Notes on positroid varieties

1 Grassmannian

1.1 Plücker coordinates

The Grassmannian $\text{Gr}(k, n)$ is the space of $k$-dimensional vector subspaces of $\mathbb{C}^n$. To a subspace $V = \text{Span}\{v_1, \ldots, v_k\} \subseteq \mathbb{C}^k$, we associate the multivector $v_1 \wedge \cdots \wedge v_k \in \bigwedge^k \mathbb{C}^n$. If we choose another basis $(w_1, \ldots, w_k)$ for $V$, we get the same multivector scaled by the determinant of the change of basis matrix. This defines an embedding

$$\Delta : \text{Gr}(k, n) \to \mathbb{P}(\bigwedge^k \mathbb{C}^n),$$

called the Plücker embedding. The homogeneous coordinates on $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$ are called Plücker coordinates.

Let $(e_j)_{1 \leq j \leq n}$ be the standard basis of $\mathbb{C}^n$ and let $v_i = \sum_{j=1}^n v_{ij} e_j$. Define the matrix $k \times n$

$$A = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{kn} \end{bmatrix},$$

so that $V$ spanned by the rows of $A$. Since $\dim V = k$, the matrix $A$ has rank $k$. For $a \in [n]$, let $A_a$ denote the $a$th column of $A$. For $I = \{1 \leq i_1 < i_2 < \cdots < i_k \leq n\} \in \binom{[n]}{k}$, let $A_I$ denote the $k \times k$ submatrix $[A_{i_1} \cdots A_{i_k}]$. Then the Plücker coordinate

$$\Delta_I(V) = \det A_I.$$  

$\text{Gr}(k, n)$ is a closed subvariety of $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$ whose ideal $I_{k,n}$ is generated by the Plücker relations

$$\sum_{s=1}^{k+1} (-1)^s \Delta_{i_1, \ldots, i_{k-1}, j_s} \Delta_{j_1, \ldots, j_{k+1}} = 0,$$

for $I = \{i_1 < \cdots < i_{k-1}\} \in \binom{[n]}{k-1}$, $J = \{j_1 < \cdots < j_{k+1}\} \in \binom{[n]}{k+1}$.

Example 1.1. The ideal of $\text{Gr}(2, 4)$ is generated by the relations

$I = \{1\}, J = \{1, 2, 3\} : \Delta_{11} \Delta_{23} - \Delta_{12} \Delta_{13} + \Delta_{13} \Delta_{12} = 0,$

$I = \{3\}, J = \{1, 2, 4\} : -\Delta_{31} \Delta_{24} + \Delta_{32} \Delta_{14} - \Delta_{12} \Delta_{34} = \Delta_{13} \Delta_{24} - \Delta_{14} \Delta_{23} - \Delta_{12} \Delta_{34}$ etc.
Therefore
\[ \text{Gr}(2,4) = \text{Proj } \mathbb{C}[\Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{23}, \Delta_{24}, \Delta_{34}] / \langle \Delta_{13}\Delta_{24} - \Delta_{14}\Delta_{23} - \Delta_{12}\Delta_{34} \rangle. \]

1.2 Column matroid of a point in the Grassmannian

A matroid $\mathcal{M}$ of rank $k$ on the set $[n]$ is a nonempty collection of $k$-element subsets of $[n]$, called bases, satisfying the exchange axiom:

For any $I, J \in \mathcal{M}$ and $i \in I$ there exists $j \in J$ such that $I - \{i\} \cup \{j\} \in \mathcal{M}$.

For $V \in \text{Gr}(k,n)$, the matroid $\mathcal{M}(V)$ of $V$ is
\[ \mathcal{M}(V) := \left\{ I \in \binom{[n]}{k} : \Delta_I(V) \neq 0 \right\}. \]

Proposition 1.2. $\mathcal{M}(V)$ is a matroid.

Proof. For $I, J \in \mathcal{M}(V)$ and $i \in I$, consider the Plücker relation for the subsets $I - \{i\}$, $J \cup \{i\}$.

Since the term $\Delta_I \Delta_J$ is nonzero, so there must be another nonzero term $\Delta_{I - \{i\} \cup \{j\}} \Delta_J \cup \{i\} - \{j\}$.

