Notes on positroid varieties

1 Grassmannian

1.1 Plücker coordinates

The *Grassmannian* $\mathrm{Gr}(k,n)$ is the space of k-dimensional vector subspaces of $\mathbb{C}^n$. To a subspace $V = \mathrm{Span}\{v_1,\dots,v_k\} \subseteq \mathbb{C}^k$, we associate the multivector $v_1 \wedge \cdots \wedge v_k \in \bigwedge^k \mathbb{C}^n$. If we choose another basis $(w_1,\dots,w_k)$ for V, we get the same multivector scaled by the determinant of the change of basis matrix. This defines an embedding

$$\Delta : \mathrm{Gr}(k,n) \hookrightarrow \mathbb{P}(\bigwedge^k \mathbb{C}^n),$$

called the Plücker embedding. The homogeneous coordinates on $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$ are called *Plücker coordinates.*

Let $(e_j)_{1\le j\le n}$ be the standard basis of $\mathbb{C}^n$ and let $v_i = \sum_{j=1}^n v_{ij}e_j$. Define the matrix $k \times n$

$$A = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{kn} \end{bmatrix},$$

so that V spanned by the rows of A. Since $\dim V = k$, the matrix A has rank k. For $a \in [n]$, let A_a denote the ath column of A. For $I = \{1 \le i_1 < i_2 < \cdots < i_k \le n\} \in \binom{[n]}{k}$, let A_I denote the $k \times k$ submatrix $\begin{bmatrix} A_{i_1} \cdots A_{i_k} \end{bmatrix}$. Then the Plücker coordinate

$$\Delta_I(V) = \det A_I.$$

$\mathrm{Gr}(k,n)$ is a closed subvariety of $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$ whose ideal $I_{k,n}$ is generated by the Plücker relations

$$\sum_{s=1}^{k+1} (-1)^s \Delta_{i_1,\dots,i_{k-1},j_s} \Delta_{j_1,\dots,\widehat{j_s},\dots,j_{k+1}},$$

for $I = \{i_1 < \cdots < i_{k-1}\} \in \binom{[n]}{k-1}$, $J = \{j_1 < \cdots < j_{k+1}\} \in \binom{[n]}{k+1}$.

Example 1.1. The ideal of $\mathrm{Gr}(2,4)$ is generated by the relations

$I = \{1\}, J = \{1,2,3\} : \Delta_{11}\Delta_{23} - \Delta_{12}\Delta_{13} + \Delta_{13}\Delta_{12} = 0,$
$I = \{3\}, J = \{1,2,4\} : -\Delta_{31}\Delta_{24} + \Delta_{32}\Delta_{14} - \Delta_{12}\Delta_{34} = \Delta_{13}\Delta_{24} - \Delta_{14}\Delta_{23} - \Delta_{12}\Delta_{34}$ etc.

Therefore

$$\mathrm{Gr}(2,4) = \mathrm{Proj}\ \mathbb{C}[\Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{23}, \Delta_{24}, \Delta_{34}]/\langle \Delta_{13}\Delta_{24} - \Delta_{14}\Delta_{23} - \Delta_{12}\Delta_{34}\rangle.$$

1.2 Column matroid of a point in the Grassmannian

A *matroid* $\mathcal{M}$ of *rank* k on the set $[n]$ is a nonempty collection of k-element subsets of $[n]$, called *bases*, satisfying the *exchange axiom*:

For any $I, J \in \mathcal{M}$ and $i \in I$ there exists $j \in J$ such that $I - \{i\} \cup \{j\} \in \mathcal{M}$.

