The cluster modular group of the dimer model

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with an appendix by Giovanni Inchiostro

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Abstract

Associated to a convex integral polygon $N$ is a dimer cluster integrable system $\mathcal{X}_N$. We compute the group $G_N$ of symmetries of $\mathcal{X}_N$, called the (2-2) cluster modular group, showing that it matches a conjecture of Fock and Marshakov. Combinatorially, non-torsion elements of $G_N$ are ways of shuffling the underlying bipartite graph, generalizing domino-shuffling. Algebro-geometrically, $G_N$ is a subgroup of the Picard group of a certain algebraic surface associated to $N$.

1 Introduction

Domino-shuffling is a technique introduced in [EKLP] to enumerate and generate domino tilings of the Aztec diamond graph, and was used to give the first proof of the arctic circle theorem [JPS]. Domino tilings are dual to the dimer model on the square grid. There are generalizations of domino-shuffling, called (2-2) cluster modular transformations for other biperiodic bipartite graphs and they form a group called the (2-2) cluster modular group. This group was studied by Fock and Marshakov [FM16], who gave an explicit conjecture for its isomorphism type. The goal of this paper is to study these generalized shufflings, and in particular, to compute the cluster modular group for any biperiodic bipartite graph.

(2-2) cluster modular transformations give rise to dynamical systems on the space of weights. Let $\Gamma$ be a bipartite graph on a torus $T$ and let $L_\Gamma := H^1(\Gamma, \mathbb{C}^\times)$ be the space of weights on $\Gamma$ (cf. section 2.2). There are two types of local rearrangements of bipartite graphs called elementary transformations (see Figure 1). Each elementary transformation has an associated birational map of weights, characterized by
the property that it preserves the dimer partition function up to a constant scaling factor (see for example [GK12, Theorem 4.7]). Given a sequence of elementary transformations such that the initial and final graphs are both $\Gamma$ (called a cluster transformation), composing the induced birational maps of weights gives a birational automorphism of $L_{\Gamma}$. The cluster transformation is trivial if this induced map on weights is the identity. The (2-2) cluster modular group is the group of cluster transformations modulo the trivial ones.

A zig-zag path in $\Gamma$ is a path that turns maximally left at white vertices and maximally right at black vertices (see Figure 3). Associated to any bipartite graph on a torus $T$ is a convex integral polygon $N$ called its Newton polygon, whose primitive edges correspond to homology classes of zig-zag paths in $T$. The (2-2) cluster modular group is determined by $N$. Elementary transformations have an appealing description in terms of homotopy of zig-zag paths (see Figure 4 and section 2.1).

Let $\tilde{\Gamma}$ be the planar biperiodic graph whose quotient under the translation action of $H_1(T, \mathbb{Z})$ is $\Gamma$, that is, the preimage of $\Gamma$ in the universal cover of $T$. We can lift a cluster transformation to an $H_1(T, \mathbb{Z})$-periodic sequence of elementary transformations from $\tilde{\Gamma}$ to itself. If we superpose $\tilde{\Gamma}$ over itself after the cluster transformation, the lift of each zig-zag path is superposed over a lift of a zig-zag path with the same homology class. Following Fock and Marshakov [FM16], we can associate an integer function $f$ on the edges of the Newton polygon $N$ as follows: for any edge $E$ of $N$, the inverse image in the universal cover of the torus of all zig-zag paths corresponding to

Figure 1: Elementary transformations along with induced birational maps of weight tori.
Figure 2: The cluster modular transformation called domino-shuffling.

Figure 3: A zig-zag path (solid red) and its representation as a path in the medial graph (dashed red).

Figure 4: Equivalence of elementary transformations and $2 - 2$ moves.
Figure 5: The Newton polygon along with the function $f$ for the cluster modular transformation in Figure 2. The yellow edge corresponds to the yellow zig-zag path in Figure 2, which is translated one step to the left during the cluster modular transformation.

$E$ (that is all zig-zag paths whose homology classes are in the direction of $E$) is an infinite collection of parallel zig-zag paths in $\tilde{\Gamma}$; let us label them by $(\alpha^i)_{i \in \mathbb{Z}}$, ordered along the direction normal to $E$ and pointing out of $N$. Consider the zig-zag path $\alpha^0$: after the cluster transformation, if we superpose $\tilde{\Gamma}$ over itself, $\alpha^0$ is superposed over a parallel zig-zag path, say $\alpha^j$. We define $f(E)$ to be $-j$, which is the distance that this zig-zag path (and therefore any parallel zig-zag path $\alpha^i$) is translated by the cluster transformation. For example, Figure 5 shows the function $f$ for domino-shuffling. The function so defined satisfies

$$\sum_{\text{Edges } E \text{ of } N} f(E) = 0. \quad (1)$$

Let us denote by $\mathbb{Z}^\text{Edges of } N$ the group of integral functions on the edges of $N$ satisfying (1). There is an ambiguity in superposing $\tilde{\Gamma}$ over itself, because we can translate by $H_1(\mathbb{T}, \mathbb{Z})$. Therefore to make $f$ a well-defined function of the cluster transformation, we need to consider it as an element of the quotient

$$\mathbb{Z}^\text{Edges of } N / H_1(\mathbb{T}, \mathbb{Z}),$$

where the embedding is given by the distance zig-zag paths in $\tilde{\Gamma}$ are translated by elements of $H_1(\mathbb{T}, \mathbb{Z})$:

$$H_1(\mathbb{T}, \mathbb{Z}) \hookrightarrow \mathbb{Z}^\text{Edges of } N \quad \gamma \mapsto \langle E \mapsto \langle E, \gamma \rangle \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the intersection form in $H_1(\mathbb{T}, \mathbb{Z})$. Figure 2 shows the relative positions of a zig-zag path corresponding to the yellow edge of $N$ in Figure 3 before and after the cluster transformation from Figure 2. Our main result is the following conjecture of Fock and Marshakov [FM16]:
Theorem 1.1 (cf. Theorem 4.6). If the Newton polygon $N$ has an interior lattice point, the $(2-2)$ cluster modular group is isomorphic to:

$$\mathbb{Z}^{\text{Edges of } N}_{0}/H_1(\mathbb{T}, \mathbb{Z}).$$

When there is no interior lattice point, the $(2-2)$ cluster modular group is a smaller finite group.

In particular, the rank of the $(2-2)$ cluster modular group depends only on the number of sides of $N$:

Corollary 1.2. When $N$ contains an interior lattice point, the rank of the $(2-2)$ cluster modular group is $|E_N| - 3$, where $|E_N|$ is the number of edges of the polygon $N$. When $N$ has no interior lattice points, the rank is zero.

Informally, while the collection of all zig-zag paths undergoes a complex sequence of moves, if we restrict attention to the set of zig-zag paths in a specific homology direction, no two zig-zag paths in this set can cross during a cluster transformation. Therefore this set of zig-zag paths as a whole undergoes a translation. The function $f$ defined above records these translations, and remarkably, if we can essentially reconstruct the entire cluster transformation from $f$.

The proof of Theorem 1.1 has two parts. In Section 3, we show that every element of $\mathbb{Z}^{\text{Edges of } N}_{0}/H_1(\mathbb{T}, \mathbb{Z})$ arises from a cluster transformation. This part of the theorem is purely combinatorial.

Translations by elements of $H_1(\mathbb{T}, \mathbb{Z})$ clearly give rise to trivial cluster transformations. The second part of the proof of the Theorem shows that these are the only trivial cluster transformations. It is difficult to directly check if the induced birational map of weights is the identity. However, integrability of the space of weights gives rise to a local reparameterization such that the induced birational map of weights is linearized. Kenyon and Okounkov [KO03] defined the spectral transform of $wt \in \mathcal{L}_\Gamma$ to be a triple $(C, S, \nu)$, where $C \subset (\mathbb{C}^\ast)^2$ is a curve called the spectral curve and $S$ is a divisor of degree $g$ equal to the genus of $C$, that is a formal linear combination of $g$ points in $C$. $C$ is the vanishing locus of a Laurent polynomial $P(z, w)$ which is a homology-class-weighted version of the partition function for dimer covers. The spectral transform is birational [F15, GGK], so we can view $(C, S, \nu)$ as a local reparameterization of $\mathcal{L}_\Gamma$. For a fixed $C$, the space of degree $g$ effective divisors is birational to the Jacobian of $C$, and in this parameterization, every cluster transformation is a translation of the divisor $S$ in (a finite cover of) the Jacobian variety of
Figure 6: The black point on the left is the divisor $S$ on the amoeba of the spectral curve. The points at infinity of the curve are in bijection with zig-zag paths and coloured according to Figure 5. The cluster transformation in Figure 2 maps the black point to the pink point. Fock [F15] shows that this map is the translation shown below the figure in the Jacobian variety of the spectral curve. This translation is determined by the function $f$ shown in Figure 5.

Therefore the question of which cluster transformations are trivial is answered by the following.

**Theorem 1.3** (cf. Theorem A.11). Assume $N$ has an interior lattice point. If $L$ is a non-trivial line bundle on the projective toric surface $X_N$ associated to $N$, then for a generic spectral curve $C$, we have $L|_C \not\cong \mathcal{O}_C$.

This is proved by Giovanni Inchiostro in the appendix.

In the last paragraph of [FM16, Section 7.3], Fock and Marshakov provide an alternate description of $\mathbb{Z}^{\text{Edges of } N}_{\text{gcd}} / H_1(\mathbb{T}, \mathbb{Z})$ as the group of divisor classes on the toric surface $X_N$ that restrict to degree 0 divisors on a generic spectral curve $C$. However this is only true as stated for polygons whose sides are all primitive, that is, no side contains a lattice point other than the end points. Recently Treumann, Williams and Zaslow [TWZ18] gave a different version of linearization of cluster modular transformations under the spectral transform, replacing the toric variety $X_N$ by a toric stack $\mathcal{X}_N$.

**Proposition 1.4** (cf. Proposition 4.8). When the Newton polygon $N$ has an interior lattice point, the (2-2) cluster modular group is isomorphic to a certain subgroup of $\text{Pic}(\mathcal{X}_N)$. 

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We end the introduction by describing the (2-2) cluster modular groups for some small Newton polygons.

**Triangles** For triangular $N$, [IU15, Proposition 11.3] tells us that there is a unique bipartite graph in $\mathbb{T}$ with Newton polygon $N$ and its lift to the plane is the honeycomb lattice. Since this graph does not admit any elementary transformations, the only cluster modular transformations are translations.

**Quadrilaterals** Corollary 1.2 tells us that the cluster modular group has rank one. The dimer models that have quadrilateral Newton polygons coincide with those that arise from Speyer’s “crosses and wrenches” construction [Spey04]. The octahedron recurrence studied there is the (essentially unique) non-torsion cluster modular transformation (on the $A$ cluster variety). Other incarnations of cluster modular transformations for quadrilateral $N$ are Hirota’s bilinear difference equation [Miwa82], the domino-shuffling algorithm [EKLP, Propp03], the shuffling studied in [BF18] for the suspended pinch-point graph and the pentagram map [FM16 Section 8.5].

The octahedron recurrence can be used to compute arctic curves [PS06, DFS14]. We observed in [G18] that part of the data needed for this technique of computing arctic curves is a cluster modular transformation along with edge-weights that are periodic under the induced birational map. We hope that understanding the cluster modular group will help generalize this method beyond the quadrilateral Newton polygon case. Since higher degree polygons have cluster modular groups with rank greater than one by Corollary 1.2, we expect a family of arctic curves, one for each non-torsion cluster modular transformation.

**Higher degree polygons** Cluster modular transformations for the $dP_2$ quiver, which has a pentagon Newton polygon, were explicitly studied in [GLVY16]. The $dP_3$ quiver with a hexagonal Newton polygon has been studied in [LMNT14, LM17, LM19]. The cube recurrence studied in [CS04, PS06] arises as the restriction to the resistor network subvariety of a cluster modular transformation on the $dP_3$ graph [GK12 Section 6.3].

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2 Background

See [GK12] for further background about the objects described in this section.

Basic notation. Let $\mathbb{T}$ be a topological torus, and let $T := H_1(\mathbb{T}, \mathbb{Z})^\vee \otimes \mathbb{Z} \mathbb{C}^\times$ be the algebraic torus with group of characters $H_1(\mathbb{T}, \mathbb{Z})$. Given a convex integral polygon $N \subset H_1(\mathbb{T}, \mathbb{R})$, that is, a convex polygon whose vertices are in $H_1(\mathbb{T}, \mathbb{Z})$, we denote by $V_N$ and $E_N$ the vertices and edges of $N$ respectively.