Example 1.3. $\mathcal{M}\left(\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}\right) = \{13, 14, 23, 24\}$.

Define the matroid stratum
\[ \text{Gr}(\mathcal{M}) := \{ V \in \text{Gr}(k,n) : \mathcal{M}(V) = \mathcal{M} \}. \]

We have the matroid stratification
\[ \text{Gr}(k,n) = \bigsqcup_{\mathcal{M}} \text{Gr}(\mathcal{M}), \]

where $\mathcal{M}$ varies over matroids of rank $k$ on $[n]$. The stratification is very poorly behaved.

1.3 Cyclically shifted Schubert cells

For each $a \in [n]$, let $<_a$ denote the following linear order on $[n]$
\[ a <_a a + 1 <_a a + 2 <_a \cdots <_a n <_a 1 <_a 2 <_a \cdots <_a a - 1. \]

For $B, C \in \binom{[n]}{k}$, we get the partial order $B \leq_a C$ if $b_i \leq_a c_i$ where $B = \{b_1 <_a b_2 <_a \cdots <_a b_k\}$ and $C = \{c_1 <_a c_2 <_a \cdots <_a c_k\}$.

For $S \subseteq [n]$, let $\text{Project}_{[j]} : \mathbb{C}^n \to \mathbb{C}^j$ denote the projection. For $I \in \binom{[n]}{k}$,
\[ X_I = \{ V \in \text{Gr}(k,n) : \dim \text{Proj}_{[j]}(V) = \#I \cap [j] \} \]

is called the Schubert cell labeled by $I$. 
Example 1.4. For example, if \((k, n) = (3, 6)\) and \(I = 135\), the Schubert cell consists of

\[
\text{Row-span} \begin{bmatrix}
0 & 0 & 0 & 1 & \ast \\
0 & 1 & \ast & 0 & \ast \\
1 & \ast & 0 & \ast & \ast \\
\end{bmatrix}.
\]

If \(I \in \binom{[n]}{k}\), then \(S_I := \{J \in \binom{[n]}{k} : I \leq J\}\) is a matroid called the Schubert matroid with minimal element \(I\). Then

\[
X_I^0 = \{V \in \text{Gr}(k, n) : I \in \mathcal{M}(V) \subseteq S_I\} = \bigsqcup_{M : I \in M \subseteq S_I} \text{Gr}(M),
\]

so the Schubert stratification

\[
\text{Gr}(k, n) = \bigsqcup_{I \in \binom{[n]}{k}} X_I^0
\]

is coarser than the matroid stratification.

Similarly \(S_{I, a} = \{J \in \binom{[n]}{k} : I \leq a J\}\) is called a cyclically shifted Schubert matroid, and the cyclically shifted Schubert cell

\[
X_{I, a}^0 = \{V \in \text{Gr}(k, n) : I \in \mathcal{M}(V) \subseteq S_{I, a}\}.
\]

2 Positroids

Open positroid varieties give an intermediate stratification that is still well behaved.

The \textit{totally nonnegative Grassmannian} \(\text{Gr} \geq 0(k, n)\) is the subset of \(\text{Gr}(k, n)\) such that \(\Delta_I \geq 0\) for all \(I \in \binom{[n]}{k}\). A matroid \(\mathcal{M}\) of rank \(k\) on \([n]\) is called a \textit{positroid} if \(\mathcal{M} = \mathcal{M}(V)\) for some \(V \in \text{Gr} \geq 0(k, n)\).

Example 2.1. For the dimer model in Figure 1, the associated point in \(\text{Gr} \geq (3, 6)\) is

\[
V = \text{Row-span} \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & -1 & 0 & 1 & 0 \\
\end{bmatrix} \in \text{Gr}(3, 6),
\]

and the positroid \(\mathcal{M}(V) = \{124, 134, 145, 234, 245, 345\}\).

2.1 Grassmann necklaces

A \textit{Grassmann necklace} is a subset \(\mathcal{I} = \{I_1, \ldots, I_n\}\) of \(\binom{[n]}{k}\) such that for \(a \in [n]\), we have

1. if \(a \in I_a\), then \(I_{a+1} = I_a - \{a\} \cup \{b\}\) (where \(b\) may be equal to \(a\)), and
2. if \(a \notin I_a\), then \(I_a = I_{a+1}\).