For $V \in \mathrm{Gr}(k,n)$, the *matroid* $\mathcal{M}(V)$ of V is

$$\mathcal{M}(V) := \left\{ I \in \binom{[n]}{k} : \Delta_I(V) \neq 0 \right\}.$$

Proposition 1.2. *$\mathcal{M}(V)$ is a matroid.*

Proof. For $I, J \in \mathcal{M}(V)$ and $i \in I$, consider the Plücker relation for the subsets $I - \{i\}, J \cup \{i\}$. Since the term $\Delta_I \Delta_J$ is nonzero, so there must be another nonzero term $\Delta_{I-\{i\}\cup\{j\}}\Delta_{J\cup\{i\}-\{j\}}$. $\square$

Example 1.3. $\mathcal{M}\left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}\right) = \{13, 14, 23, 24\}$.

Define the *matroid stratum*

$$\mathrm{Gr}(\mathcal{M}) := \{V \in \mathrm{Gr}(k,n) : \mathcal{M}(V) = \mathcal{M}\}.$$

We have the *matroid stratification*

$$\mathrm{Gr}(k,n) = \bigsqcup_{\mathcal{M}} \mathrm{Gr}(\mathcal{M}),$$

where $\mathcal{M}$ varies over matroids of rank k on $[n]$. The stratification is very poorly behaved.

1.3 Cyclically shifted Schubert cells

For each $a \in [n]$, let $<_a$ denote the following linear order on $[n]$

$$a <_a a+1 <_a a+2 <_a \cdots <_a n <_a 1 <_a 2 <_a \cdots <_a a-1.$$

For $B, C \in \binom{[n]}{k}$, we get the partial order $B \leq_a C$ if $b_i \leq_a c_i$ where $B = \{b_1 <_a b_2 <_a \cdots <_a b_k\}$ and $C = \{c_1 <_a c_2 <_a \cdots <_a c_k\}$.

For $S \subseteq [n]$, let $\mathrm{Project}_{[j]} : \mathbb{C}^{[n]} \to \mathbb{C}^{[j]}$ denote the projection. For $I \in \binom{[n]}{k}$,

$$X_I^\circ = \{V \in \mathrm{Gr}(k,n) : \dim \mathrm{Project}_{[j]}(V) = \#I \cap [j]\}$$

is called the *Schubert cell* labeled by I.

Example 1.4. For example, if $(k,n) = (3,6)$ and $I = 135$, the Schubert cell consists of

$$\text{Row-span}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 1 & * & 0 & * \\ 1 & * & 0 & * & 0 & * \end{bmatrix}.$$

If $I \in \binom{[n]}{k}$, then $\mathcal{S}_I := \left\{ J \in \binom{[n]}{k} : I \leq J \right\}$ is a matroid called the *Schubert matroid* with minimal element I. Then

$$\begin{aligned} X_I^\circ &= \{V \in \mathrm{Gr}(k,n) : I \in \mathcal{M}(V) \subseteq \mathcal{S}_I\} \\ &= \bigsqcup_{\mathcal{M}: I \in \mathcal{M} \subseteq \mathcal{S}_I} \mathrm{Gr}(\mathcal{M}), \end{aligned}$$

so the *Schubert stratification*

$$\mathrm{Gr}(k,n) = \bigsqcup_{I \in \binom{[n]}{k}} X_I^\circ$$

is coarser than the matroid stratification.

Similarly $\mathcal{S}_{I,a} = \{J \in \binom{[n]}{k} : I \leq_a J\}$ is called a *cyclically shifted Schubert matroid*, and the *cyclically shifted Schubert cell*

$$X_{I,a}^\circ = \{V \in \mathrm{Gr}(k,n) : I \in \mathcal{M}(V) \subseteq \mathcal{S}_{I,a}\}.$$

2 Positroids

Open positroid varieties give an intermediate stratification that is still well behaved.

The *totally nonnegative Grassmannian* $\mathrm{Gr}_{\geq 0}(k,n)$ is the subset of $\mathrm{Gr}(k,n)$ such that $\Delta_I \geq 0$ for all $I \in \binom{[n]}{k}$. A matroid $\mathcal{M}$ of rank k on $[n]$ is called a *positroid* if $\mathcal{M} = \mathcal{M}(V)$ for some $V \in \mathrm{Gr}_{\geq 0}(k,n)$.