Let $\Sigma \subset H_1(\mathbb{T}, \mathbb{Z})^\vee \otimes \mathbb{Z} \mathbb{R}$ denote the dual fan of $N$. Let $\Sigma(r)$ denote the $r$-dimensional faces of $\Sigma$. Let $u_\rho$ be the primitive integral vector along the ray $\rho \in \Sigma$. Let $E_\rho$ denote the edge of $N$ that is dual to $\rho$. Let $|E_\rho|$ be its integral length, defined as the number of primitive integral vectors in $E_\rho$.

2.1 Combinatorial objects

Bipartite torus graphs. A bipartite graph is a graph whose vertices are colored black or white, such that each edge is incident to a black and a white vertex. A bipartite torus graph is a bipartite graph $\Gamma$ embedded in $\mathbb{T}$ such that the faces of $\Gamma$, that is, the connected components of $\mathbb{T} - \Gamma$, are contractible. We denote by $B(\Gamma)$ and $W(\Gamma)$ the black and white vertices of $\Gamma$ respectively.

Zig-zag paths and minimality. A zig-zag path in $\Gamma$ is an oriented path in $\Gamma$ that turns maximally left at white vertices and maximally right at black vertices. We usually represent a zig-zag path by an oriented path in the medial graph that passes consecutively through the edges of the zig-zag path (see Figure 3). $\Gamma$ is said to be minimal if, in the preimage $\tilde{\Gamma}$ of $\Gamma$ in the universal cover $H_1(\mathbb{T}, \mathbb{R})$ of $\mathbb{T}$, zig-zag paths...
Figure 8: The Newton polygon and zig-zag paths for the graph in Figure 7.
have no self-intersections and there are no parallel bigons, that is, pairs of zig-zag paths oriented the same way intersecting at two points. The unique convex integral polygon $N(\Gamma) \subset H_1(\mathbb{T}, \mathbb{R})$ whose primitive integral edges are given by the homology classes of zig-zag paths in counterclockwise cyclic order is called the Newton polygon of $\Gamma$. We usually abbreviate $N(\Gamma)$ to $N$ when the graph is clear from context.

We label the edges of $N$ by rays of the dual fan: the edge corresponding to $\rho \in \Sigma(1)$ is denoted by $E_\rho$. We denote by $Z_\rho$ the set of zig-zag paths whose homology classes are the primitive vectors contained in the edge $E_\rho$.

**Elementary transformations.** There are two local rearrangements of bipartite torus graphs called *elementary transformations*:

1. Spider moves (Figure 1a);

2. Shrinking/expanding 2-valent white vertices (Figure 1b).

We say that two bipartite torus graphs $\Gamma_1$ and $\Gamma_2$ are *topologically equivalent* if there is a sequence of elementary transformations that converts the graph $\Gamma_1$ into $\Gamma_2$. Applying either of the elementary transformations twice gives back the original graph, and therefore this is an equivalence relation on bipartite torus graphs. Elementary transformations are local and do not change homology classes of zig-zag paths. Therefore they leave the Newton polygon invariant and so

$$\{\text{Minimal bipartite torus graphs}\}/\text{topological equivalence} \xrightarrow{\Gamma \mapsto N(\Gamma)} \{\text{Convex integral polygons in } H_1(\mathbb{T}, \mathbb{R})\}, \quad (2)$$

is a well-defined function.

**Theorem 2.1** (Goncharov and Kenyon, 2012 [GK12, Theorem 2.5]). *The function in (2) which associates to a graph its Newton polygon is a bijection.*

In other words, for each convex integral polygon in $H_1(\mathbb{T}, \mathbb{R})$, there is a family of minimal bipartite torus graphs associated to $N$, and any two members of a family are related by elementary transformations.

**Triple point diagrams.** A *triple point diagram* in a disk $\mathbb{D}$ is a collection of oriented curves called *strands*, defined up to isotopy, such that:

1. Three strands meet at each intersection point.
2. The end points of each strand are distinct boundary points.

3. The orientations on the strands induce consistent orientations on the complementary regions.

Each strand starts and ends in ∂D, so if there are n strands, there are 2n points in ∂D, whose orientations alternate “in” and “out” around ∂D. A triple point diagram is minimal if strands have no self intersections and parallel bigons.

There is a local move called a 2-2 move on triple point diagrams (see Figure 9).

**Theorem 2.2** (Thurston, 2004 [Thur04], Postnikov, 2006 [Post06]). Suppose we have a disk D with 2n points in its boundary alternately labeled “in” and “out”.

1. For any of the n! matchings of “in” and “out” points, there is a minimal triple point diagram that realizes the matching.

2. Any two minimal triple point diagrams with the same boundary matching of “in” and “out” points are related by 2-2 moves.

In the course of proving Theorem 2.2, Thurston proves the following result that we will require later.

**Proposition 2.3** (Thurston, 2004 [Thur04], Section 2). Let α, β, γ be three strands that correspond to three consecutive points on the boundary of D. Then there is a triple crossing diagram (called standard in [Thur04]) in which α, β and γ meet at a triple point just adjacent to the boundary (that is, this is the first triple point of each of these strands as we look along the strand starting at this boundary point).
**Triple point diagrams in** $\mathbb{T}$. A triple point diagram in $\mathbb{T}$ is a collection of oriented curves called *strands* in $\mathbb{T}$, determined up to isotopy, such that:

1. Three strands meet at each intersection point.

2. No strand is a homologically trivial loop in $\mathbb{T}$.

3. The orientations on the strands induce consistent orientations on the complementary regions.

A triple point diagram in $\mathbb{T}$ is *minimal* if the lift of any strand to the plane has no self-intersections and the lifts of any two strands to the universal cover form no parallel bigons.

**Equivalence of triple point diagrams and bipartite torus graphs in** $\mathbb{T}$. We recall the equivalence between minimal triple point digarams in $\mathbb{T}$ and minimal bipartite torus graphs from [GK12]:

1. To convert a minimal bipartite torus graph to a triple point diagram, we first expand all black vertices with degree greater than or equal to 4 by moves inverse to shrinking a degree 2 white vertex to get a graph in which all black vertices have degree 3. Then we draw all zig-zag paths so that the black complementary regions are now triangles. Finally we shrink all these black triangle regions into points to get a triple point diagram.

2. To construct a bipartite graph from a triple point diagram, we start by resolving each triple point into a counterclockwise triangle. Put a black vertex in each complementary region that is oriented counterclockwise and a white vertex in each complimentary region that is oriented clockwise. Edges between black and white vertices are given by the vertices of the resolved triple point diagram. The faces of the bipartite graph will be the regions where the orientations alternate.

Under this correspondence, we have:

\[
\text{Minimal bipartite torus graphs } \leftrightarrow \text{Minimal triple point diagrams in } \mathbb{T},
\]

\[
\text{Zig-zag paths } \leftrightarrow \text{Strands},
\]

\[
\text{Elementary transformations } \leftrightarrow (2-2) \text{ moves}.
\]

**2.2 The dimer model**

In this section, we introduce the dimer model, mostly following [GK12].
Weights on bipartite torus graphs. We associate to \( \Gamma \) the torus of weights
\[
\mathcal{L}_\Gamma := H^1(\Gamma, \mathbb{C}^\times).
\]
A 1-cocycle representing \( wt \in H^1(\Gamma, \mathbb{C}^\times) \) is called an edge-weight. For \( L \in H_1(\Gamma, \mathbb{Z}) \), we denote the pairing of cohomology and homology by \( wt(L) \).

For a face \( f \) of \( \Gamma \), we denote by \( \partial f \) the counterclockwise oriented boundary of \( f \). We define
\[
X_f := wt(\partial f).
\]
They satisfy the unique relation \( \prod_f X_f = 1 \), arising from the relation \( \sum_f \partial f = 0 \) in \( H_1(\Gamma, \mathbb{Z}) \).

Conjugated surface and the \( \epsilon \) form. Given a bipartite torus graph \( \Gamma \), by puncturing each face, we obtain a ribbon graph. From this ribbon graph, we can construct a new ribbon graph \( \hat{\Gamma} \) by twisting the ribbon structure at all white vertices. Gluing in the faces of \( \hat{\Gamma} \), we obtain a surface \( \hat{S} \) of genus \( g \) called the conjugated surface, where \( g \) is the number of interior lattice points in \( N \). Since \( \hat{\Gamma} \cong \Gamma \) graphs, we can define a skew-symmetric bilinear form \( \epsilon : H_1(\Gamma, \mathbb{Z}) \times H_1(\Gamma, \mathbb{Z}) \) as follows:

If \( L_1, L_2 \in H_1(\Gamma, \mathbb{Z}) \), using \( \hat{\Gamma} \cong \Gamma \), we can view them as loops in \( H_1(\hat{\Gamma}, \mathbb{Z}) \). Using the embedding \( \hat{\Gamma} \hookrightarrow \hat{S} \), they are loops in \( \hat{S} \). Let \( \langle \cdot, \cdot \rangle_{\hat{S}} \) denote the intersection form on \( \hat{S} \). Define \( \epsilon(L_1, L_2) := \langle L_1, L_2 \rangle_{\hat{S}} \).

Mutations. Elementary transformations \( s : \Gamma_1 \to \Gamma_2 \) bipartite torus graphs induce birational maps of weights \( \mu_s : \mathcal{L}_{\Gamma_1} \to \mathcal{L}_{\Gamma_1} \) described below. In both cases, there is a canonical identification, which we also call \( s \), of \( H_1(\Gamma_1, \mathbb{Z}) \) with \( H_1(\Gamma_2, \mathbb{Z}) \).

1. Spider move at face \( f \): We define \( \mu_s \) by:
\[
\mu_s(wt)(L) = wt(s^{-1}(L)) \left( 1 + wt(f) \frac{\text{sign} \epsilon(s^{-1}(L), \partial f)}{\epsilon(s^{-1}(L), \partial f)} \right). 
\]
See Figure 1a for how the weights of the faces involved transform.

2. Shrinking/expanding degree two white vertices: see Figure 1b. We define \( \mu_s \) by:
\[
\mu_s(wt)(L) = wt(s^{-1}(L)).
\]
The dimer cluster variety $\mathcal{X}_N$. Suppose $N$ is a convex integral polygon in $H_1(\mathbb{T}, \mathbb{R})$. By theorem 2.1 there is a family of minimal bipartite torus graphs with Newton polygon $N$ that are related by elementary transformations. Associated with each graph $\Gamma$ in the family is its torus of weights $\mathcal{L}_\Gamma$. Gluing the $\mathcal{L}_\Gamma$ using the birational maps induced by the elementary transformations, we obtain a scheme $\mathcal{X}_N$ called the dimer cluster variety.

The (2-2) cluster modular group. We say that two bipartite torus graphs $\Gamma_1$ and $\Gamma_2$ are isotopic if there is an isotopy in $\mathbb{T}$ relating $\Gamma_1$ and $\Gamma_2$. A (2-2) cluster transformation $t : \Gamma_0 \to \Gamma_n$ is a sequence:

$$\Gamma_0 \xrightarrow{s_0} \Gamma_1 \xrightarrow{s_1} \cdots \xrightarrow{s_{n-1}} \Gamma_n,$$

where each $s_i$ is an elementary transformation or an isotopy in $\mathbb{T}$. A (2-2) cluster transformation $t$ induces a birational map $\mu_t$ of weight tori by composition:

$$\mu_t := \mu_{s_{n-1}} \circ \cdots \circ \mu_{s_0} : \mathcal{L}_{\Gamma_0} \to \mathcal{L}_{\Gamma_n}.$$

A (2-2) cluster transformation $\Gamma \to \Gamma$ is called trivial if the induced birational map of weight tori is the identity. The groupoid $\mathcal{G}_N$ whose objects are minimal bipartite torus graphs $\Gamma$ with Newton polygon $N$ and morphisms are (2-2) cluster transformations modulo trivial (2-2) cluster transformations is called the (2-2) cluster modular groupoid of $\mathcal{X}_N$. The fundamental group $G_N$ of $\mathcal{G}_N$ is called the (2-2) cluster modular group and its elements are called (2-2) cluster modular transformations. Although we need a base point $\Gamma$ to define the fundamental group $G_N$, a different choice of base point gives an isomorphic group. Elements of $G_N$ are also called discrete flows in [FM16].