Therefore \(I_a = I_{a+1}\) can occur in two ways, either with \(a\) in both or \(a\) not in either.
Proposition 2.2 (Postnikov, 2006). For each $a \in [n]$ and a matroid $M$ of rank $k$ on $[n]$, let $I_a$ be the $<_a$-lexicographically minimal basis in $M$. Then $I(M) = (I_1, \ldots, I_n)$ is a Grassmann necklace of type $(k,n)$.

Example 2.3. Consider the matroid $\{124, 134, 145, 234, 345\}$. The Grassmann necklace is

$I = (124, 234, 345, 145, 145, 124)$.

If $I$ is a Grassmann necklace, then define the matroid

$M(I) = \left\{ J \in \binom{[n]}{k} : I_a \leq_a J \text{ for all } a \in [n] \right\}$

where $I_a = \bigcap_{a \in [n]} S_{I_a,a}$.

Theorem 2.4 (Oh, 2011). $I \mapsto M(I)$ is a bijection between Grassmann necklaces and positroids with inverse $M \mapsto I(M)$.

Corollary 2.5. For a matroid $M$, $I(M)$ is the unique Grassmann necklace such that $I \subseteq M \subseteq M(I(M))$.

Proof. $I_a$ is the $<_a$-lex minimal element of $M(I(M))$. Since $M \subseteq M(I(M))$, $I_a$ is the $<_a$-lex minimal element of $M$. Therefore $I = I(M)$. \qed

The positroid $M(I(M))$ is called the positroid envelope of $M$.

Example 2.6. Consider the matroid $\{124, 134, 145, 234, 345\}$ with Grassmann necklace $I = (124, 234, 345, 145, 145, 124)$ from Example 2.3. The positroid envelope is the complement of

$\{123 \cup \emptyset \cup \emptyset \cup \{456\} \cup \{56, 125, 135, 15*, 25, 35\} \cup \{6, 123\}\}$,

that is $\{124, 145, 234, 245, 345\}$.

2.2 Open positroid varieties (Knutson, Lam and Speyer, 2013)

For a positroid $M$ with Grassmann necklace $I$, the open positroid variety is defined as the intersection of cyclically shifted Schubert cells

$\Pi^o(M) = \bigcap_{a \in [n]} X^a_{I_a,a}$.

Proposition 2.7. We have

$\Pi^o(M) = \text{Proj} \mathbb{C}[\Delta_I : I \in M, \Delta^{-1}_{I_a} : a \in [n]]/I_{k,n}$

$= \bigcup_{\mathcal{N}} \text{Gr}({\mathcal{N}})$,

where the sum is over matroids $\mathcal{N}$ with positroid envelope $M$.
Proof. We have

\[ \Pi^\circ(\mathcal{M}) = \bigcap_{a \in [n]} X_{I_a,a}^g \]

\[ = \bigcap_{a \in [n]} \{ V \in \text{Gr}(k, n) : I_a \in \mathcal{M}(V) \subseteq S_{I_a,a} \} \]

\[ = \{ V \in \text{Gr}(k, n) : I \subseteq \mathcal{M}(V) \subseteq \mathcal{M}(I) = \cap_{a \in [n]} S_{I_a,a} \} \quad \text{(using Oh’s theorem)} \]

\[ = \{ V \in \text{Gr}(k, n) : \Delta_I(V) = 0 \text{ for } I \notin \mathcal{M}(I), \Delta_{I_a}(V) \neq 0 \text{ for all } a \in [n] \} \]

\[ = \bigcup_{\mathcal{N}} \text{Gr}(\mathcal{N}), \]

where the sum is over matroids \( \mathcal{N} \) with positroid envelope \( \mathcal{M} \).

We have the stratification

\[ \text{Gr}(k, n) = \bigsqcup_{\text{positroids } \mathcal{M}} \Pi^\circ(\mathcal{M}), \]

which is a coarsening of the matroid stratification.

