Generalized shufflings in the dimer model

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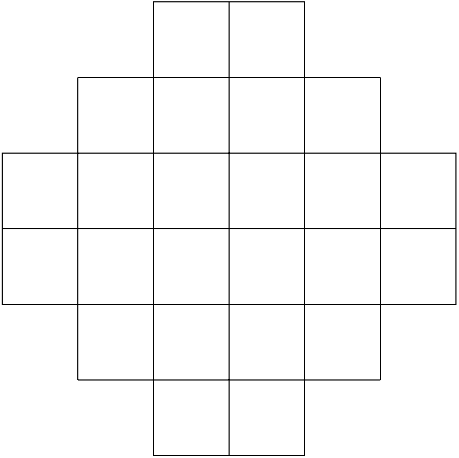
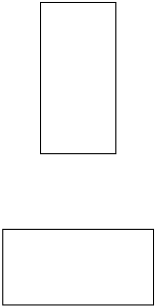
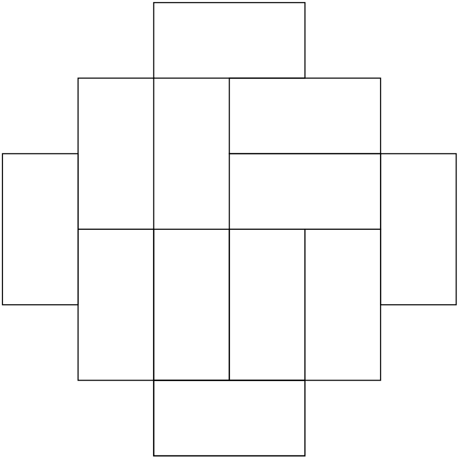
Domino tilings of the aztec diamond

