The final exam is on Wed., Dec. 13, 1:30pm-3:30pm, in the usual classroom. The exam will cover the entire course up to and including class on Fri., Dec. 10. You may use a non-programmable calculator to do arithmetic on the exam. Please explain your steps to receive full credit. You may use two sheets of notes. Exam booklets will be provided. Vector and matrix norms are the infinity-norm.

1. True or False? Give a reason to justify your answer.

   a) If two floating point numbers with \( n \) significant digits are added, then the result also has \( n \) significant digits.

   b) \( D_+D_-y_i = D_-D_+y_i \)

   c) \( D_0f(x) = f'(x) + O(h^2) \)

   d) If \( Ax = 0 \), then \( A = 0 \) or \( x = 0 \).

   e) If \( A \) is invertible, then \( ||A||^{-1} \leq ||A^{-1}|| \).

   f) \( \rho(B) \leq ||B|| \) for any matrix \( B \)

   g) The spectral radius of a matrix satisfies the properties required to be a matrix norm.

   h) If the pivot elements arising in Gaussian elimination are nonzero, then \( A \) is invertible.

   i) In solving an \( n \times n \) system of linear equations by Gaussian elimination, if \( n \) increases by a factor of 5, then the operation count increases by a factor of approximately 25.

   j) In computing the solution of a linear system \( Ax = b \), if the residual norm \( ||r|| \) is small, then the error norm \( ||e|| \) is also small.

   k) In solving the linear system \( Ax = b \), where \( A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \), one step of Gauss-Seidel reduces the norm of the error as much as two steps of Jacobi.

   l) In solving a linear system \( Ax = b \) by an iterative method \( x_{k+1} = Bx_k + c \), if \( ||B|| < 1 \), then \( \lim_{k \to \infty} x_k = x \) for any initial guess \( x_0 \).

   m) In solving a linear system \( Ax = b \) by an iterative method such as Jacobi or Gauss-Seidel, if the matrix \( A \) has dimension \( n \times n \), then the exact solution is obtained after \( n \) iterations.

   n) In solving a 2-point boundary value problem using a finite-difference scheme with mesh size \( h \), if Jacobi’s method is used to solve the linear system, then the spectral radius of the iteration matrix satisfies \( \lim_{h \to 0} \rho(B_J) = 0 \).

   o) Suppose a two-dimensional boundary value problem is solved using a finite-difference scheme and the resulting linear system is solved by Jacobi’s method with stopping criterion \( ||r_k|| \leq 10^{-2} \). If the mesh size \( h \) is decreased, then the number of iterations needed to satisfy the stopping criterion is increased.

   p) Jacobi and Gauss-Seidel converge linearly, but optimal SOR converges quadratically.

   q) If \( \lambda = 0 \) is an eigenvalue of \( A \), then \( A \) is not invertible.

   r) The “inverse power method” is used to find the inverse of a matrix.

   s) Wilkinson’s example shows that the coefficients of a polynomial can depend sensitively on the roots.

   t) When the power method is applied to find the largest eigenvalue and corresponding eigenvector of a matrix, the vectors \( v^{(k)} \) are normalized at each step in order to accelerate convergence of the method.
u) If \( p_n(x) \) is the interpolating polynomial of degree \( n \) for a given function \( f(x) \) at points \( x_i = a + ih \), where \( h = \frac{b-a}{n} \) and \( i = 0 : n \), then \( \lim_{n \to \infty} p_n(x) = f(x) \) for all \( x \in [a, b] \).

v) If \( p_n(x) \) is the Taylor polynomial of degree \( n \) for \( f(x) \) about \( x = a \), then \( p_n^{(n+1)}(a) = 0 \).

w) Polynomial interpolation at the uniform points on the interval \( a \leq x \leq b \) gives a good approximation near the endpoints of the interval and a bad approximation near the center of the interval.

x) Suppose \( f(x) \) is approximated by a cubic spline interpolant \( s(x) \) on the interval \( a \leq x \leq b \) with interpolation points \( x_i = a + ih \), where \( h = \frac{b-a}{n} \) and \( i = 0 : n \). Then if \( n \) is doubled, the error defined by \( \max_{a \leq x \leq b} |f(x) - s(x)| \) is reduced by a factor of approximately \( 1/16 \).

2. State one advantage of . . .

a) . . . Newton’s method over the secant method.

b) . . . Gaussian elimination with pivoting over Gaussian elimination without pivoting.

c) . . . optimal SOR over Gauss-Seidel.

d) . . . Chebyshev points over uniform points.

e) . . . cubic spline interpolation over Taylor approximation.

3. Consider the following approximation for the first derivative, \( f'(x) \approx \frac{-3f(x)+4f(x+h)-f(x+2h)}{2h} \).

a) Apply the method to compute \( f'(1) \), where \( f(x) = e^x \) with step sizes \( h = 1, \frac{1}{2}, \frac{1}{4} \). Present the results in a table with the following format. column 1: \( h \) (step size), column 2: approximation, column 3: error, column 4: error/h, column 5: error/h^2.

b) The error has the form: \( \text{error} = cf^{(m)}(x)h^n + \cdots \). Find the constants \( c, m, n \). Are the results of part (a) and (b) consistent with each other? Explain.