2.3 Plabic graphs

Let \( \Gamma = (B \cup W, E, F) \) be a bipartite graph embedded in a disk with \( n \) vertices on the boundary of the disk, all white and labeled by \([n]\) in clockwise order. We further assume that each boundary vertex has degree 0 or 1. Let \( k = \#W - \#B \).

2.4 Decorated permutations

A zig-zag path in \( \Gamma \) is a path that turns maximally right at black vertices and maximally left at white vertices. \( \Gamma \) is said to be minimal if zig-zag paths:

1. No zig-zag path is a closed loop,
2. No zig-zag paths have self intersections, except a boundary leaf, and
3. No two zig-zag paths form a parallel bigon.

Note that anti-parallel bigons are allowed.

A decorated permutation \( \pi' = (\pi, \col) \) is a permutation \( \pi \in [n] \) along with a coloring function \( \col : \{ i \in [n] : \pi(i) = i \} \to \{ \bullet, \circ \} \).

To \( \Gamma \), we associate a decorated permutation \( \pi'(\Gamma) \) as follows:

1. \( \pi(i) \) is defined to be the label of the end-point of the zig-zag path that starts at \( i \), and
2. If \( \pi(i) = i \), minimality of \( \Gamma \) implies that boundary vertex \( i \) is of the following two types:
   a. It is a degree 0 white vertex, in which case we define \( \col(i) = \circ \), or
   b. It is a degree 1 white vertex incident to a leaf black vertex, in which case we define \( \col(i) = \bullet \).
Figure 1: A bipartite graph Γ with target face labels.

Example 2.8. For the graph Γ in Figure 1, the decorated permutation is \( \pi^*(\Gamma) = (3, 5, 1, \circ, 4, 2, \bullet) \).

Theorem 2.9 (Postnikov, 2006 and Thurston, 2004). For any decorated permutation \( \pi' \), there is a family of minimal graphs \( \Gamma \) such that \( \pi^*(\Gamma) = \pi' \). Two minimal graphs \( \Gamma_1 \) and \( \Gamma_2 \) are move equivalent if and only if \( \pi^*(\Gamma_1) = \pi^*(\Gamma_2) \).

2.5 Dimers and the positroid of a plabic graph

A dimer cover or almost perfect matching \( M \) of \( \Gamma \) is a subset of the edges such that each interior vertex is incident to exactly one edge in \( M \). Let \( \partial M \subseteq [n] \) denote the subset of boundary vertices that are not used by \( M \), so \( \#\partial M = k \).

Define the matroid \( M(\Gamma) \) of \( \Gamma \) to be

\[
M(\Gamma) = \left\{ I \in \binom{[n]}{k} : I = \partial M \text{ for some dimer cover } M \right\}.
\]

Theorem 2.10. For any graph \( \Gamma \) as above, \( M(\Gamma) \) is a positroid.

2.6 Cluster algebra of a plabic graph

For each zig-zag path from \( i \) to \( j \), we place a label \( j \) in each face of \( \Gamma \) that is to the left of the zig-zag path. For \( f \in F \), let \( f^\leftarrow \) denote the label of face \( f \).

Let \( Q(\Gamma) \) denote the quiver obtained as follows:

1. The boundary faces are the frozen vertices.
2. Interior faces are the mutable vertices.
3. For each edge in $\Gamma$ separating faces $f_1$ and $f_2$, at least one of which is mutable, we draw an arrow between $f_1$ and $f_2$ oriented such that the white vertex of the edge is on the left.

4. Remove two cycles.

Let $\mathcal{A}(Q(\Gamma), (\Delta_{I(f)})_{f \in F})$ denote the cluster algebra of with quiver $Q(\Gamma)$ and initial cluster variables $(\Delta_{I(f)})_{f \in F}$.

### 2.7 Grassmann necklaces from plabic graphs

Given a graph $\Gamma$, let $f_a$ denote the boundary face of $\Gamma$ between boundary vertices $a-1$ and $a$. Then

$$I(\Gamma) = (\uparrow I(f_1), \ldots, \uparrow I(f_n))$$

is a Grassmann necklace. In other words, the Grassmann necklace consists of the face labels of the frozen vertices of $Q(\Gamma)$.