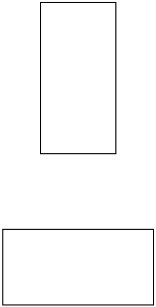




Dominoes

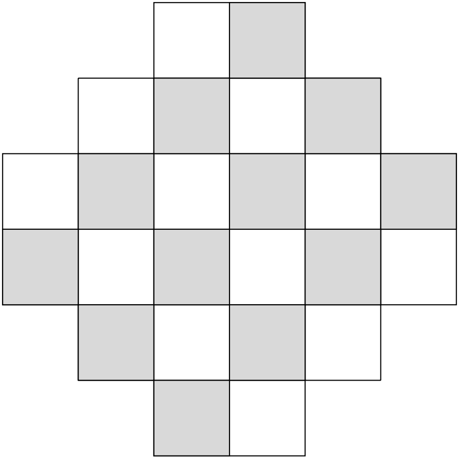
Domino tilings of the aztec diamond



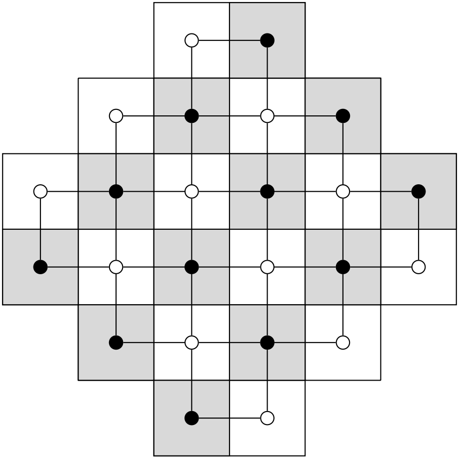


Dominoes

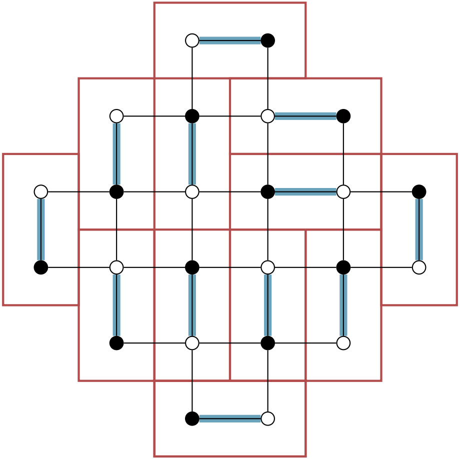
Bipartite coloring



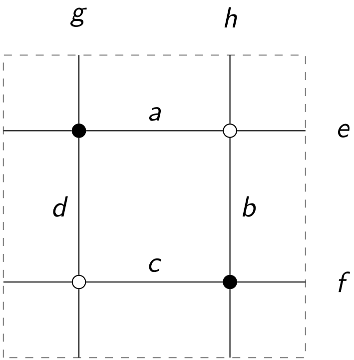
Dual bipartite graph



Dominoes = dimers



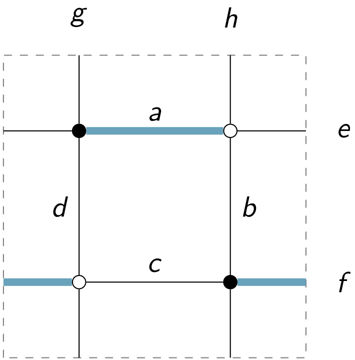
Weighted bipartite graphs on the torus



Γ with weight wt $\in \mathbb{R}_{>0}, \mathbb{C}^{\times}$

Boltzmann probability measure on dimers

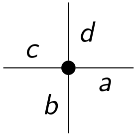
$$\mathrm{wt}(M) := \prod_{e \text{ edge in } M} \mathrm{wt}(M)$$
$$\mathbb{P}(M) := \frac{\mathrm{wt}(M)}{Z}$$

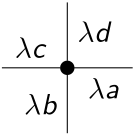


dimer M with $\mathrm{wt}(M) = af$

Gauge transformations

$$\mathrm{wt}_1 \sim \mathrm{wt}_2$$



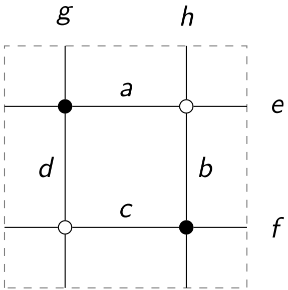


$$\lambda \in \mathbb{R}_{>0}, \mathbb{C}^{\times}$$

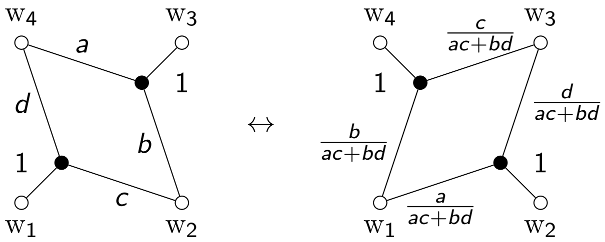
$\mathbb{P}$ is invariant under gauge transformations

Space of weights on Γ

$$\begin{aligned}
\mathcal{L}_\Gamma &:= \mathbb{C}^\times\text{-weights modulo gauge transformations} \\
&= H^1(\Gamma, \mathbb{C}^\times) \\
&\cong (\mathbb{C}^\times)^{\#\text{faces}+1} \\
\mathcal{L}_\Gamma(\mathbb{R}_{>0}) &:= \mathbb{R}_{>0}\text{-weights modulo gauge transformations}
\end{aligned}$$

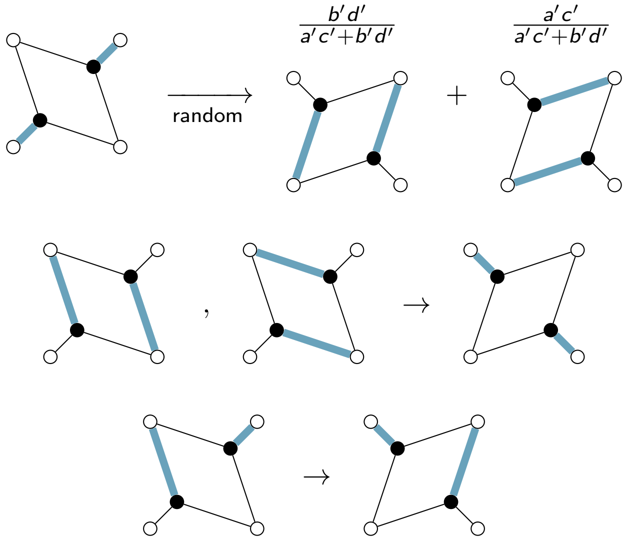


Spider move/urban renewal/cluster mutation [Kuperberg]