Dimer covers. A dimer cover or perfect matching of $\Gamma$ is a collection of edges of $\Gamma$ such that each vertex of $\Gamma$ is incident to exactly one edge in the collection. By orienting each edge from its black vertex to its white vertex, we can view each dimer as a 1-chain in $\Gamma$. Fix a dimer cover $M_0$ which we call the reference dimer cover. Then we can associate to each dimer cover $M$ a homology class $[M - M_0] \in H_1(\mathbb{T}, \mathbb{Z})$ and weight $\text{wt}([M - M_0])$. The Newton polygon $N$ has the following description in terms of dimer covers.

**Proposition 2.4 ([GK12, Theorem 3.12]).** Suppose $\Gamma$ is a minimal bipartite torus graph with Newton polygon $N$. Up to a translation in $H_1(\mathbb{T}, \mathbb{R})$, we have:

$$N = \text{Convex-hull} \{[M - M_0] : M \text{ is a dimer cover of } \Gamma \}.$$
**Kasteleyn theory.** Let \( R \) be a fundamental rectangle for \( T \). Let \( \gamma_z, \gamma_w \) be the oriented sides of \( R \) generating \( H_1(T, \mathbb{Z}) \), as shown in Figure 7. To each edge \( e \) of \( \Gamma \), we associate a character
\[
\varphi(e) = z^{(e, \gamma_w)} w^{(e, -\gamma_z)},
\]
where we consider the edge \( e \) to be oriented from its black vertex to its white vertex and \((\cdot, \cdot)\) is the intersection index.

\( \kappa \in H^1(\Gamma, \mathbb{C}^\times) \) is called a **Kasteleyn sign** if:

1. \( \kappa(L) = \pm 1 \) for all \( L \in H_1(\Gamma, \mathbb{Z}) \).
2. \( \kappa(\partial f) = (-1)^{l/2 + 1} \) if \( f \) is a face of \( \Gamma \) containing \( l \) edges in its boundary.

The **Kasteleyn matrix**
\[
K(z, w) : \mathbb{C}[z^{\pm 1}, w^{\pm 1}]^{B(\Gamma)} \to \mathbb{C}[z^{\pm 1}, w^{\pm 1}]^{W(\Gamma)}
\]
is defined as
\[
K(z, w)_{w,b} = \sum_{e \in E(\Gamma) \text{ incident to } w,b} \kappa(e) \varphi(e) z^{(e, \gamma_w)} w^{(e, -\gamma_z)},
\]
where \( \kappa, \varphi \) are any 1-cocycles representing their cohomology classes.

**Theorem 2.5** (Kasteleyn 1963, [Kast63]).

\[
\frac{\det K(z, w)}{\text{wt}(M_0) z^{(M_0, \gamma_w)} w^{(M_0, -\gamma_z)}} = \sum_{\text{dimer cover of } \Gamma} \text{sign}(M) \text{wt}([M - M_0])(z, w)^{M - M_0},
\]
where \( \text{sign}(M) \in \{\pm 1\} \) is a sign that depends on the homology class \([M - M_0]\) and \( \kappa \).

\[
P(z, w) := \frac{\det K(z, w)}{\text{wt}(M_0) z^{(M_0, \gamma_w)} w^{(M_0, -\gamma_z)}},
\]
is called the **characteristic polynomial**, and its vanishing locus \( C_0 := \{(z, w) \in (\mathbb{C}^\times)^2 : P(z, w) = 0\} \) is called the (open) **spectral curve**. Note that while the Kasteleyn matrix depends on the choice of 1-cocycles representing the cohomology classes \( \varphi, \kappa \) and the choice of a reference matching \( M_0 \), the spectral curve is independent on these choices. By Proposition 2.4, the Newton polygon of \( P(z, w) \) coincides with the Newton polygon of \( \Gamma \).
**A construction of Fock and Marshakov.** We follow [FM16, Section 7.3]. Let $Z_0^{\Sigma(1)}$ be the group of integer valued functions $f$ on $\Sigma(1)$ such that $\sum_{\rho \in \Sigma(1)} f(\rho) = 0$. Let $\langle \cdot, \cdot \rangle_T : H_1(T, \mathbb{Z}) \times H_1(T, \mathbb{Z}) \to \mathbb{Z}$ be the intersection pairing in $T$. We have an embedding

$$j : H_1(T, \mathbb{Z}) \hookrightarrow Z_0^{\Sigma(1)}$$

$$\gamma \mapsto \left( \sum_{\alpha \in Z_\rho} \langle [\alpha], \gamma \rangle_T \right)_{\rho \in \Sigma(1)}$$

Let $\Gamma$ be a bipartite torus graph and let $T$ be its triple point diagram. A cluster transformation $\Gamma \rightarrow \Gamma$ is equivalent to a sequence of triple point diagrams

$$T = T_0 \rightarrow T_1 \rightarrow \cdots \rightarrow T_{n-1} \rightarrow T_n \cong T,$$  \hspace{1cm} (4)

where $T_{i+1}$ is obtained from $T_i$ by either performing a 2-2 move or $T_{i+1}$ is related to $T_i$ by an isotopy in $T$. Let $\{\alpha^i\}$ be the set of strands in $T$. The sequence (4) can be interpolated by a one parameter family of curves $\alpha^i(t)$ in $T$, where $t \in [0,1]$ such that $\alpha^i(0) = \alpha^i$ and such that the intersections remain triple at all but $n - 1$ parameter values where we have a quadruple intersection in the course of a 2-2 move. Using the isomorphism of triple point diagrams $T = T_0 \cong T_n$, we glue the end points of the parameter interval $[0,1]$ to get an $S^1$. During the course of the sequence (4), each strand $\alpha$ in $T$ traces out a 2-chain $S_\alpha := \{(u, t) : u \in \alpha(t), t \in S^1\}$ in $T \times S^1$.

Let $Z_\rho = \{\alpha^i_\rho\}_{i=1}^{|E_\rho|}$ be the strands in $T$ corresponding to $\rho \in \Sigma(1)$. The cluster transformation maps each strand $\alpha \in Z_\rho$ bijectively to another strand in $Z_\rho$, and therefore $\partial(\sum_i S_{\alpha^i_\rho}) = 0$. Moreover, $\sum_\rho \sum_i S_{\alpha^i_\rho}$ is a 2-boundary: it is the boundary of the 3-chain in $T \times S^1$ traced out by the regions of $T$ corresponding to white vertices of $\Gamma$. Therefore we have

$$\sum_\rho \sum_i [S_{\alpha^i_\rho}] = 0, \text{ in } H_2(T \times S^1, \mathbb{Z}).$$  \hspace{1cm} (5)

Let $(\gamma_z, \gamma_w)$ be the standard basis for $H_1(T, \mathbb{Z})$ from Figure 7 and $\gamma_t$ be a generator of $H_1(S^1, \mathbb{Z})$. By the Künneth formula, $H_2(T \times S^1, \mathbb{Z}) \cong \Lambda^2 Z[\gamma_z, \gamma_w]$. If a strand $\alpha^i_\rho \in Z_\rho$ with $[\alpha^i_\rho] = X_\rho \gamma_z + Y_\rho \gamma_w$ is translated by $a_\rho \gamma_z + b_\rho \gamma_w$ during the sequence (4), then

$$[S_{\alpha^i_\rho}] = (X_\rho \gamma_z + Y_\rho \gamma_w) \wedge (a_\rho \gamma_z + b_\rho \gamma_w + \gamma_t)$$

$$= (b_\rho X_\rho - a_\rho Y_\rho) \gamma_z \wedge \gamma_w + X_\rho \gamma_z \wedge \gamma_t + Y_\rho \gamma_w \wedge \gamma_t.$$  \hspace{1cm} (6)
Define a function 
\[
g : \Sigma(1) \to \mathbb{Z} \\
\rho \mapsto |E_\rho|(b_\rho X_\rho - a_\rho Y_\rho).
\]

Informally, each zig-zag path in \(\Gamma\) is translated in the universal cover to a parallel zig-zag path by the cluster transformation. \(g(\rho)\) is the number of steps in the direction of \(\rho\) that any zig-zag path in \(Z_\rho\) is translated. Writing (5) in coordinates using (6), we get
\[
\left( \sum_{\rho \in \Sigma(1)} |E_\rho|(b_\rho X_\rho - a_\rho Y_\rho) \right) \gamma_z \wedge \gamma_w + \left( \sum_{\rho \in \Sigma(1)} X_\rho \gamma_z + Y_\rho \gamma_w \right) \wedge \gamma_t = 0.
\]

We have \(\left( \sum_{\rho \in \Sigma(1)} X_\rho \gamma_z + Y_\rho \gamma_w \right) = 0\) because this is the sum of counterclockwise oriented edges of the Newton polygon. Since \(\sum_{\rho \in \Sigma(1)} |E_\rho|(b_\rho X_\rho - a_\rho Y_\rho) = 0\), we get \(g \in \mathbb{Z}_0^{\Sigma(1)}\).

The above construction gives us a group homomorphism \(\psi\) defined as the composition
\[
\{\text{Cluster transformations } \Gamma \to \Gamma\} \to \mathbb{Z}_0^{\Sigma(1)} \to \mathbb{Z}_0^{\Sigma(1)}/jH_1(\Gamma, \mathbb{Z}).
\]

3 Surjectivity of \(\psi\)

In this section we show that the group homomorphism \(\psi\) defined in (7) is surjective. Given an element of \(f \in \mathbb{Z}_0^{\Sigma(1)}\), we will construct a cluster transformation \(t_f\) such that \(\psi(t_f) = f\).

A construction of Goncharov and Kenyon. We recall the construction of a minimal bipartite torus graph with Newton polygon \(N\) from [GK12]. We require that the graph has two additional properties that are not explicitly mentioned in [GK12], but are immediate consequences of the construction. Suppose the torus \(\mathbb{T}\) is constructed by gluing opposite sides of a rectangle \(R\). We label the north, west, south and east sides of \(R\) by \(\partial R_N, \partial R_W, \partial R_S, \partial R_E\) respectively. For each ray \(\rho \in \Sigma(1)\), let \(X_\rho \gamma_z + Y_\rho \gamma_w\) the primitive edge vector in the direction of \(E_\rho\), where \(\gamma_z, \gamma_w\) are the generators of \(H_1(\mathbb{T}, \mathbb{Z})\) that are given by the sides of \(R\) oriented as in Figure 7. For each \(\rho \in \Sigma(1)\), draw loops \(\{\alpha_{\rho i}^{E_\rho}\}_{i=1}^{|E_\rho|}\) in \(\mathbb{T}\), each with homology class \(X_\rho \gamma_z + Y_\rho \gamma_w\) so
that the total number of intersections of any loop with the boundary of \( R \) is minimal. Isotope the loops in \( \mathbb{T} \) so that:

1. The intersections of the loops with each side of \( R \) alternate in orientation, “in” and “out”.
2. The west-most point on \( \partial R_N \) is an “out” point.
3. We do not introduce any new intersection points of loops with \( \partial R \) during the isotopy.

Using Theorem 2.2 we can isotope the loops in \( R \) to obtain a minimal triple crossing diagram in \( R \) with the same boundary matching. Using the procedure outlined in Section 2.1, we convert it to a minimal bipartite torus graph.

**Proposition 3.1** ([GK12]). For a convex integral polygon \( N \), there is a minimal bipartite torus graph \( \Gamma \) with Newton polygon \( N \) satisfying:

1. The west-most intersection point of a strand with \( \partial R_N \) is an “out” point.
2. The number of intersections of each zig-zag path with the boundary of \( R \) is the smallest possible for a minimal triple point diagram with Newton polygon \( N \).

We require the following lemma that is contained in the proof of [GK12, Theorem 2.5]. We include the proof because it is short and illustrative of the type of arguments we will make later.