4. Consider solving the equation \( f(x) = x^2 - 5 = 0 \) by the bisection method.

a) Explain why \( 0 \leq x \leq 4 \) is a suitable starting interval.

b) Take 3 steps of the bisection method, i.e. compute \( x_0, x_1, x_2 \).

c) Approximately how many steps are needed to ensure that the error is less than \( 10^{-4} \)?

5. Suppose fixed-point iteration is applied to the function \( g(x) = x^2 - \frac{1}{2}x + \frac{1}{2} \). Find the fixed points and in each case determine whether the iteration converges for starting values sufficiently close to the fixed point.

6. The component form of SOR for a \( 2 \times 2 \) system is given below. Correct any errors.

\[
\begin{align*}
\frac{a_{11} x_1^{(k+1)}}{a_{12} x_2^{(k+1)}} &= a_{11} x_1^{(k)} - \omega (a_{11} x_1^{(k)} + a_{12} x_2^{(k)} - b_1) \\
\frac{a_{22} x_2^{(k+1)}}{a_{22} x_2^{(k+1)}} &= a_{22} x_2^{(k)} - \omega (a_{21} x_1^{(k)} + a_{22} x_2^{(k+1)} - b_2)
\end{align*}
\]

7. Consider the linear system \( 2x_1 - x_2 = 1, -x_1 + 2x_2 - x_3 = 0, -x_2 + 2x_3 = 1 \), with solution \( x_1 = x_2 = x_3 = 1 \). a) Write out Jacobi’s method in component form. Take one step starting from the zero vector. Compute the error norms ||\( e_0 || ||e_1 || \). b) Repeat for Gauss-Seidel.

8. Let \( A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \), \( A_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \).

a) For which of these does Jacobi’s method converge?

b) For which of these does Gauss-Seidel converge?

9. Which of the following matrices are positive definite? a) \( \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \), b) \( \begin{pmatrix} 4 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 4 \end{pmatrix} \)
10. Let \( A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix} \).

a) Find a vector \( x \) such that \( ||Ax|| = ||A|| \).

b) If \( Ax = b \) is solved by Jacobi’s method, does the iteration converge for any initial guess?

c) Answer the same question for Gauss-Seidel.

d) Find the optimal SOR parameter \( \omega \).

e) Find an approximate \( e \)-value of \( A \) by taking one step of the power method.

f) Find an approximate \( e \)-value of \( A \) by taking one step of the inverse power method.

11. Prove the following results or give a counterexample.

a) If \( A \) is positive definite, then \( A \) is invertible.

b) If \( A \) is positive definite, then the diagonal elements of \( A \) are positive.

c) If \( A \) is positive definite, then the eigenvalues of \( A \) are positive.

d) If \( A \) is invertible, then \( A^TA \) is symmetric and positive definite.

12. The 2-point boundary value problem \(-y'' + y = x, y(0) = 1, y(1) = 0 \) for \( 0 \leq x \leq 1 \) is solved by the finite-difference scheme \(-D_x^2 D_y w_i + w_i = x_i \) for \( i = 1 : n \), with step size \( h = 1/(n+1) \), where \( x_i = ih \) and \( w_0 = 1, w_{n+1} = 0 \). Using \( n = 3 \), write down the linear system \( A_h w_h = f_h \).

13. Consider the Poisson equation \(-\Delta \phi = f \) with boundary condition \( \phi = g \), on the unit square \( 0 \leq x, y \leq 1 \). Let the domain be discretized with mesh size \( h = \frac{1}{4} \). Then there are nine unknown values in the interior of the domain, \( w_{ij} \), for \( i, j = 1, 2, 3 \). Suppose the equation is discretized using the finite-difference scheme discussed in class, \(- (D_x^2 D_y w_{ij} + D_y^2 D_x w_{ij}) = f_{ij} \), and the linear system is written as \( A_h w_h = f_h \), where the elements of \( w_h \) have the ordering shown in the figure (this is called the red-black ordering, like a checkerboard, and it is different than the ordering considered in class). Write down the matrix \( A_h \) in this case.

14. The formulas \( \rho(B_J) = \cos \pi h, \rho(B_{GS}) = \cos^2 \pi h, \rho(B_{ws}) = \frac{1-\sin \pi h}{1+\sin \pi h} \) were given in class for the spectral radius of the iteration matrix of a certain finite-difference scheme applied to a boundary value problem. Graph each formula \( \rho(B) \) as a function of \( \pi h \) on the same plot for \( 0 \leq \pi h \leq \pi \). Label each formula. What do the graphs imply about the rate of convergence of the methods?

15. Apply the spectral method (see hw7) to solve \( Ax = b \), where \( A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}, b = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \).

16. Determine whether \( \rho(A) = ||A|| \). a) \( \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \) b) \( \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix} \) c) \( \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \)

17. Let \( A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \). Find \( \max_{x \neq 0} R_A(x) \) and \( \min_{x \neq 0} R_A(x) \). Make sure to justify your answer.

