi\hbar)^n} \int \overline{f}(z) g(z) exp(-\frac{|z|^2}{\hbar}) dz d\overline{z}$$

forms a Hilbert space and there is a Bargman quantization as well.Define:

$$a = \frac{1}{\sqrt{2}}(q + ip),\ a^* = \frac{1}{\sqrt{2}}(q - ip),\ aa^* - a^*a = \hbar$$

In the Bargman quantization:

$$(a^*)^{\mathbf{B}} = m_z,\ a^{\mathbf{B}} = \frac{\partial}{\partial z}$$

Reformulation of CCR

CCR:

$$aa^* - a^*a = 1$$

Two realizations:

Standard realization:

$$a = P \text{ and } a^* = Q \text{ acting on } L^2(\mathbb{R}, dt)$$

Fock realization:

$$a = \frac{\partial}{\partial z},\ a^* = m_z$$

Reformulation of CCR:

$$u(s) := e^{isp},\ v(t) := e^{itq}$$

CCR is equivalent to:

$$u(s)v(t) = e^{ist\hbar}v(t)u(s)$$

Uniqueness theorem

Theorem. *(Stone-von Neumann uniqueness theorem). Let p and q be self adjoint operators in a Hilbert space H satisfying the canonical commutation relation in the Heisenberg form. Then H is isomorphic to a direct sum of several copies of $L^2(\mathbb{R}, dt)$ where the operators p and q act by the standard realization. In particular, any irreducible realization of CCR is equivalent to the standard one.*

Proposition.

a) *In any realization of CCR there exists a vacuum vector.*

b) *For any irreducible realization the vacuum vector is unique (up to a scalar factor).*

c) *All irreducible realizations of CCR are equivalent to each other. There exists a orthonormal basis $\{v_k\}_{k\geq 0}$ such that the operators a, $N = a^*a$,a^* are given by:*

$$av_k = \sqrt{k} \cdot v_{k-1},\ Nv_k = k \cdot v_k,\ a^* \cdot v_k = \sqrt{k+1} \cdot v_{k+1}$$

Classification of unitary irreducible representations of H

Theorem. *Any unitary irreducible representation of the Heisenberg group H is equivalent to exactly one of the following:*

a) *the 1-parametric family of equivalence classes of infite-dimensional unirreps π_λ, $\lambda \neq 0$ given by*

$$\pi_\lambda(g_{a,b,c}) = \pi_\lambda(exp(cZ)exp(bY)exp(aX)) = e^{2\pi i\lambda c}v(\lambda b)u(\frac{a}{\hbar}),$$ *and*

b) *the 2-parametric family of 1-dimensional representation $\pi_{\mu,\nu}$ given by*

$$\pi_{\mu,\nu} = e^{2\pi i(a\mu+c\nu)}$$