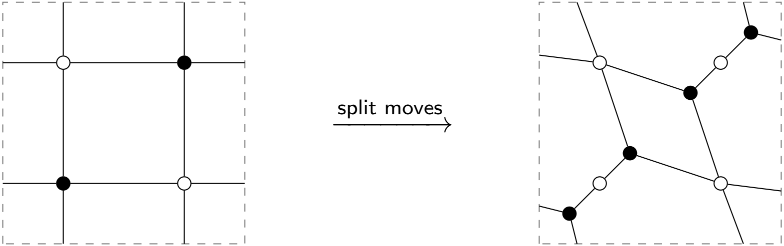
The transformation of weights is characterized by compatibility with the Boltzmann probability measure

Shuffling

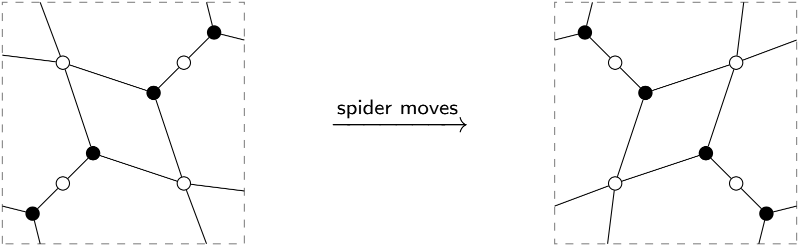


etc.

Domino shuffling

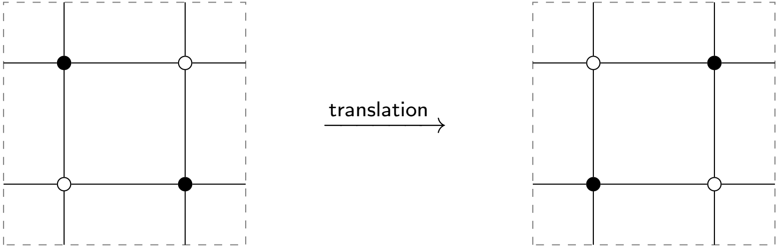


Domino shuffling



Domino shuffling

Domino shuffling



The cluster modular group

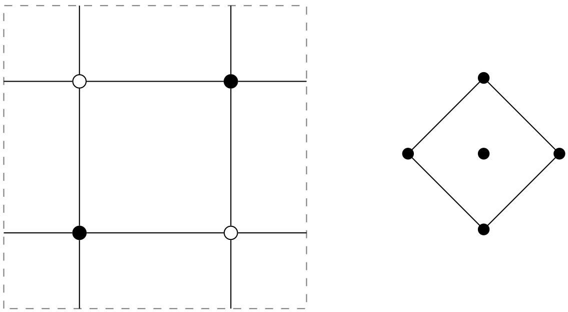
A sequence of spider moves and translations such that the initial and final graphs are both Γ is called a cluster modular transformation.

A cluster modular transformation is trivial if the induced map of weights is the identity.

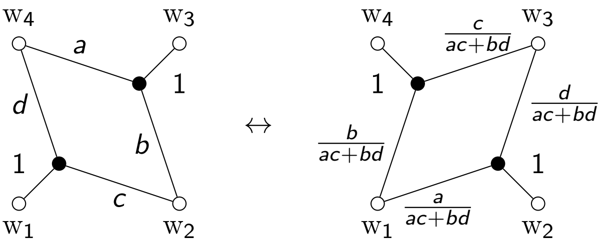
The cluster modular group (based at Γ) G_Γ is the group of cluster modular transformations modulo the trivial ones.

For example, we will see that the cluster modular group for the graph in domino shuffling is $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and it is generated by the domino shuffling sequence and the translation $(\frac{1}{2}, \frac{1}{2})$.

