1. True or False. Determine whether the following statements are true or false and provide an explanation for your answer (5 points each).

(a) Let $F(x, y, z)$ be a vector field whose components have continuous second partial derivatives in $\mathbb{R}^3$, then

$$\nabla \cdot (\nabla \times F) = 0$$

(b) If $F = \langle P, Q \rangle$ and $P_y = Q_x$ in an open region $D$, then $F$ is conservative.

(c) If $f$ has continuous partial derivatives on $\mathbb{R}^3$ and $C$ is any circle, then $\oint_C \nabla f \cdot dr = 0$.

(d) $\int_{-C} f(x, y) \; ds = - \int_C f(x, y) \; ds$.

(e) If $S$ is a sphere and $F$ is a constant vector field, then $\iint_S F \cdot dS = 0$.

(f) Suppose a solid $E$ and its boundary surface $S$ satisfy the conditions of the divergence theorem, then

$$\iiint_E (f \nabla g - g \nabla f) \cdot n \; dV = \iint_S (f \nabla^2 g - g \nabla^2 f) \; dS$$

(g) $\int_{-C} f(x, y) \; dx = - \int_C f(x, y) \; dx$.

(h) The integral

$$\int_0^{2\pi} \int_0^2 \int_r^2 \; dz \; dr \; d\theta$$

represents the volume enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 2$.

(i) If $C$ is a simple closed curve in $\mathbb{R}^2$, $n$ its unit normal vector, $D$ the region it encloses, and $F$ a vector field whose component functions have continuous first partial derivatives, then

$$\oint_C F \cdot n \; ds = \iint_D \nabla \cdot F \; dA$$

(j) If $C$ is a smooth simple closed curve in the $xy$-plane, then the area of the domain inside $C$ is given by

$$\oint_C x \; dy$$

where the integration is counterclockwise.
2. Consider the two dimensional potential fluid flow with velocity field,
\[ \mathbf{v}(x, y) = \langle v_1(x, y), v_2(x, y) \rangle = \nabla \Phi(x, y) \]
where \( \Phi(x, y) \) is a smooth scalar function.

(a) Show that \( \nabla \times \mathbf{v} = 0 \) for any choice of the function \( \Phi(x, y) \), and explain why this implies that the circulation, \( \oint_C \mathbf{v} \cdot d\mathbf{r} \), is equal to zero for any simple closed curve \( C \).

(b) A fluid is called incompressible if \( \nabla \cdot \mathbf{v} = 0 \). Show that in that case the velocity potential satisfies Laplace’s equation,
\[ \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0. \]

(c) Compute the velocity field of the potential flow with \( \Phi(x, y) = x^2 - y^2 \), and show that the fluid is incompressible.

(d) Consider the motion of a fluid particle in the velocity field from part (c). Let \( \mathbf{r}(t) \) be the position of the particle at time \( t \), and \( \mathbf{r}(0) = (2, 3) \). Write the system of differential equations describing the position of the particle. Show that the fluid particle is moving on the hyperbola \( xy = 6 \).

3. Compute the flux, \( \iint_{\partial E} \mathbf{F} \cdot d\mathbf{S} \), of the vector field
\[ \mathbf{F}(x, y, z) = \frac{x^3}{3} \mathbf{i} + \frac{y^3}{3} \mathbf{j} + 4 \mathbf{k} \]
through the boundary \( \partial E \) of the domain \( E \) bounded by the paraboloid \( z = x^2 + y^2 \) and the plane \( z = 1 \).

4. Let \( S \) be a simple closed surface with outward unit normal \( \mathbf{n} \), and let \( \mathbf{r} = \langle x, y, z \rangle \) be a parametric representation of \( S \).

(a) Show that the volume of the solid contained in the surface \( S \) is given by the integral
\[ V = \frac{1}{3} \int_S \mathbf{r} \cdot \mathbf{n} \, dS. \]

(b) Calculate the volume of a cone of height \( H \) and base area \( A \) by putting the vertex of the cone at the origin and using the formula found in part (a).

5. Let the vector field \( \mathbf{F} \) in \( \mathbb{R}^3 \) be defined by
\[ \mathbf{F}(x, y, z) = \langle z^2, 2yz, 2xz + y^2 \rangle. \]

(a) Show that the vector field \( \mathbf{F} \) is conservative, and find a scalar function \( f(x, y, z) \) such that \( \mathbf{F} = \nabla f \).

(b) If \( \mathbf{F} \), as given above, is the force acting on a particle, find the work of this force in moving the particle from the initial \( (t = 0) \) to the final \( (t = 1) \) points of the curve
\[ \mathbf{r}(t) = \left\langle \sqrt{t}, 3^t - 1, 3 \sin \frac{2t}{\pi} \right\rangle, \quad t \in [0, 1] \]
6. Evaluate \( \oint_C P \, dx + Q \, dy \) where \( P(x, y) = \tan y\), \( Q(x, y) = 3x + x \sec^2 y \) and \( C \) is the circle \( (x - 2)^2 + (y - 5)^2 = 4 \).

7. Find the area of the portion of the plane \( x + 2y + 3z = 1 \) that is inside the cylinder \( x^2 + y^2 = 2 \).

8. Find the area of the parametric surface given by

\[
(u, v) = \langle u \cos v, u \sin v, u \rangle, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi,
\]

and describe the surface.

9. Find the area of the parametric surface given by

\[
r(u, v) = \langle \sin u \cos v, \sin u \sin v, \sin u \rangle, \quad 0 \leq u \leq \frac{\pi}{4}, \quad 0 \leq v \leq 2\pi,
\]

and describe the surface.

10. Use the divergence theorem to compute the flux across \( S \) of the vector field given by

\( \mathbf{F} = \langle x, y, z \rangle \), where \( S \) is the hemisphere \( x^2 + y^2 + z^2 = 4, \ z \geq 0 \) (without the bottom). \textbf{Caution:} the divergence theorem is only part of what you need, you will still need to calculate a surface integral!

**Problems from the text**, page 1172: 11, 13, 15, 17, 19, 27, 29, 32, 37, 38, and 39