**Lemma 3.2.** Suppose \( T \) is a triple point diagram in \( \mathbb{T} \). The relative order along the boundary of \( R \) of strands associated to the same ray of \( \Sigma \) is fixed. The relative order of two incoming or outgoing strands associated to different edges of \( N \) can be interchanged by 2-2 moves and isotopy.
Proof. Suppose $\alpha$ and $\gamma$ are two consecutive “out” strands in $T$ that correspond to different rays of $\Sigma$. Then by the alternating property, there is an “in” strand $\beta$ of $T$ between them. Since $\alpha$ and $\gamma$ belong to different rays of $\Sigma$, they must cross at a triple point inside $R$. By Proposition 2.3, there is a triple point diagram $T'$ in which the three strands $\alpha, \beta, \gamma$ meet at a triple point just adjacent to the boundary. By Theorem 2.2, we can use 2-2 moves and isotopy to convert $T$ into $T'$. Then we isotope this triple point across the boundary $\partial R$, which permutes boundary intersections of $\alpha$ and $\gamma$, as illustrated in Figure 10. \qed

Change of basis for $H_1(\Gamma, \mathbb{Z})$. Let $X_\rho \gamma_z + Y_\rho \gamma_w$ be the homology class of a zig-zag path in $Z_\rho$ in the basis $(\gamma_z, \gamma_w)$ of $H_1(T, \mathbb{Z})$ from Figure 7. Changing the basis, or equivalently, changing the fundamental rectangle $R$ of $T$ corresponds to the action of $SL(2,\mathbb{Z})$ on $H_1(T, \mathbb{Z})$. $SL(2,\mathbb{Z})$ is generated by

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$  

Let $g \cdot R$ denote the fundamental parallelogram with boundary formed by the vectors $g \cdot \gamma_z$ and $g \cdot \gamma_w$. We describe the action of some elements of $SL_2(\mathbb{Z})$ explicitly.

1. In the basis $(S \cdot \gamma_z, S \cdot \gamma_w)$, the vector $a\gamma_z + b\gamma_y$ has coordinates

$$S^{-1} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ -a \end{pmatrix}.$$  

Therefore the new coordinates are obtained from the old coordinates by rotating clockwise by $\frac{\pi}{2}$.

2. In the basis $(T \cdot \gamma_z, T \cdot \gamma_w)$, the vector $a\gamma_z + b\gamma_y$ has coordinates

$$T^{-1} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a + b \\ b \end{pmatrix}.$$  

Therefore $T$ is a shear.

3. Define

$$U := -TST = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$  

In the basis $(U \cdot \gamma_z, U \cdot \gamma_w)$, the vector $a\gamma_z + b\gamma_y$ has coordinates

$$U^{-1} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ a + b \end{pmatrix}.$$  

Therefore $U$ is also shear.
Proof of surjectivity. The main result of this section is:

**Theorem 3.3.** The group homomorphism

\[ \psi : \{ \text{Cluster transformations} \, \Gamma \rightarrow \Gamma \} \rightarrow \mathbb{Z}^{\Sigma_0(1)}/jH_1(\mathbb{T}, \mathbb{Z}), \]

defined in [7] is surjective.

**Proof.** Let \( \rho, \sigma \in \Sigma(1) \) be two consecutive rays in counterclockwise cyclic order. Since the functions \( \delta_{\rho} - \delta_{\sigma} \) generate \( \mathbb{Z}^{\Sigma_0(1)} \), it suffices to show that there is a cluster transformation \( t \) such that \( \psi(t) = \delta_{\rho} - \delta_{\sigma} \).

Let \( (X_{\rho}, Y_{\rho}) \) and \( (X_{\sigma}, Y_{\sigma}) \) be the homology classes of strands in \( Z_{\rho}, Z_{\sigma} \) respectively in the basis \( (\gamma_z, \gamma_w) \). Changing the basis by repeatedly using \( T \) or \( U \), we may assume that \( (X_{\rho}, Y_{\rho}) \) is neither horizontal nor vertical. Then, rotating if necessary using \( S \), we can assume that \( X_{\rho}, Y_{\rho} > 0 \). Now making another change of basis by repeatedly using \( T \) or \( U \), we may assume that \( (X_{\sigma}, Y_{\sigma}) \) is not horizontal or vertical either. For example, if \( (X_{\rho}, Y_{\rho}) = (0, -1) \) and \( (X_{\sigma}, Y_{\sigma}) = (1, 0) \), we can do the following change of basis:

\[
(0, -1), (1, 0) \mapsto (-1, -1), (1, 0) \mapsto (1, 1), (-1, 0) \mapsto (1, 2), (-1, -1).
\]

The strategy of the proof is similar to the proof of lemma 3.2. We create a simple configuration of strands near the boundary of \( R \) using isotopy and 2–2 moves, and then push this configuration past \( \partial R \).

Using Proposition 3.1, we obtain a minimal triple point diagram \( \mathfrak{T} \) in a fundamental rectangle \( R \) of \( \mathbb{T} \) such that:

1. \( (X_{\rho}, Y_{\rho}) \in \mathbb{Z}^2_{>0} \).
2. \( X_{\sigma}, Y_{\sigma} \neq 0 \).

Since in what follows we will have occasion to deal with strands in both \( \mathbb{T} \) and \( R \), let us call strands in \( \mathbb{T} \) zig-zag loops and reserve the term “strand” for strands in \( R \), to avoid confusing the two notions. The strands in \( R \) are the components of the intersections of zig-zag loops with the interior of \( R \). Let \( U_{\rho} \) denote the set of strands whose zig-zag loops correspond to the edge \( E_{\rho} \) of \( N \). By minimality of \( \mathfrak{T} \), two strands in \( U_{\rho} \) do not intersect and therefore the partition of the boundary intersection points by the strands in \( U_{\rho} \) is a “parallel crossing”. Therefore there is a (strict) linear order \( <_{\rho} \) on \( U_{\rho} \), where strands are ordered from smallest to largest in
the direction of the ray $\rho$. Let us denote by $\alpha$ the $<_\rho$-largest strand in $U_\rho$. Similarly let $\beta$ be the $<_\sigma$-smallest strand in $U_\sigma$. Since $(X_\rho, Y_\rho) \in \mathbb{Z}^2_{>0}$, the strand $\alpha$ is the north-west-most among all strands corresponding to $\rho$.

**Lemma 3.4.** The strand $\alpha$ has its “in” boundary point on $\partial R_W$ and its “out” boundary point on $\partial R_N$.

**Proof.** Since $X_\rho, Y_\rho > 0$, there is a strand associated to $\rho$ that intersects $\partial R_N$ and a strand associated to $\rho$ that intersects $\partial R_W$. By assumption, $\alpha$ is the north-west-most strand associated to $\rho$, and therefore both of its end points are in $\partial R_N \cup \partial R_W$. Its end points cannot both be on the same side of the boundary of $R$, because the zig-zag loop containing $\alpha$ has smallest possible number of intersections with $\partial R$ (property 2 in Proposition 3.1). Since $X_\rho, Y_\rho > 0$, its “in” boundary point must be on $\partial R_W$ and its “out” boundary point must be on $\partial R_N$ (again by property 2 in Proposition 3.1).

**Lemma 3.5.** Starting from $\mathcal{X}$ and using 2-2 moves and isotopy in $\mathbb{T}$, we can obtain a new triple point diagram $\mathcal{S}$ in $\mathbb{T}$, such that:

1. The strands in $U_\rho$ have been cyclically shifted in the direction of $\rho$ (so that $\alpha$ is now $<_\rho$-smallest).

2. The strands in $U_\sigma$ have been cyclically shifted in the direction of $-\sigma$ (so that $\beta$ is now $<_\sigma$-largest).

3. The linear orders of strands corresponding to all other rays are unchanged.

**Proof.** By using lemma 3.2, we can permute the boundary points to make the intersection points of $\alpha$ with $\partial R$ the north-most “in” point in $\partial R_W$ and the west-most “out” point in $\partial R_N$. By property 1 in Proposition 3.1, the west-most intersection point of a strand in $\mathcal{X}$ with $\partial R_N$ is an “out” point. Therefore the end-points of $\alpha$ are the north-most intersection point in $\partial R_W$ and the west-most intersection point in $\partial R_N$ respectively. Now we have to deal with four cases, depending on which quadrant $(X_\sigma, Y_\sigma)$ lies in.

1. $X_\sigma, Y_\sigma > 0$.

Since $N$ is a closed polygon, there must exist a ray $\tau \in \Sigma(1)$ such that if $(X_\tau, Y_\tau)$ are the coordinates of a zig-zag path in $Z_\tau$, we have $Y_\tau < 0$. Making a change of basis using $\mathbb{T}$, we can further assume $X_\tau < 0$ without affecting the assumptions already in place. Since $X_\sigma, Y_\sigma > 0$, the strand $\beta$ is the southeast-most among all strands associated to $\sigma$. By an argument similar to the
(a) Initial configuration.  

(b) Configuration after isotopy.

Figure 11: Isotoping the local configuration of strands past the northwest corner of $R$ in case 1.

proof of Lemma $3.4$, $\beta$ has its “out” point on $\partial R_N$ and “in” point on $\partial R_W$. Permuting boundary points using Lemma $3.2$, we make the intersections of $\beta$ with $\partial R$ the south-most “in” point in $\partial R_E$ and the east-most “out” point in $\partial R_N$.

Since the total homology of all zig-zag loops is zero, the total intersection number of the loops with any side of $R$ is zero, that is, we have an equal number of “in” and “out” points in any side of $R$, alternating in orientation as we move along the side. By our assumptions on $\alpha$ and $\Sigma$, the intersection point of $\alpha$ with $\partial R_N$ is the west-most point in $\partial R_N$ and its orientation is “out”. Therefore, the east-most point in $\partial R_N$ is an “in” point, which means there is an “in” point to the east of $\beta$ in $\partial R_N$. For the same reason, there is an “out” point south of $\beta$ in $R_W$. Permuting boundary intersections using Lemma $3.2$, we can make the south-east-most strand $\gamma$ corresponding to $\tau$, which by the argument in Lemma $3.4$, has a boundary point on each of these sides, pass through both these points. Using Theorem $2.2$, we can make $\gamma$ and $\beta$ run parallel to the boundary. Again using Theorem $2.2$, we can make the three strands $\alpha, \beta, \gamma$ meet just adjacent to the northeast corner of $R$ to obtain the local picture shown in Figure $11a$. We isotope the triple point across the corner to obtain the configuration in Figure $11b$. This achieves the shift of cyclic orders for $\rho, \sigma$ without changing the cyclic orders of strands corresponding to other rays.

2. $X_\sigma, Y_\sigma < 0$.

The strand $\beta$ is the north-west-most among all strands associated to $\sigma$. By the
argument in Lemma 3.4, it has an “in” boundary point on $\partial R_N$ and an “out” boundary point on $\partial R_N$. Permuting boundary intersections using lemma 3.2 we can make the strand $\beta$ the west-most “in” strand in $\partial R_N$ and the north-most “out” strand in $\partial R_W$. Now we use Theorem 2.2 to make $\alpha, \beta$ run parallel to the boundary to obtain the local picture shown in Figure 12a near the north-west corner of $R$. We then isotope to get the configuration in Figure 12b.

3. $X_\sigma < 0, Y_\sigma > 0$.

   We can use $T \in SL(2, \mathbb{Z})$ to make $X_\sigma > 0$, reducing to case 1.

4. $X_\sigma > 0, Y_\sigma < 0$.

   This case cannot occur because of convexity of $N$.

To prove the theorem, we need to find a sequence of 2-2 moves and isotopy $t : \mathcal{T} \to \mathcal{T}$ such that $\psi(t) = \delta_\rho - \delta_\sigma$. Using lemma 3.5 we obtain a triple point diagram $\mathcal{G}$ in $\mathcal{T}$. Now we use lemma 3.2 to permute the boundary points so that if a pair of “in” and “out” points in $\mathcal{T}$ is connected by a strand that corresponds to a ray $\tau$, then the corresponding “in” and “out” points of $\mathcal{G}$ are also connected by a strand corresponding to $\tau$. If $\tau \neq \rho, \sigma$, then this is the same strand as in $\mathcal{T}$. When $\tau$ either $\rho$ or $\sigma$, this is a cyclically shifted strand. Let $\mathcal{U}$ be the triple point diagram in $R$ thus obtained from $\mathcal{G}$. Now we apply Theorem 2.2 to convert $\mathcal{U}$ to $\mathcal{T}$ using a sequence of 2-2 moves and isotopy in $R$. We define $t$ to be the sequence of 2-2 moves and isotopy $\mathcal{T} \to \mathcal{G} \to \mathcal{U} \to \mathcal{T}$. By construction, $\psi(t) = \delta_\rho - \delta_\sigma$, and the theorem is proved.
4 Trivial seed cluster transformations

By Theorem 3.3, $\psi$ is surjective. To complete the proof of Theorem 1.1, we need to find the kernel of $\psi$.