Example 2.1. For the dimer model in Figure 1, the associated point in $\mathrm{Gr}_{\geq}(3,6)$ is

$$V = \text{Row-span}\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{bmatrix} \in \mathrm{Gr}(3,6),$$

and the positroid $\mathcal{M}(V) = \{124, 134, 145, 234, 245, 345\}$.

2.1 Grassmann necklaces

A *Grassmann necklace* is a subset $\mathcal{I} = \{I_1, \ldots, I_n\}$ of $\binom{[n]}{k}$ such that for $a \in [n]$, we have

1. if $a \in I_a$, then $I_{a+1} = I_a - \{a\} \cup \{b\}$ (where b may be equal to a), and
2. if $a \notin I_a$, then $I_a = I_{a+1}$.

Therefore $I_a = I_{a+1}$ can occur in two ways, either with a in both or a not in either.

Proposition 2.2 (Postnikov, 2006)**.** *For each $a \in [n]$ and a matroid $\mathcal{M}$ of rank k on $[n]$, let I_a $<_a$-lexicographically minimal basis in $\mathcal{M}$. Then $\mathcal{I}(\mathcal{M}) = (I_1, \ldots, I_n)$ is a Grassmann necklace of type (k, n).*

Example 2.3. Consider the matroid $\{124, 134, 145, 234, 345\}$. The Grassmann necklace is

$$\mathcal{I} = (124, 234, 345, 145, 145, 124).$$

If $\mathcal{I}$ is a Grassmann necklace, then define the matroid

$$\begin{aligned}\mathcal{M}(\mathcal{I}) &= \left\{ J \in \binom{[n]}{k} : I_a \leq_a J \text{ for all } a \in [n] \right\} \\ &= \bigcap_{a \in [n]} \mathcal{S}_{I_a, a}.\end{aligned}$$

Theorem 2.4 (Oh, 2011)**.** *$\mathcal{I} \mapsto \mathcal{M}(\mathcal{I})$ is a bijection between Grassmann necklaces and positroids with inverse $\mathcal{M} \mapsto \mathcal{I}(\mathcal{M})$.*

Corollary 2.5. *For a matroid $\mathcal{M}$, $\mathcal{I}(\mathcal{M})$ is the unique Grassmann necklace such that $\mathcal{I} \subseteq \mathcal{M} \subseteq \mathcal{M}(\mathcal{I}(\mathcal{M}))$.*

Proof. I_a is the $<_a$-lex minimal element of $\mathcal{M}(\mathcal{I}(\mathcal{M}))$. Since $\mathcal{M} \subseteq \mathcal{M}(\mathcal{I}(\mathcal{M}))$, I_a is the $<_a$-lex minimal element of $\mathcal{M}$. Therefore $\mathcal{I} = \mathcal{I}(\mathcal{M})$. □

The positroid $\mathcal{M}(\mathcal{I}(\mathcal{M}))$ is called the *positroid envelope* of $\mathcal{M}$.

Example 2.6. Consider the matroid $\{124, 134, 145, 234, 345\}$ with Grassmann necklace

$$\mathcal{I} = (124, 234, 345, 145, 145, 124)$$

from Example 2.3. The positroid envelope is the complement of

$$\{123\} \cup \varnothing \cup \varnothing \cup \{456\} \cup \{*56, 125, 135, 15*, *25, *35\} \cup \{**6, 123\},$$

that is $\{124, 145, 234, 245, 345\}$.