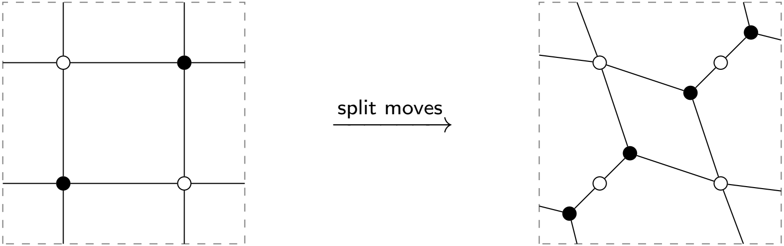
Zig-zag paths and the Newton polygon



Spider move and zig-zag paths



Domino shuffling and and zig-zag paths



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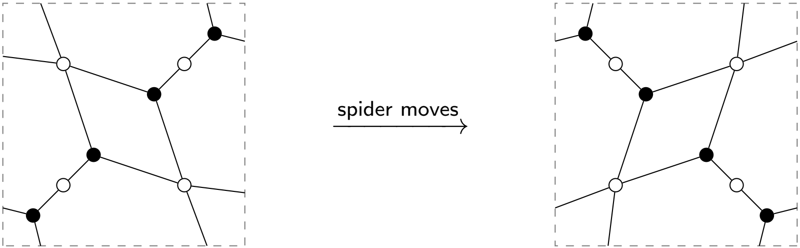

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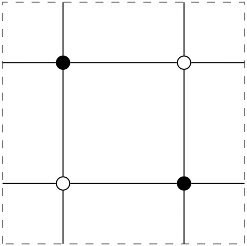
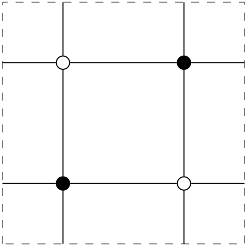
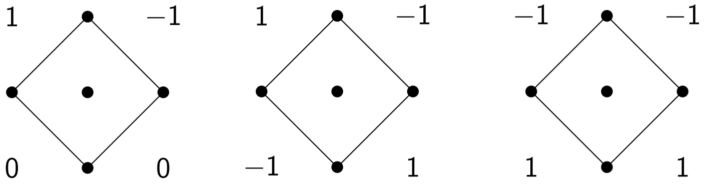


$\xrightarrow{\text{translation}}$



Fock-Marshakov conjecture [2016]



domino shuffling   translation by $(1,0)$ translation by $(0,1)$

Theorem (George-Inchiostro [2020])

The homomorphism

$$\{\textit{cluster modular transformations}\} \to \mathbb{Z}_0^{\textit{Edges of } N} / H_1(\textit{torus}, \mathbb{Z})$$

gives an isomorphism $G_\Gamma \cong \mathbb{Z}_0^{\textit{Edges of } N} / H_1(\textit{torus}, \mathbb{Z})$.

Consequences

Theorem (George-Inchiostro [2020])

The homomorphism

$$\{\textit{cluster modular transformations}\} \to \mathbb{Z}_0^{\textit{Edges of } N} / H_1(\textit{torus}, \mathbb{Z})$$

gives an isomorphism $G_\Gamma \cong \mathbb{Z}_0^{\textit{Edges of } N} / H_1(\textit{torus}, \mathbb{Z})$.

1. G_Γ depends only on the Newton polygon N of Γ.
2. G_Γ has rank $\#$Edges of $N - 3$.

Example (reality check)

Every triangular Newton polygon corresponds to a fundamental domain of the honeycomb graph. By our theorem, the cluster modular group has rank zero, so it is a finite group. It is the group of translation symmetries of the graph. This makes sense because the honeycomb graph has no square faces to do spider moves.

Idea behind proof

The proof has two parts:

1. Combinatorial part: We use the theory of triple crossing diagrams (Dylan Thurston [2004], Alexander Postnikov [2006]) to show that every function on the edges of the Newton polygon is realized by a sequence of spider moves and translations i.e. the map of Fock-Marshakov is surjective.
2. Algebro-geometric part: We use results of Fock [2015] on the spectral transform of Kenyon-Okounkov [2006] to check which cluster modular transformations are trivial.

The spectral transform [Kenyon-Okounkov 2006]

The spectral transform is a rational map

$$\mathcal{L}_\Gamma \to \{(C, \mathcal{F}, \text{some combinatorial data})\}$$

where

1. C is an algebraic curve called the spectral curve.
2. $\mathcal{F}$ is a line bundle on C (of degree $=$ genus of C).

Theorem (Fock 2015)

The spectral transform is birational. Each cluster modular transformation is a certain explicit translation in the Jacobian variety of C (the group of line bundles on C).

Therefore the complicated map on weights induced by a cluster modular transformation becomes linear in the spectral transform coordinates!

Thank you!