**Example 2.11.** The Grassmann necklace of the graph $\Gamma$ in Figure 1 is $I(\Gamma) = (124, 234, 345, 145, 145, 124)$. Given a decorated permutation $(\pi, \text{col})$, we define for $a \in [n]$

$$I_a = \{ j \in [n] : j <_a \pi^{-1}(j) \text{ or } (\pi(j) = j \text{ and } \text{col}(j) = \circ) \}.$$ 

**Example 2.12.** For the decorated permutation $\pi' = (3, 5, 1, \circ, 2, \bullet)$ in Example 2.8, we have

$I_1 = \{ j \in [6] : j < \pi^{-1}(j) \} \cup \{ 4 \} = 124$ etc.

Note that $I_1 = \uparrow I(f_1)$.

To recover $\pi'$, we have to look at two cases:

1. If $a \in I_a$, we have $I_{a+1} = I_a - \{ a \} \cup \{ b \}$. Let $\pi(a) = b$. If $a = b$, let col$(a) = \circ$.
2. If $a \notin I_a = I_{a+1}$, we let $\pi(a) = a$ and col$(a) = \bullet$.

**Example 2.13.** Consider the Grassmann necklace in Example 2.11. We have

1. $1 \in I_1 = 124$ and $I_2 = 234 = 124 - \{ 1 \} \cup \{ 3 \}$, so $\pi(1) = 3$.
2. $2 \in I_2 = 234$ and $I_3 = 345$, so $\pi(2) = 5$.
3. $3 \in I_3 = 345$ and $I_4 = 145$, so $\pi(3) = 1$.
4. $4 \in I_4 = 145$ and $I_5 = I_4$, so $\pi(4) = 4$ and col$(4) = \circ$.
5. $5 \in I_5 = 145$ and $I_6 = 124$, so $\pi(5) = 2$.
6. $6 \notin I_6 = 124$, so $\pi(6) = 6$ and col$(6) = \bullet$.

Therefore $\pi' = (3, 5, 1, \circ, 2, \bullet)$ which agrees with Example 2.8.

**Proposition 2.14** (Postnikov, 2006). The above constructions give compatible bijections between positroids, plabic graphs, decorated permutations and Grassmann necklaces.
2.8 Back to positroid varieties

For a positroid $\mathcal{M}$ with Grassmann necklace $\mathcal{I}$, define the positroid variety $\Pi(\mathcal{M}) := \Pi^\circ(\mathcal{M})$, the Zariski closure of the open positroid variety $\Pi^\circ(\mathcal{M})$ in $\text{Gr}(k, n)$.

**Theorem 2.15.**

$$\Pi(\mathcal{M}) = \{ V \in \text{Gr}(k, n) : \Delta_I(V) = 0 \text{ for all } I \notin \mathcal{M} \} = \text{Proj } \mathbb{C}[\Delta_I : I \in \mathcal{M}]/I_{k,n}.$$

**Example 2.16.** If $\mathcal{M} = \binom{[n]}{k}$ is the uniform matroid, then

$$\Pi_{k,n} := \Pi(\mathcal{M}) = \text{Gr}(k, n),$$

$$\Pi^\circ_{k,n} := \Pi^\circ(\mathcal{M}) = \{ V \in \text{Gr}(k, n) : \Delta_{I_a}(V) \neq 0 \text{ where } I_a = \{a, a+1, \ldots, a+k-1\} \}.$$  

**Theorem 2.17** (Knutson, Lam and Speyer, 2013). $\Pi(\mathcal{M})$ is an irreducible, normal, Cohen-Macaulay variety with rational singularities.

Let $\tilde{\Pi}^\circ(\mathcal{M})$ denote the affine cone over $\Pi^\circ(\mathcal{M})$.

**Theorem 2.18** (Galashin and Lam, 2019). Suppose $\mathcal{M}$ is a positroid and $\Gamma$ is a plabic graph with $\mathcal{M}(\Gamma) = \mathcal{M}$. Then

$$\tilde{\Pi}^\circ(\mathcal{M}) \cong \text{Spec } \mathcal{A} \left( Q(\Gamma), \left( \Delta_{-I(f)} \right)_{f \in F} \right).$$