2.2 Open positroid varieties (Knutson, Lam and Speyer, 2013)

For a positroid $\mathcal{M}$ with Grassmann necklace $\mathcal{I}$, the *open positroid variety* is defined as the intersection of cyclically shifted Schubert cells

$$\Pi^\circ(\mathcal{M}) = \bigcap_{a \in [n]} X^\circ_{I_a, a}.$$

Proposition 2.7. *We have*

$$\begin{aligned}\Pi^\circ(\mathcal{M}) &= \mathit{Proj}\ \mathbb{C}[\Delta_I : I \in \mathcal{M}, \Delta_{I_a}^{-1} : a \in [n]]/I_{k,n} \\ &= \bigsqcup_{\mathcal{N}} \mathrm{Gr}(\mathcal{N}),\end{aligned}$$

where the sum is over matroids $\mathcal{N}$ with positroid envelope $\mathcal{M}$.

Proof. We have

$$\begin{aligned}
\Pi^\circ(\mathcal{M}) &= \bigcap_{a\in[n]} X^\circ_{I_a,a} \\
&= \bigcap_{a\in[n]} \{V \in \mathrm{Gr}(k,n) : I_a \in \mathcal{M}(V) \subseteq \mathcal{S}_{I_a,a}\} \\
&= \{V \in \mathrm{Gr}(k,n) : \mathcal{I} \subseteq \mathcal{M}(V) \subseteq \mathcal{M}(\mathcal{I}) = \cap_{a\in[n]}\mathcal{S}_{I_a,a}\} \quad \text{(using Oh's theorem)} \\
&= \{V \in \mathrm{Gr}(k,n) : \Delta_I(V) = 0 \text{ for } I \notin \mathcal{M}(\mathcal{I}), \Delta_{I_a}(V) \neq 0 \text{ for all } a \in [n]\} \\
&= \bigsqcup_{\mathcal{N}} \mathrm{Gr}(\mathcal{N}),
\end{aligned}$$

where the sum is over matroids $\mathcal{N}$ with positroid envelope $\mathcal{M}$. □

We have the stratification

$$\mathrm{Gr}(k,n) = \bigsqcup_{\text{positroids } \mathcal{M}} \Pi^\circ(\mathcal{M}),$$

which is a coarsening of the matroid stratification.

2.3 Plabic graphs

Let $\Gamma = (B \cup W, E, F)$ be a bipartite graph embedded in a disk with n vertices on the boundary of the disk, all white and labeled by $[n]$ in clockwise order. We further assume that each boundary vertex has degree 0 or 1. Let $k = \#W - \#B$.

2.4 Decorated permutations

A *zig-zag path* in Γ is a path that turns maximally right at black vertices and maximally left at white vertices. Γ is said to be *minimal* if zig-zag paths:

1. No zig-zag path is a closed loop,
2. No zig-zag paths have self intersections, except a boundary leaf, and
3. No two zig-zag paths form a parallel bigon.

Note that anti-parallel bigons are allowed.

A *decorated permutation* $\pi^{\cdot} = (\pi, \mathrm{col})$ is a permutation $\pi \in [n]$ along with a coloring function $\mathrm{col} : \{i \in [n] : \pi(i) = i\} \to \{\bullet, \circ\}$.

To Γ, we associate a decorated permutation $\pi^{\cdot}(\Gamma)$ as follows:

1. $\pi(i)$ is defined to be the label of the end-point of the zig-zag path that starts at i, and
2. If $\pi(i) = i$, minimality of Γ implies that boundary vertex i is of the following two types:
   (a) It is a degree 0 white vertex, in which case we define $\mathrm{col}(i) = \circ$, or
   (b) It is a degree 1 white vertex incident to a leaf black vertex, in which case we define $\mathrm{col}(i) = \bullet$.

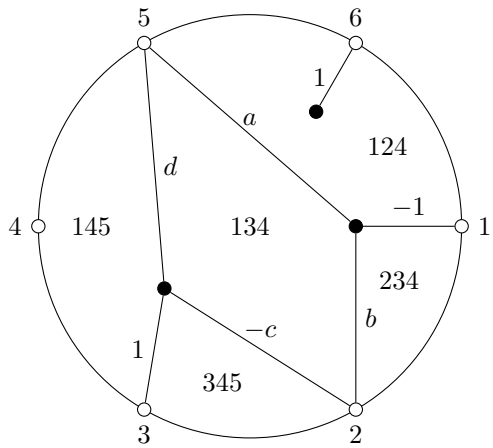

Figure 1: A bipartite graph Γ with target face labels.