**Projective toric surfaces.** A *toric surface* is a normal algebraic variety of dimension 2 containing the algebraic torus $(\mathbb{C}^*)^2$ as a dense open subset, such that the action of $(\mathbb{C}^*)^2$ on itself extends. A convex integral polygon $N$ gives rise to a projective toric surface $X_N$, along with an ample divisor $D_N$, such that the linear system $|D_N|$ is identified curves defined by vanishing of Laurent polynomials with Newton polygon $N$. A generic curve $C \in |D_N|$ has genus $g$ equal to the number of interior lattice points in $N$. The complement of the algebraic torus in $X_N$ is a union of $\mathbb{P}^1$s, called *lines at infinity*, parameterized by the edges of $N$, and intersecting according to the combinatorics of $N$. We denote the line at infinity corresponding to $E_\rho \in E_N$ by $D_\rho$. For $C \in |D_N|$, we have $|C \cap D_\rho| = |E_\rho|$, where the points in $C \cap D_\rho$ are counted with multiplicity.

**The spectral transform.** We follow [GK12, Section 7]. A *spectral data* is a triple $(C, S, \nu)$ where:

1. $C$ is a curve in $|D_N|$.
2. $S$ is a degree $g$ effective divisor in $C$, where $g$ is the number of interior lattice points of $N$.
3. $\nu = \{\nu_\rho\}_{\rho \in \Sigma(1)}$ is a collection of bijections $\nu_\rho : Z_\rho \simto C \cap D_\rho$ (recall that $|E_\rho| = |Z_\rho| = |C \cap D_\rho|$).

Let $S_N$ be the moduli space parameterizing the spectral data related to $N$.

Fix a minimal bipartite graph $\Gamma$ with Newton polygon $N$, and a white vertex $w$ of $\Gamma$. There is a rational map, called the *spectral transform*, defined by Kenyon and Okounkov [KO03]:

$$\kappa_{\Gamma,w} : \mathcal{X}_N(\mathbb{C}) \dashrightarrow S_N$$

$$wt \mapsto (C, S, \nu),$$

as follows:
1. $C$ is the closure of $C_0$ in $X_N$, and is called the spectral curve. By Theorem 2.5, $C \in |D_N|$. The points in $C \setminus C_0 = \bigcup_{\rho \in \Sigma(1)} C \cap D_\rho$ are called the points at infinity.

2. $S$ is a degree $g$ effective divisor in $C_0$ defined as follows: Consider the following exact sequence of sheaves given by the Kasteleyn operator:

$$0 \to \bigoplus_{b \in B(\Gamma)} \mathcal{O}_{(C \times)^2} \xrightarrow{K(z,w)} \bigoplus_{w \in W(\Gamma)} \mathcal{O}_{(C \times)^2} \to \text{coker } K(z,w) \to 0.$$  

When $C_0$ is smooth, which is true when $wt$ is generic, $\text{coker } K(z,w)$ is the push-forward of a line bundle $L$ on $C_0$. The image of the section $\delta_w$ of $\bigoplus_{w \in W(\Gamma)} \mathcal{O}_{(C \times)^2}$ in $\text{coker } K(z,w)$ restricts to a section of $L$. $S$ is defined to be the divisor of this section. It is a degree $g$ effective divisor (see [KO03, Theorem 1] for a proof when $X_N = \mathbb{P}^2$ and [GGK] for the general case.)

3. $\nu$ is the bijection between zig-zag paths and points at infinity defined by the following property: $\nu(\alpha)$ is the point in $C \cap D_\rho$ where $K(z,w)|_{\alpha}$ is singular, where $K(z,w)|_{\alpha}$ denotes the Kasteleyn matrix of the zig-zag path $\alpha$ viewed as a bipartite graph in $T$. The coordinates of $\nu(\alpha)$ are determined by $wt(\alpha)$.

We have:

**Theorem 4.1** (Fock, 2015 [F15], George, Goncharov and Kenyon [GGK]). The spectral transform is a birational map.

**The discrete Abel map.** Let $\widetilde{\Gamma}$ be the preimage of $\Gamma$ in the universal cover of $T$. Let $\text{Div}_\infty(C)$ denote the divisors at infinity of $C$, that is $\mathbb{Z}$-linear combinations of the points at infinity. Following Fock [F15], we define the discrete Abel map

$$\mathbf{d}_0 : \text{Vertices of } \widetilde{\Gamma} \to \text{Div}_\infty(C),$$

using the following rules: For a choice of white vertex $w$ of $\widetilde{\Gamma}$, we have the normalization $\mathbf{d}_0(w) = 0$, and for any path $\gamma$ from $v_1$ to $v_2$, we have

$$\mathbf{d}_0(v_2) - \mathbf{d}_0(v_1) = \sum_{\text{Zig-zag paths } \alpha} \langle \alpha, \gamma \rangle \nu(\alpha), \quad (8)$$

where $\langle \cdot, \cdot \rangle$ is the intersection form on the universal cover of $T$. $\mathbf{d}_0$ can be effectively computed by the following procedure: If $bw$ is an edge, with zig-zag paths $\alpha, \beta$ containing $bw$, then

$$\mathbf{d}_0(w) = \mathbf{d}_0(b) - \nu(\alpha) - \nu(\beta).$$
We have an embedding
\[ H_1(\mathbb{T}, \mathbb{Z}) \hookrightarrow Div_\infty(C) \]
\[ \gamma \mapsto \text{div}_C(z, w)^\gamma = \sum_{\text{zig-zag paths } \alpha} \langle \alpha, \gamma \rangle \nu(\alpha), \] (9)
where \((z, w)^\gamma\) denotes the character of \(T = (\mathbb{C}^\times)^2\) associated to \(\gamma \in H_1(\mathbb{T}, \mathbb{Z})\). \(d\) is \(H_1(\mathbb{T}, \mathbb{Z})\)-equivariant:
\[ d_0(v + \gamma) = d_0(v) + \gamma, \]
so that although \(d_0(v)\) is not well-defined for a vertex \(v\) of \(\Gamma\), the divisor class \([d_0(v)]\) is the same for all lifts of \(v\) to \(\tilde{\Gamma}\) and therefore well-defined. Therefore we define
\[ d: V(\Gamma) \to \text{Cl}(C) \]
\[ v \mapsto [d_0(\tilde{v})], \]
where \(\tilde{v}\) is any lift of \(v\) in \(\tilde{\Gamma}\), and \(\text{Cl}(C)\) is the divisor class group of \(C\).

**Example.** Consider the bipartite torus graph in Figure 7 with zig-zag paths labeled as in Figure 8. The discrete Abel map normalized so that \(d(w) = 0\) is as follows.
\[ d(w_1) = [-\nu(\alpha_1) + \nu(\gamma)], \]
\[ d(b_1) = [\nu(\alpha_2) + \nu(\beta)], \]
\[ d(b_2) = [\nu(\beta) + \nu(\gamma)], \]
where the zig-zag paths are labeled as in Figure 8.

**Elementary transformations and induced discrete Abel maps.** We now describe how \(\nu\) changes under isotopy and elementary transformations and use this to define induced discrete Abel maps.

1. Suppose \(s: \Gamma \to \Gamma\) is an automorphism of \(\Gamma\) induced by an isotopy in \(\mathbb{T}\). \(s\) induces a bijection of the set of zig-zag paths \(Z\) with itself that preserves \(Z_\rho\). Let
\[ \mu_s: \mathcal{L}_\Gamma \to \mathcal{L}_\Gamma \]
\[ wt \mapsto wt \circ s^{-1}, \]
be the induced birational map of weights. If \(\alpha\) is a zig-zag path in the graph \(\Gamma\) after the isotopy, the point at infinity \(\nu_s(\alpha)\) associated with it is determined
Figure 13: Induced discrete Abel maps.
by $\mu_s(wt(\alpha)) = wt(s^{-1}(\alpha))$. Therefore we have $\nu_s(\alpha) := \nu(s^{-1}(\alpha))$.

If $d$ is a discrete Abel map on $\Gamma$, we define an induced discrete Abel map $d_s$ on $\Gamma$ by the rule: $d_s(v) := d(s^{-1}(v))$.

2. Let $s : \Gamma_1 \to \Gamma_2$ be an elementary transformation and let $Z_1$ and $Z_2$ denote the zig-zag paths of $\Gamma_1$ and $\Gamma_2$ respectively. Let $\mu_s : \mathcal{L}_{\Gamma_1} \to \mathcal{L}_{\Gamma_2}$ be the induced map of weights. $s$ induces a bijection $Z_1 \sim \to Z_2$ between zig-zag paths of $\Gamma_1$ and $\Gamma_2$ such that $\mu_s(wt(\alpha)) = wt(s^{-1}(\alpha))$ for all zig-zag paths $\alpha \in Z_2$. Therefore we have $\nu_s(\alpha) := \nu(s^{-1}(\alpha))$.

Suppose $d_1$ is a discrete Abel map on a graph $\Gamma_1$. An elementary transformation $s : \Gamma_1 \to \Gamma_2$ induces a discrete Abel map $d_2$ on $\Gamma_2$ as follows: the elementary transformation only changes $\Gamma_1$ in a disc. The induced discrete Abel map $d_2$ is defined to be equal to $d_1$ outside the disc, and extended to the interior of the disc using $(8)$ and $\nu_s$ (see Figure 13).

If $t$ is a sequence of graph isomorphisms and elementary transformations, we get an induced $\nu_t$ and $d_t$ by composing.

**2-2) Cluster modular transformations and the spectral transform.** Let $t : \Gamma \to \Gamma$ be a (2-2) cluster modular transformation and let $\mu_t$ denote the induced birational automorphism of $X_N$. Suppose $d$ is a discrete Abel map on $\Gamma$. Let $\nu_t$ be the induced bijection between zig-zag paths and points at infinity and $d_t$ the induced discrete Abel map. Any two discrete Abel maps on $\Gamma$ differ only by their normalization, so $d - d_t$ is a degree zero divisor class of $C$.

**Theorem 4.2** (Fock, 2015 \cite{F15} Proposition 1). The following diagram commutes:

\[
\begin{array}{ccc}
X_N & \xrightarrow{\kappa_{\Gamma,w}} & S_N \\
\downarrow{\mu_t} & & \downarrow{\nu_t} \\
X_N & \xrightarrow{\kappa_{\Gamma,w}} & S_N
\end{array}
\]

where the map on the left is $(C, S, \nu) \mapsto (C, S_t, \nu_t)$, where $S_t$ is the (generically) unique degree $g$ effective divisor satisfying

\[S_t = S + d(w) - d_t(w), \text{ in } Cl^g(C),\]

where $Cl^g(C)$ is the group of degree $g$ divisor classes.
\(\psi\) and the discrete Abel map. In this section, we prove the following proposition.

**Proposition 4.3.** The birational automorphism \(\mu_t\) of \(X_N\) induced by a cluster transformation \(t\) factors through \(\psi\):

\[
\begin{align*}
\{\text{Cluster transformations } \Gamma \to \Gamma\} & \xrightarrow{\psi} \mathbb{Z}^\Sigma_{(1)}_0 / jH_1(\mathbb{T}, \mathbb{Z}) \\
\downarrow_{t \to \mu_t} & \downarrow \Bir(X_N) \\
& ,
\end{align*}
\]

where \(\Bir(X_N)\) is the group of birational automorphisms of \(X_N\).

**Proof.** We show that the induced discrete Abel map \(d_t\) and the induced bijection \(\nu_t\) for a cluster transformation \(t\) are both determined by \(\psi(t)\). By Theorem 4.2, the induced birational map \(\mu_t\) is determined by \(d_t\) and \(\nu_t\).