18. Construct the divided difference table for the following data set and find the Newton form of the interpolating polynomial \( p_2(x) \). \((x_0, x_1, x_2) = (2, 4, 5), (y_0, y_1, y_2) = (-1, 4, 8)\). Use the polynomial to estimate the missing data for \( x = 3 \).
19. Let \( f(x) = 1 + x + x^2 \) and take \( x_0 = -1, x_1 = 0, x_2 = 1 \). Construct the divided difference table and find the Newton form of the interpolating polynomial \( p_2(x) \).

20. Let \( f(x) = 1/x \) and take \((x_0, x_1, x_2, x_3) = (1, 2, 3, 4)\). Find the Newton form of the interpolating polynomial \( p_3(x) \). Verify that \( p_3(x) \) interpolates \( f(x) \) at the given points.

21. The outdoor temperature \( T(t) \) is recorded at two-hour intervals starting at 8am and ending at 4pm, but the 12pm measurement was accidentally omitted. The recorded temperatures are \( T(8am) = 30^\circ\text{F}, T(10am) = 40^\circ\text{F}, T(2pm) = 50^\circ\text{F}, T(4pm) = 60^\circ\text{F} \). Use a cubic interpolating polynomial to estimate the missing temperature.

22. The thermal conductivity of air as a function of temperature is given in the table below. Estimate the thermal conductivity of air when \( T = 240 \text{ K} \) using the Newton form of the interpolating polynomial.

<table>
<thead>
<tr>
<th>temperature (K)</th>
<th>100</th>
<th>200</th>
<th>300</th>
<th>400</th>
<th>500</th>
<th>600</th>
</tr>
</thead>
<tbody>
<tr>
<td>thermal conductivity ((\text{mW/m} \cdot \text{K}))</td>
<td>9.4</td>
<td>18.4</td>
<td>26.2</td>
<td>33.3</td>
<td>39.7</td>
<td>45.7</td>
</tr>
</tbody>
</table>

23. Let \( f(x) \) be a given function and let \( p_1(x) \) be the Taylor polynomial of degree 1 for \( f(x) \) about a given point \( x = a \).
   a) Show that \( f(x) = p_1(x) + \int_a^x (x-t)f''(t)dt \). (hint: integrate by parts.)
   b) Show that \(|f(x) - p_1(x)| \leq \frac{1}{8}Mh^2\), where \( M = \max_{a \leq t \leq x} |f''(t)| \) and \( h = |x-a| \). (note: this shows that linear Taylor approximation is 2nd order accurate.)

24. Determine whether \( s(x) \) is a natural cubic spline.
   a) \( s(x) = \begin{cases} 0 & 0 \leq x \leq 1 \\ x^3 - 3x^2 + 3x - 1 & 1 \leq x \leq 2 \end{cases} \)
   b) \( s(x) = \begin{cases} -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1 & -1 \leq x \leq 0 \\ \frac{3}{2}x^3 - \frac{3}{2}x^2 + 1 & 0 \leq x \leq 1 \end{cases} \)

25. Find the natural cubic spline \( s(x) \) satisfying \( s(0) = 0, s(1/2) = 1, s(1) = 0 \). Your answer will be 2 cubic polynomials, \( s_0(x) \) defined on the interval \( 0 \leq x \leq \frac{1}{2} \) and \( s_1(x) \) defined on the interval \( \frac{1}{2} \leq x \leq 1 \).

26. Compute the integral \( \int_0^1 \sin \pi x \, dx \) using the trapezoidal rule and present the results in a table with the following columns. column 1: \( h \) (take \( h = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \)), column 2: \( T(h) \), column 3: \(|I - T(h)|\), column 4: \(|I - T(h)|/h\), column 5: \(|I - T(h)|/h^2\), where \( I \) is the exact value and \( T(h) \) is given by the trapezoidal rule. What is the order of accuracy? Explain.

27. The local form of the midpoint rule is \( \int_0^b f(x) \, dx \approx cf(\frac{1}{2}b) \), where \( c \) is a constant.
   a) Find the value of \( c \) which ensures that the midpoint rule is exact for \( f(x) = 1 \). Show that the resulting method is also exact for \( f(x) = x \).
   b) Is the midpoint rule more accurate or less accurate than the trapezoid rule?

28. Consider the formula \( \int_0^{2h} f(x) \, dx \approx c_0 f(0) + c_1 f(h) + c_2 f(2h) \), where \( c_0, c_1, c_2 \) are constants.
   a) Find values of \( c_0, c_1, c_2 \) which ensure that the formula is exact for \( f(x) = 1, x, x^2 \).
   b) Is the formula exact for \( f(x) = x^3, x^4, x^5 \)? (note: this formula is called Simpson’s rule.)

29. Evaluate the integral \( \int_0^{2\pi} e^{-x} \sin x \, dx \) using the following methods with mesh size \( h = 2\pi, \pi, \frac{1}{2}\pi \) in parts (a), (b), (c).
   a) trapezoid rule , b) midpoint rule , c) calculus (i.e. integration by parts, etc.)
   Comment on the accuracy of each method.