Example 2.8. For the graph Γ in Figure 1, the decorated permutation is $\pi^{\cdot}(\Gamma) = (3, 5, 1, \overset{\circ}{4}, 2, \overset{\bullet}{6})$.

Theorem 2.9 (Postnikov, 2006 and Thurston, 2004)**.** *For any decorated permutation $\pi^{\cdot}$, there is a family of minimal graphs Γ such that $\pi^{\cdot}(\Gamma) = \pi^{\cdot}$. Two minimal graphs Γ_1 and Γ_2 are move equivalent if and only if $\pi^{\cdot}(\Gamma_1) = \pi^{\cdot}(\Gamma_2)$.*

2.5 Dimers and the positroid of a plabic graph

A *dimer cover* or *almost perfect matching* M of Γ is a subset of the edges such that each interior vertex is incident to exactly one edge in M. Let $\partial M \subseteq [n]$ denote the subset of boundary vertices that are not used by M, so $\#\partial M = k$.

Define the matroid $\mathcal{M}(\Gamma)$ of Γ to be

$$\mathcal{M}(\Gamma) = \left\{ I \in \binom{[n]}{k} : I = \partial M \text{ for some dimer cover } M \right\}.$$

Theorem 2.10. *For any graph Γ as above, $\mathcal{M}(\Gamma)$ is a positroid.*

2.6 Cluster algebra of a plabic graph

For each zig-zag path from i to j, we place a label j in each face of Γ that is to the left of the zig-zag path. For $f \in F$, let $\overset{\bullet\leftarrow}{I}(f) \in \binom{[n]}{k}$ denote the label of face f.
Let $Q(\Gamma)$ denote the quiver obtained as follows:

1. The boundary faces are the frozen vertices.
2. Interior faces are the mutable vertices.

3. For each edge in Γ separating faces f_1 and f_2, at least one of which is mutable, we draw an arrow between f_1 and f_2 oriented such that the white vertex of the edge is on the left.

4. Remove two cycles.

Let $\mathcal{A}\left(Q(\Gamma), \left(\Delta_{\overset{\bullet\leftarrow}{I}(f)}\right)_{f\in F}\right)$ denote the cluster algebra of with quiver $Q(\Gamma)$ and initial cluster variables $\left(\Delta_{\overset{\bullet\leftarrow}{I}(f)}\right)_{f\in F}$.

2.7 Grassmann necklaces from plabic graphs

Given a graph Γ, let f_a denote the boundary face of Γ between boundary vertices $a-1$ and a. Then

$$\mathcal{I}(\Gamma) = (\overset{\bullet\leftarrow}{I}(f_1), \ldots, \overset{\bullet\leftarrow}{I}(f_n))$$

is a Grassmann necklace. In other words, the Grassmann necklace consists of the face labels of the frozen vertices of $Q(\Gamma)$.

Example 2.11. The Grassmann necklace of the graph Γ in Figure 1 is $\mathcal{I}(\Gamma) = (124, 234, 345, 145, 145, 124)$.

Given a decorated permutation (π, col), we define for $a \in [n]$

$$I_a = \{j \in [n] : j <_a \pi^{-1}(j) \text{ or } (\pi(j) = j \text{ and } \mathrm{col}(j) = \circ)\}.$$

Example 2.12. For the decorated permutation $\pi^{\cdot} = (3, 5, 1, \overset{\circ}{4}, 2, \overset{\bullet}{6})$ in Example 2.8, we have

$$I_1 = \{j \in [6] : j < \pi^{-1}(j)\} \cup \{4\} = 124 \text{ etc.}$$

Note that $I_1 = \overset{\bullet\leftarrow}{I}(f_1)$.