Let \(t : \Gamma = \Gamma_0 \to \Gamma_1 \to \cdots \to \Gamma_n \to \Gamma_n \cong \Gamma\), be a cluster transformation where \(\Gamma_{i+1}\) is obtained from \(\Gamma_i\) by an elementary transformation or \(\Gamma_{i+1}\) is isomorphic to \(\Gamma_i\) by an isotopy in \(\mathbb{T}\). Let \(\rho \in \Sigma(1)\). Let \(d_0\) be a discrete Abel map on \(\tilde{\Gamma}\) with the normalization \(d_0(\tilde{w}) = 0\), where \(\tilde{w}\) is a chosen lift of \(w\). Let \(d_0\big|_{C \cap D_\rho}\) denote restriction of the divisor \(d_0\) to points at infinity associated to \(\rho\). The set of all zig-zag paths in the universal cover of \(\mathbb{T}\) associated to \(\rho\) subdivides the universal cover into a collection of strips \(S(a)\) indexed by \(a \in \mathbb{Z}^{C \cap D_\rho} \cap \mathbb{R}\):

\[
S(a) := d_0\big|_{C \cap D_\rho}^{-1}(a).
\]

A elementary transformation or isotopy \(s : \Gamma_1 \to \Gamma_2\) lifts to an \(H_1(\mathbb{T}, \mathbb{Z})\)-periodic collection of elementary transformations or isotopy \(\tilde{s} : \tilde{\Gamma}_1 \to \tilde{\Gamma}_2\) in the universal cover of \(\mathbb{T}\). Different lifts differ by \(H_1(\mathbb{T}, \mathbb{Z})\), but our construction will be independent of these choices. Suppose \(d_0\) is a discrete Abel map on \(\tilde{\Gamma}_1\). For \(a \in \mathbb{Z}^{C \cap D_\rho}\), let \(S_1(a)\) denote the corresponding strip. We define an induced discrete Abel map \(d_{s,0}\) as follows:

1. If \(\tilde{s} : \tilde{\Gamma}_1 \to \tilde{\Gamma}_2\) is induced by an isotopy in \(\mathbb{T}\), we define \(d_{s,0}(v) := d_0(\tilde{s}^{-1}(v))\).

2. If \(\tilde{s} : \tilde{\Gamma}_1 \to \tilde{\Gamma}_2\) is an elementary transformation, we define it as in Figure 13.

These are simply \(H_1(\mathbb{T}, \mathbb{Z})\)-periodic versions of the induced discrete Abel map on \(\mathbb{T}\) defined earlier. By construction, we have the following property: if \(S_1(a)\) is the strip whose right boundary is a zig-zag path \(\alpha\) of \(\tilde{\Gamma}_1\), then \(S_2(a)\) is the strip whose right
boundary is \( \tilde{s}(\alpha) \).

Let \( d_{t,0} \) be the discrete Abel map induced by the cluster transformation \( \tilde{t} \), obtained by composing. Suppose during \( t \) a strand in \( Z_\rho \) is translated by \( a_\rho \gamma_z + b_\rho \gamma_w \). Then the above property implies that the strip \( S_t(a) \) of \( \Gamma \) is obtained from \( S(a) \) by translating by \( a_\rho \gamma_z + b_\rho \gamma_w \). Therefore

\[
(d_{t,0} - d_0) \big|_{C \cap D_\rho} = \sum_{\alpha \in Z_\rho} \langle \alpha, \pi_\rho \rangle \nu(\alpha),
\]

(10)

where \( \pi_\rho \) is any path between a vertex in \( S_t(a) \) and a vertex in \( S(a) \). Choosing a different path does not affect (10). Moreover since (10) is unaffected if \( a_\rho \gamma_z + b_\rho \gamma_w \) modified by a vector in the span of \( X_\rho \gamma_z + Y_\rho \gamma_w \), it is determined by the projection on \( (Y_\rho, -X_\rho) \) and therefore by \( \psi(t)(\rho) = |E_\rho|(b_\rho X_\rho - a_\rho Y_\rho) \).

Summing over all \( \rho \in \Sigma(1) \) and taking divisor classes, we get

\[
d_t - d = \left[ \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho} \langle \alpha, \pi_\rho \rangle \nu(\alpha) \right].
\]

(11)

A different choice of lift \( \tilde{\Gamma} \) of \( \Gamma_i \) would modify (10) by an element of \( H_1(\mathbb{T}, \mathbb{Z}) \), and therefore leave (11) unchanged.

We find \( \nu_t(\alpha) \) from \( \psi(t)(\rho) \) as follows: if \( \alpha \) is a zig-zag path in \( Z_\rho \), consider a lift \( \tilde{\alpha} \) of it to the universal cover of \( \mathbb{T} \). Suppose \( \tilde{\alpha} \) is the right boundary of a strip \( S \). Since the effect of \( t \) on strips is to translate them by \( a_\rho \gamma_z + b_\rho \gamma_w \), where \( \psi(t)(\rho) = |E_\rho|(b_\rho X_\rho - a_\rho Y_\rho) \), we get that

\[
\nu_t(\alpha) = \nu(\beta),
\]

(12)

where \( \beta \in Z_\rho \) is the zig-zag path such that one of its lifts to the universal cover is the right boundary of the strip \( S - (a_\rho \gamma_z + b_\rho \gamma_w) \). Now that we have found \( \nu_t \) and \( d_t - d \), Theorem 4.2 gives us the birational map \( \mu_t \).

\section*{A shuffling algorithm of Borodin and Ferrari.}

The cluster transformation \( t \) shown in Figure 14 for the graph \( \Gamma \) in Figure 7 was introduced and studied by Borodin and Ferrari in [BF18]. Suppose \( wt \in L_\Gamma \). Let \( C_\alpha := wt(\alpha) \) denote the monodromy of \( wt \) around a zig-zag path \( \alpha \). Then \( \{C_{\alpha_1}, C_{\alpha_2}, C_\beta, C_\gamma\} \) is a set of coordinates for \( L_\Gamma \).
Figure 14: A shuffling algorithm of Borodin and Ferrari. In the first step, we do a spider move at the face labeled $f_2$ in Figure 7 and in the second step we all contract degree 2 black vertices and translate. We have drawn one of the white vertices (the vertex $w$ in the original graph) larger than the others to illustrate the translation. The zig-zag paths associated with $\rho$ are translated right by $\frac{2\pi}{2}$ during $t$. The strips associated with $\rho$ are labeled at the top.
A cocycle representing $wt$ in this basis is shown in Figure 15. The Kasteleyn matrix and spectral curve are:

$$K(z, w) = \begin{pmatrix} b_1 & b_1 \\ b_1 & b_1 \end{pmatrix} \begin{pmatrix} C_{\alpha_1} + w & C_{\alpha_1} C_{\alpha_2} C_{\beta} z \\ -1 + C_{\gamma} w & C_{\alpha_1} + w \end{pmatrix} \begin{pmatrix} w \\ w_1 \end{pmatrix},$$

$$C = \{ C_{\alpha_1} C_{\alpha_2} + C_{\alpha_1} w + C_{\alpha_2} w + w^2 + C_{\alpha_1} C_{\alpha_2} C_{\beta} z - C_{\alpha_1} C_{\alpha_2} C_{\beta} C_{\gamma} w z = 0 \}.$$  (13)

From 13, we recover the Newton polygon shown in Figure 8. Since the Newton polygon has no interior lattice points, $C$ is a genus 0 curve and therefore the divisor $S$ in the spectral transform is irrelevant. There are two points at infinity of $C$ associated to the ray $\rho = \mathbb{R}_{\geq 0}(1, 0)$:

$$P_1 : z \rightarrow 0, w = -C_{\alpha_1},$$
$$P_2 : z \rightarrow 0, w = -C_{\alpha_2},$$

and only one point at infinity of $C$ associated to each of the other rays:

$$P_3 : z \rightarrow -\frac{1}{C_{\beta}}, w \rightarrow 0;$$
$$P_4 : z \rightarrow \infty, w = \frac{1}{C_{\gamma}},$$
$$P_5 : z, w \rightarrow \infty, w/z = C_{\alpha_1} C_{\alpha_2} C_{\beta} C_{\gamma}.$$  

The bijection $\nu$ is:

$$(\alpha_1, \alpha_2, \beta, \gamma, \delta) \mapsto (P_1, P_2, P_3, P_4, P_5).$$
Figure 16: The bipartite torus graph involved in domino-shuffling showing the labels of the faces and the loops $a$ and $b$.

The spectral transform is:

$$\mathcal{L}_\Gamma \xrightarrow{\kappa_{\Gamma,w}} S_N$$

$$(C_{\alpha_1}, C_{\alpha_2}, C_\beta, C_\gamma) \mapsto (C, \nu),$$

and the induced map $\mu_t : \mathcal{L}_\Gamma \to \mathcal{L}_\Gamma$ is:

$$(C_{\alpha_1}, C_{\alpha_2}, C_\beta, C_\gamma) \mapsto (C_{\alpha_2}, C_{\alpha_1}, C_\beta, C_\gamma),$$

and the induced bijection $\nu_t$ is:

$$(\alpha_1, \alpha_2, \beta, \gamma, \delta) \mapsto (P_2, P_1, P_3, P_4, P_5).$$

Since $t$ exchanges the weights of the zig-zag paths $\alpha_1$ and $\alpha_2$, and also exchanges the points at infinity associated with these zig-zag paths, it follows that the diagram in Theorem 4.2 commutes in this example.

**Domino-shuffling.** Consider the graph $\Gamma$ in Figure 16. $a$ and $b$ are two cycles in $H_1(\Gamma, \mathbb{Z})$ whose projections to $\mathbb{T}$ generate $H_1(\mathbb{T}, \mathbb{Z})$. Let $X_i := wt(f_i), i = 1, 2, 3$ and let $A := wt(a), B := wt(b)$. Then $(X_1, X_2, X_3, A, B)$ gives coordinates on $\mathcal{L}_\Gamma$. A cocycle representing $wt$ is shown in Figure 17. The Kasteleyn matrix and the
Figure 17: A cocycle representing $ut$, along with Kasteleyn signs and the characters $\phi(e)$ (red) for the graph in Figure 16.

The spectral curve are:

$$K(z,w) = \left( \begin{array}{cc} b_1 & b_2 \\ \frac{1}{B} - 1 + Bw & 1 - \frac{X_1 X_3}{Bw} \frac{1}{AX_2 z} \end{array} \right) \begin{array}{c} w \\ w_1 \end{array},$$

$$C = \left( 1 + X_1 + \frac{A^2}{X_2} + X_1 X_3 \right) - Bw - \frac{X_1 X_3}{Bw} - \frac{A}{X_2 z} - AX_1 z. \quad (14)$$

There is only one zig-zag path in each direction, so $\nu$ is trivial. The divisor $S = (p,q)$ has only one point, which is found by simultaneously solving $\text{adj} K(z,w)_{b,w} = 0$, for all black vertices $b$ of $\Gamma$. In this case, we get

$$(p,q) = \left( \frac{1}{AX_1 X_2}, \frac{1}{B} \right).$$

The spectral transform is:

$$L_{\Gamma} \xrightarrow{\kappa_{\Gamma, w}} S_N \xrightarrow{\kappa_{\Gamma, w}} (X_1, X_2, X_3, A, B) \mapsto (C, (p, q)). \quad (15)$$

Consider the cluster transformation in Figure 2. The induced map $\mu_t : L_{\Gamma} \to L_{\Gamma}$ is:
\[
X_1 \mapsto X_2 \frac{(1 + X_1)^2}{(1 + X_1 X_2 X_3)^2},
\]
\[
X_2 \mapsto X_1^{-1},
\]
\[
X_3 \mapsto X_1 X_2 X_3,
\]
\[
A \mapsto \frac{AX_1 (1 + X_1 X_2 X_3)}{1 + X_1},
\]
\[
B \mapsto \frac{BX_2}{(1 + X_1)(1 + X_1 X_2 X_3)}. \tag{16}
\]

Let \(S_t = (p_t, q_t)\) be the induced divisor. We can find the induced divisor in two different ways, verifying Theorem 4.2.

1. We have \(d_t(w) = [\nu(\alpha) - \nu(\beta)] = [\nu(\gamma) - \nu(\delta)]\), where the second equality comes from \(\text{div}_C z = -\nu(\alpha) + \nu(\beta) + \nu(\gamma) - \nu(\delta)\). The unique point that satisfies

\[
S_t = S + d(w) - d_t(w),
\]

must be

\[
p_t = \frac{1 + X_1 X_2 X_3}{AX_2(1 + X_1)}, \quad q_t = \frac{1 + X_1 X_2 X_3}{BX_2(1 + X_1)}, \tag{17}
\]

because the rational function

\[
R(z, w) = \frac{Bw(-1 + AX_1 X_2 z)}{-AX_1 X_2 X_3 z + Bw(-1 + A(1 + X_1) X_2 z)}
\]

has zeros at \((p, q)\) and \(\nu(\delta)\) and poles at \((p_t, q_t)\) and \(\nu(\gamma)\), as can be checked using the coordinates of the points at infinity:

\[
\nu(\gamma) : z, w \to 0, \quad \frac{w}{z} = \frac{-X_1 X_2 X_3}{AB},
\]

\[
\nu(\delta) : z \to \infty, w \to 0, \quad z w = \frac{-X_3}{AB}.
\]

2. We can find the composition \(\kappa_{\Gamma, w} \circ \mu_t\) using (15) and (16) to get (17).

\textbf{Triviality of cluster transformations.} From Theorem 4.2, a cluster transformation \(t\) is trivial if and only if \(\nu_t = \nu\) and \(d(w) - d_t(w) = 0\) in \(\text{Pic}^0(C)\) for a generic curve \(C \in |D_N|\). As a reality check, we observe that translation by \(\gamma \in H_1(T, \mathbb{Z})\) is trivial: it induces \(\nu_t = \nu\) and \(d_t(w) = d(w) - [\text{div} (z, w) \gamma] = d(w)\). Therefore by Proposition 4.3 if \(\psi(t) = 0\), then \(t\) is a trivial cluster transformation, so we have:
Lemma 4.4. \( \ker \psi \subseteq \{ \text{Trivial cluster transformations} \} \).

We will show that in non-degenerate situations, this inclusion is an equality. We start with the following simple consequence of Theorem 4.2.

Lemma 4.5. Let \( t \) be a cluster transformation such that \( \psi(t) \) is a non-zero torsion element of \( \mathbb{Z}^{\Sigma(1)}_0 / jH_1(\mathbb{T}, \mathbb{Z}) \). Then \( \mu_t \) is non-trivial.

Proof. From (12), we see that \( \nu_t \neq \nu \). Therefore by Theorem 4.2, \( \mu_t \) is non-trivial.

The main theorem of the paper is:

Theorem 4.6. If \( g \neq 0 \), the cluster modular group is:

\[
G_N \cong \mathbb{Z}^{\Sigma(1)}_0 / jH_1(\mathbb{T}, \mathbb{Z}).
\]

If \( g = 0 \), then:

\[
G_N \cong \mathbb{Z}^{\Sigma(1)}_0 / \{ f \in \mathbb{Z}^{\Sigma(1)}_0 : f(\rho) \text{ is divisible by } |E_{\rho}| \text{ for all } \rho \in \Sigma(1) \}.
\]

Proof. When \( g = 0 \), \( S = \emptyset \), so \( \mu_t \) is determined by the action of \( t \) on \( \nu \). Therefore \( t \) is trivial if an only if \( \nu_t = \nu \), which happens if and only if \( \psi(t)(\rho) \) is divisible by \( |E_{\rho}| \) for all \( \rho \in \Sigma(1) \).

When \( g \neq 0 \), if \( t \) is a cluster transformation such that \( \psi(t) \neq 0 \), then either:

1. \( \psi(t) \) is a non-zero torsion element: It is non-trivial by Corollary 4.5.

2. \( \psi(t) \) is not a torsion element: Consider the cluster transformation \( t^n \) obtained by iterating \( t \), where

\[
n = k \prod_{\rho \in \Sigma(1)} |E_{\rho}|, \quad k \in \mathbb{Z}.
\]

Then from Theorem 4.2 applied to \( t^n \), we see that the induced map of spectral data by \( t^n \) is given by \( (C, S, \nu) \mapsto (C, S', \nu) \), where \( S' \) is the generically unique degree \( g \) effective divisor satisfying

\[
S' = S + D|_C,
\]

where

\[
D = n \sum_{\rho} \frac{\psi(t)(\rho)}{|E_{\rho}|} D_{\rho}
\]
is a divisor at infinity of $X_N$. For sufficiently large $k$, $\mathcal{O}_{X_N}(D)$ is a line bundle on $X_N$ [CLS11, Proposition 4.2.7]. Since $\psi(t)$ is not a torsion element, $D$ is not a torsion element of the divisor class group of $X_N$ either; indeed if $lD$ is a principal divisor for some $l \in \mathbb{Z}$, then $lD$ is the divisor of a character $(z, w)^{\gamma}$ for some $\gamma \in H_1(\mathbb{T}, \mathbb{Z})$. However this means that $\psi(t^{ln}) \in jH_1(\mathbb{T}, \mathbb{Z})$, contradicting the assumption that $\psi(t)$ is not a torsion element.

Therefore $\mathcal{O}_{X_N}(D) \not\cong \mathcal{O}_{X_N}$, and so by Theorem [A.11] we get $\mathcal{O}_{X_N}(D)|_C \not\cong \mathcal{O}_C$ for a generic spectral curve $C$. Therefore by (18), $t^n$ is not a trivial cluster transformation. Since $\mu_{t^n} = \mu_t^n$, $t$ is also not a trivial cluster transformation.

Therefore $\ker \psi = \{\text{Trivial cluster transformations } \Gamma \to \Gamma\}$. By Theorem [3.3], $\psi$ is surjective, so by the first isomorphism theorem, the cluster modular group

$$G_N \cong \mathbb{Z}_{0}^{\Sigma(1)} \sslash jH_1(\mathbb{T}, \mathbb{Z}).$$

\[\square\]

### 4.1 Picard group of the toric stack

Associated to $N$ is a stacky fan $\Sigma = (\Sigma, \hat{\Sigma}, \beta)$ where:

1. $\Sigma$ is the normal fan of $N$ in $H_1(\mathbb{T}, \mathbb{Z})^\vee \otimes \mathbb{R}$;

2. $\hat{\Sigma}$ is a fan in an auxiliary lattice $\mathbb{Z}^{[E_N]}$, formed by the walls of the positive orthant;

3. Let $\{e_{\rho}\}$ be the standard basis of $Z^{E_{N}}$. $\beta : \mathbb{Z}^{[E_N]} \to H_1(\mathbb{T}, \mathbb{Z})^\vee$ is the homomorphism defined by $\beta(e_{\rho}) = |E_{\rho}|u_{\rho}$. Note that $\beta$ gives a combinatorial correspondence between cones of $\hat{\Sigma}$ and $\Sigma$.

Just as the normal fan $\Sigma$ of $N$ can be used to construct toric surface $X_N$, the stacky fan $\hat{\Sigma}$ gives rise to a stacky toric surface $X_N$. $X_N$ has coarse moduli space the toric variety $X_N$; let us denote by $\pi : \mathcal{X}_N \to X_N$ the projection. Let $\mathcal{O}_{\mathcal{X}_N} \left(\frac{1}{|E_\rho|}D_\rho\right)$ be the unique line bundle on $\mathcal{X}_N$ satisfying

$$\mathcal{O}_{\mathcal{X}_N} \left(\frac{1}{|E_\rho|}D_\rho\right)^{\otimes |E_\rho|} \cong \pi^* \mathcal{O}_{X_N}(D_\rho).$$

Just as the class group of $X_N$ is generated by the toric divisors, the Picard group of $\mathcal{X}_N$ is generated by the line bundles $\mathcal{O}_{\mathcal{X}_N} \left(\frac{1}{|E_\rho|}D_\rho\right)$.
Theorem 4.7 (Borisov and Hua, 2009 [BH09, Proposition 3.3]). The following is an isomorphism of groups:

\[ \mathbb{Z}^E_N / M \rightarrow \text{Pic} (\mathcal{X}_N) \]

\[ f \mapsto \mathcal{O}_{\mathcal{X}_N} \left( \sum_{\rho} \frac{f(E_{\rho})}{|E_{\rho}|} D_{\rho} \right) \]

Proposition 4.8. This isomorphism identifies \( \mathbb{Z}^{\Sigma(1)}_0 / jH_1(\mathbb{T}, \mathbb{Z}) \) with the subgroup of Pic (\( \mathcal{X}_N \)) of line bundles \( \mathcal{O}_{\mathcal{X}_N}(D) \) where \( D = \sum_{\rho} b_{\rho} D_{\rho} \), satisfying

\[ \sum_{\rho} |E_{\rho}| b_{\rho} = 0. \]

There is a version of Theorem 4.2 which illuminates this correspondence.

Theorem 4.9 (Treumann, Williams and Zaslow, 2018 [TWZ18, Proposition 1.2]). Let \( t \) be a seed cluster transformation. Let \( \mathcal{C} = C \times_{X_N} \mathcal{X}_N \) and let \( \mathcal{C} \hookrightarrow \mathcal{X}_N \) be the embedding. We have:

\[ \mathcal{O}_{\mathcal{C}}(S_t) \cong \mathcal{O}_{\mathcal{C}}(S) \otimes i^{*} \mathcal{O}_{\mathcal{X}_N} \left( \sum_{\rho} \frac{\psi(E_{\rho})}{|E_{\rho}|} D_{\rho} \right). \]

A Appendix - Toric surfaces ruled by lines, Giovanni Inchiostro

The goal of this appendix is to relate the Picard group of a projective toric surface \( X \), with an equivariant embedding \( X \rightarrow \mathbb{P}^n \), with that of a generic hyperplane section. This will be achieved in Theorem A.11. Since one can study such a projective toric surface by looking at its associated polygon, we will use the combinatorics of the polygons to prove Theorem A.11.

In this appendix, all polygons will be convex, integral and in \( \mathbb{R}^2 \).

Definition A.1. We define a building block polygon to be a polygon \( \Delta \subseteq \mathbb{R}^2 \) with a single interior lattice point, and with at most five lattice points.

A building block polygon has either three or four edges (see Figure 18).

Our first goal is to show that given any polygon \( P \) with an interior lattice point, one can find a building block polygon \( \Delta \) with \( \Delta \subseteq P \) (Proposition A.4). We will find

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\[\Delta\] as a polygon with the least number of lattice points among all polygons which are both contained in \(P\) and have at least one interior lattice point.

We begin with a few preparatory lemmas.

**Lemma A.2.** Consider a polygon \(P\) with an interior lattice point. Then there is a polygon \(Q \subseteq P\) which has an interior lattice point and at most four edges.

**Proof.** Let \(x\) be an interior lattice point of \(P\). Pick a polygon \(Q \subseteq P\) which is minimal among all polygons contained in \(P\) that contain \(x\) as an interior lattice point, with the partial order being the inclusion. We aim at showing that \(Q\) has either three or four edges. If not, let \(a_1, \ldots, a_n\) be the vertices of \(Q\) with \(n > 4\), labeled in clockwise order. Consider the segments joining \(a_1\) with \(a_3\), and \(a_3\) with \(a_5\) respectively. They divide \(Q\) into three smaller polygons, each with fewer lattice points, and these segments intersect only at \(a_3\), since by assumption \(a_5\) is distinct from \(a_3\) and \(a_1\). Therefore \(x\) is an interior point of either the polygon with vertices \(a_1, a_3, a_4, \ldots, a_n\) or the polygon with vertices \(a_1, a_2, a_3, a_5, a_6, \ldots, a_n\). This contradicts minimality of \(Q\). \(\square\)

**Lemma A.3.** Consider a polygon \(P\) with an interior lattice point. Then there is a polygon \(Q \subseteq P\) which has a exactly one interior lattice point.

**Proof.** Consider a subpolygon \(Q\) of \(P\), which has at least two interior lattice points, \(x\) and \(y\). Up to shrinking \(Q\), we can assume, by Lemma A.2, that either \(Q\) is a triangle or a quadrilateral.
If $Q$ is a triangle, consider the line through $x$ and $y$. It must meet an edge $\ell$ of $Q$. Let $a$ and $b$ be the vertices of $\ell$. Up to swapping $x$ and $y$, we can assume that the distance between $x$ and $\ell$ is less than the distance between $y$ and $\ell$. Then the triangle with vertices $y$, $a$ and $b$, with an interior point (namely $x$) has fewer interior lattice points than $Q$ ($y$ is not an interior point).

If $Q$ is a quadrilateral, consider the two diagonals of $Q$. They have a single intersection point, so there must be a diagonal $\ell$ which does not contain both $x$ and $y$. Then $\ell$ divides $Q$ into two smaller polygons, and one of them must have an interior point.

Therefore, if we consider a polygon which is minimal with respect to inclusion and has an interior lattice point, it must have a exactly one interior lattice point. \hfill \square

**Proposition A.4.** Given any polygon $P$ with an interior lattice point, one can find a building block polygon $\Delta$ such that $\Delta \subseteq P$.

*Proof.* Consider a polygon $Q \subseteq P$. From Lemma A.3 and Lemma A.2, we can assume, up to shrinking $Q$, that $Q$ has at most four edges, and a single interior point $x$. If $Q$ has four edges and five points, we are done, otherwise there is an edge $\ell$ with a point $y \in \ell$ which is not a vertex. Let $a, b$ be the two vertices of $Q$ not contained in $\ell$. Then the segments $\overline{ya}$ and $\overline{yb}$ intersect only at $y$, and divide $Q$ into three smaller polygon. One of them must contain $x$ in its interior. Therefore if $Q$ is minimal and has four edges, it must be a building block polygon.