To recover $\pi^{\cdot}$, we have to look at two cases:

1. If $a \in I_a$, we have $I_{a+1} = I_a - \{a\} \cup \{b\}$. Let $\pi(a) = b$. If $a = b$, let $\mathrm{col}(a) = \circ$.

2. If $a \notin I_a = I_{a+1}$, we let $\pi(a) = a$ and $\mathrm{col}(a) = \bullet$.

Example 2.13. Consider the Grassmann necklace in Example 2.11. We have

1. $1 \in I_1 = 124$ and $I_2 = 234 = 124 - \{1\} \cup \{3\}$, so $\pi(1) = 3$.

2. $2 \in I_2 = 234$ and $I_3 = 345$, so $\pi(2) = 5$.

3. $3 \in I_3 = 345$ and $I_4 = 145$, so $\pi(3) = 1$.

4. $4 \in I_4 = 145$ and $I_5 = I_4$, so $\pi(4) = 4$ and $\mathrm{col}(4) = \circ$.

5. $5 \in I_5 = 145$ and $I_6 = 124$, so $\pi(5) = 2$.

6. $6 \notin I_6 = 124$, so $\pi(6) = 6$ and $\mathrm{col}(6) = \bullet$.

Therefore $\pi^{\cdot} = (3, 5, 1, \overset{\circ}{4}, 2, \overset{\bullet}{6})$ which agrees with Example 2.8.

Proposition 2.14 (Postnikov, 2006)**.** *The above constructions give compatible bijections between positroids, plabic graphs, decorated permutations and Grassmann necklaces.*

2.8 Back to positroid varieties

For a positroid $\mathcal{M}$ with Grassmann necklace $\mathcal{I}$, define the *positroid variety* $\Pi(\mathcal{M}) := \overline{\Pi^\circ(\mathcal{M})}$, the Zariski closure of the open positroid variety $\Pi^\circ(\mathcal{M})$ in $\mathrm{Gr}(k,n)$.

Theorem 2.15.

$$\begin{aligned}\Pi(\mathcal{M}) &= \{V \in \mathrm{Gr}(k,n) : \Delta_I(V) = 0 \textit{ for all } I \notin \mathcal{M}\} \\ &= \textit{Proj } \mathbb{C}[\Delta_I : I \in \mathcal{M}]/I_{k,n}.\end{aligned}$$

Example 2.16. If $\mathcal{M} = \binom{[n]}{k}$ is the uniform matroid, then

$$\begin{aligned}&\Pi_{k,n} := \Pi(\mathcal{M}) = \mathrm{Gr}(k,n), \\ &\Pi^\circ_{k,n} := \Pi^\circ(\mathcal{M}) = \{V \in \mathrm{Gr}(k,n) : \Delta_{I_a}(V) \neq 0 \text{ where } I_a = \{a, a+1, \ldots, a+k-1\}\}.\end{aligned}$$

Theorem 2.17 (Knutson, Lam and Speyer, 2013)**.** $\Pi(\mathcal{M})$ *is an irreducible, normal, Cohen-Macaulay variety with rational singularities.*

Let $\widetilde{\Pi}^\circ(\mathcal{M})$ denote the affine cone over $\Pi^\circ(\mathcal{M})$.

Theorem 2.18 (Galashin and Lam, 2019)**.** *Suppose $\mathcal{M}$ is a positroid and Γ is a plabic graph with $\mathcal{M}(\Gamma) = \mathcal{M}$. Then*

$$\widetilde{\Pi}^\circ(\mathcal{M}) \cong \textit{Spec } \mathcal{A}\left(Q(\Gamma), \left(\Delta_{\overleftarrow{I}(f)}\right)_{f \in F}\right).$$