If instead $Q$ is a triangle, assume it has more than five lattice points. Then there are two lattice points $p, q$ which are on the boundary of $Q$, but are not vertices. If
they belong to the same edge $\ell$, let $a$ be the vertex of $Q$ not contained in $\ell$. Then the segments $\ell_1 := \overline{ap}$ and $\ell_2 := \overline{aq}$ meet only at $a$, and divide $Q$ into smaller polygons. Then there must be one among $\ell_1$ and $\ell_2$ which does not contain $x$, and which is the side of a smaller polygon contained in $Q$ and with an interior point. Similarly if $p$ and $q$ do not belong to the same edge, let $a$ be the vertex not contained in the edge containing $p$. Then the segments $\ell_1 := \overline{ap}$ and $\ell_2 := \overline{qp}$ intersect only at $p$ and divide $Q$ into smaller polygons. One of them must have an interior point.

**Figure 21: Picture for Proposition A.4**

Consider the polygon $\overline{abcy}$  
Consider the triangle $\overline{abp}$  
Consider the polygon $\overline{abpq}$

**Lemma A.5.** Consider the projective toric surface $X$ corresponding to the polygon $P$, and let $Q \subseteq P$ be a subpolygon of $P$, with corresponding projective toric surface $Y$. The two polygons give projective embeddings $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$. There is a linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^m$ which gives an equivariant rational map $X \dashrightarrow Y$, which is an isomorphism on $(\mathbb{C}^*)^2$.

**Proof.** Consider the characters $\chi_0, \ldots, \chi_n$ corresponding to the lattice points in $P$, and let $\chi_0, \ldots, \chi_m$ be those corresponding to the lattice points in $Q$, with $m < n$. Then one can consider the map $\Phi : (\mathbb{C}^*)^2 \rightarrow \mathbb{P}^n$ sending $p \mapsto [\chi_0(p), \ldots, \chi_n(p)]$. The variety $X$ is the closure of the image of $\Phi$, and $Y$ is the closure of the map $\Phi : (\mathbb{C}^*)^2 \rightarrow \mathbb{P}^m$ sending $p \mapsto [\chi_0(p), \ldots, \chi_m(p)]$. Then the projection from the last $n - m$ coordinates gives the desired rational map.

**Theorem A.6.** Consider a projective toric surface corresponding to a building block polygon $P$, with the lattice points of $P$ corresponding to the characters $\chi_0, \ldots, \chi_n$. Then there is no line $\ell \subseteq \mathbb{P}^n$ passing through the identity of $T := (\mathbb{C}^*)^2 \subseteq X$.

**Proof.** We can write any line in $\mathbb{P}^n$ as the intersection of $n - 1$ linearly independent hyperplanes $H_1, \ldots, H_{n-1}$. When we restrict these to the torus $T$, they can be written
as \((H_i)|_T = \sum_j a_{i,j}x^iy^j\) where \((x, y)\) are the coordinates on \((\mathbb{C}^*)^2\). Now, if \(f\) is the equation of \(\ell|_T\), then we can factor \((H_i)|_T = fg_i\) for \(g_i \in \mathbb{C}[x^\pm, y^\pm]\). We can then consider the Newton polygon \(P_f\) associated to \(f\): if we write \(f := \sum c_{i,j}x^iy^j\), \(P_f\) is the convex hull of the points \((i, j)\) such that \(c_{i,j} \neq 0\). Similarly, if we denote with \(P_i\) the one associated to \(g_i\), from [Ost76, Theorem VI] we have that \(P_f + P_i\) has vertices corresponding to the points \(\chi_{i,j}\). In particular, \(P_f + P_i\) is a subpolygon of \(P\).

Now, since the hyperplanes \(H_i\) are linearly independent, the set \(\bigcup_i P_i\) has at least \(n - 1\) elements, say \(x_1, \ldots, x_{n-1}\). Moreover, since \(f\) vanishes on the identity, \(f\) has at least two non-zero coefficients, so \(P_f\) contains a segment \(s\). Therefore the segments \(x_i + s\) belong to \(P\). Checking the four building block polygons, one can see that this is not possible.

**Corollary A.7.** Any projective toric surface \(X\) whose polygon \(P\) has an interior lattice point is not ruled by lines.

**Proof.** From Proposition [A.4] there is a building block polygon \(Q \subseteq P\) corresponding to a toric surface \(Y\). From Lemma [A.5], there is a rational map \(\pi : X \dashrightarrow Y\), which sends lines to lines. So if \(X\) is ruled by lines, there is a line \(\ell\) passing through the identity of the torus in \(X\). Then \(\pi(\ell)\) would be a line through the identity, contradicting Theorem [A.6].

We are finally ready to prove the main theorem of this appendix:

**Proposition A.8.** Consider a projective toric surface \(X\), equivariantly embedded into \(\mathbb{P}^n\), with a non-trivial line bundle \(L\). Assume that the polygon of the embedding \(X \to \mathbb{P}^n\) has an interior lattice point. Then there is an hyperplane section \(C \subseteq X\) such that \(C\) is irreducible and smooth, and \(L|_C\) is not trivial.

**Proof.** We have:

**Lemma A.9** ([Lop91, Lemma II.2.4]). An irreducible non-degenerate surface \(S \subseteq \mathbb{P}^n, n \geq 3\), has an \((n - 1)\)-dimensional family of reducible hyperplane sections if and only if \(S\) is either ruled by lines, or is the Veronese surface, or its general projection in \(\mathbb{P}^4\), or its general projection in \(\mathbb{P}^3\) (the Steiner surface).

The Veronese surface is toric and corresponds to the Newton polygon with vertices

\[\text{Conv}\{(0, 0), (2, 0), (0, 2)\},\]
and so has no interior points. Its general projections have hyperplane sections of zero genus. Since the generic hyperplane sections of $X$ have genus 1, it is not the Veronese or its projections. Therefore by Lemma A.9 and Corollary A.7 we can find a generic pencil of hyperplane sections with all the members irreducible, and with generic member which is smooth. Such a pencil $\Pi$ gives a rational map $X \dashrightarrow \mathbb{P}^1$, we can blow-up the toric surface $\pi : Y \rightarrow X$ to resolve the indeterminacy locus, and have a morphism $f : Y \rightarrow \mathbb{P}^1$ with fibers the members of the pencil $\Pi$.

Such a morphism is flat since it is dominant with target a smooth curve, proper since the source is proper and the target separated, and generically smooth since the generic member of $\Pi$ is smooth. We want to show that $f_*(\mathcal{O}_Y) = \mathcal{O}_{\mathbb{P}^1}$. For that, first observe that $f_*(\mathcal{O}_Y)$ is a torsion free sheaf, since $Y$ is integral. Therefore, since the local rings of $\mathcal{O}_{\mathbb{P}^1}$ are DVRs and torsion free modules over a DVR are free, the sheaf $f_*(\mathcal{O}_Y)$ is locally free: it is a vector bundle. To check its rank, observe that there is a fiber $Y_p$ of $f$ at a point $p$ which is smooth and connected (the smooth member of $\Pi$). Therefore $h^0(\mathcal{O}_{Y_p}) = 1$, and from [Vak17, 28.1.1] there is an open subset $U \subseteq \mathbb{P}^1$ such that for $x \in U$ we have $h^0(\mathcal{O}_{Y_x}) = 1$. Then from [Vak17, 28.1.5] this is the rank of $f_*(\mathcal{O}_Y)$ at $x$. In particular, the latter is a line bundle. But from the definition of push forward, $H^0(\mathbb{P}^1, f_*(\mathcal{O}_Y)) = H^0(Y, \mathcal{O}_Y) \cong \mathbb{C}$: we have that $f_*(\mathcal{O}_Y)$ is a line bundle on $\mathbb{P}^1$ with a single global section. From the description of the line bundles on $\mathbb{P}^1$ we have the desired isomorphism $f_*(\mathcal{O}_Y) \cong \mathcal{O}_{\mathbb{P}^1}$.

Now, assume that for every member $C$ of $\Pi$ we have $L|_C \cong \mathcal{O}_C$. The members of $\Pi$ are the fibers of $f$, thus for every fiber $F$ of $f$ we have $\pi^*(L)|_F \cong \mathcal{O}_F$. Then from [Vak17, Proposition 28.1.11], there is a line bundle $G$ on $\mathbb{P}^1$ such that $\pi^*(L) \cong f^*(G)$. This is the first paragraph of the proof of the Seesaw theorem [Mum74].

In particular, there is a fiber $F$ of $f$ such that $\pi^*(L)|_F$ is not trivial. Then from Lemma A.10, there is an open subset $U \subseteq \mathbb{P}^1$ where for every $p \in U$ we have $\pi^*(L)|_{Y_p}$ is not trivial, and $Y_p$ is smooth (since being smooth is an open condition).

The following Lemma is well known, we provide a proof for completeness.

Lemma A.10. Consider a flat proper morphism $X \rightarrow B$ with integral fibers, and let $L$ be a line bundle on $X$. Then the set $\{b \in B \text{ such that } L|_{X_b} \cong \mathcal{O}_{X_b} \}$ is closed.

Proof. From the upper-semicontinuity theorems [Vak17, 28.1.1] the set $b \in B$ where $h^0(L|_{X_b}) > 0$ and $h^0(L^{-1}|_{X_b}) > 0$ is closed. It suffices to prove that if one has a line bundle $G$ on an integral proper (over $\mathbb{C}$) scheme $Y$, then $h^0(G) > 0$ and $h^0(G^{-1}) > 0$ imply $G \cong \mathcal{O}_Y$. This is the first paragraph of the proof of the Seesaw theorem [Mum74].
Theorem A.11. With the same assumptions of Proposition A.8, assume that the embedding $X \to \mathbb{P}^n$ is non-degenerate. Then if $C$ is the generic hyperplane section, we have $L|_C \not\cong \mathcal{O}_C$.

Proof. If we denote with $(\mathbb{P}^n)\vee$ the projective dual projective space of $\mathbb{P}^n$, we can construct the generic hyperplane section $H \subseteq \mathbb{P}^n \times (\mathbb{P}^n)\vee$ as
\[
\{(x, H) \in \mathbb{P}^n \times (\mathbb{P}^n)\vee : x \in H\}.
\]
We have the closed embedding $X \hookrightarrow \mathbb{P}^n$ which in turn gives the closed embedding $X \times (\mathbb{P}^n)\vee \hookrightarrow \mathbb{P}^n \times (\mathbb{P}^n)\vee$. We can construct the fibred product $C := H \times_{\mathbb{P}^n \times (\mathbb{P}^n)\vee} X \times (\mathbb{P}^n)\vee$. Observe that $C \to H$ is a closed embedding as well, since being a closed embedding is stable under base change. Moreover, $H \to (\mathbb{P}^n)\vee$ is proper. So the composition $\pi : C \to (\mathbb{P}^n)\vee$ is proper as well.

We understand the space $C$ via its morphism $\pi : C \to (\mathbb{P}^n)\vee$: a fiber of $\pi$ over the point of $(\mathbb{P}^n)\vee$ corresponding to the hyperplane $H$ is the intersection $H \cap X$, i.e. it is a hyperplane section in $X$. To check that the morphism $\pi$ is flat, it suffices to check that all the fibers have the same Hilbert polynomial [Vak17, 24.7.A, (d)]. But for every hyperplane section $H$, we have an embedding $\mathcal{O}_\mathbb{P}^n(-1) \to \mathcal{O}_\mathbb{P}^n$, which gives the following exact sequence where $C := X \cap H$, since $X$ is non-degenerate:
\[
0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_C \to 0.
\]
Then by definition of Hilbert polynomial, we see that the Hilbert polynomial of $C$ does not depend on $H$. Therefore all the fibers of $\pi$ have the same Hilbert polynomial, so $\pi$ is flat.

We can take the pull-back of $L$ to $X \times (\mathbb{P}^n)\vee$ and to $C$ to get a line bundle $G$ on $C$ which along each fiber $C = H \cap X$ of $\pi$ restricts to $L|_C$. From Proposition A.8, there is a smooth fiber $F$ of $\pi$ such that $G|_F \not\cong \mathcal{O}_F$. We can replace $(\mathbb{P}^n)\vee$ with the locus $U \subseteq (\mathbb{P}^n)\vee$ where $\pi$ is smooth (which is open, and contains the fiber $F$). Then Lemma A.10 applies, giving the desired result. \hfill \Box

References


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