chapter 7 : differential equations

Let \( x(t) \) be the position of a particle moving on the \( x \)-axis at time \( t \).

1st order ODE

\[
\frac{dx}{dt} = f(x) \quad \text{velocity is a function of position}
\]

\( x(0) \) : initial position

The problem is to find the position \( x(t) \) for \( t > 0 \). This is an example of an initial value problem (IVP), in contrast with the boundary value problems (BVP) we considered before.

ex

1. \[
\frac{dx}{dt} = x, \quad x(0) = 1 \quad \Rightarrow \quad x(t) = e^t
\]

2. \[
\frac{dx}{dt} = x^2, \quad x(0) = 1 \quad \Rightarrow \quad x(t) = \frac{1}{1 - t}
\]

3. \[
\frac{dx}{dt} = \sin x, \quad x(0) = 1 \quad \Rightarrow \quad x(t) = ?
\]

The simplest numerical method is Euler’s method.

choose \( \Delta t \) : time step

define \( w_n \) : numerical solution at time \( t_n = n\Delta t \)

\[
\frac{w_{n+1} - w_n}{\Delta t} = f(w_n) \quad \Rightarrow \quad w_{n+1} = w_n + f(w_n)\Delta t
\]

given \( w_0 \), we can compute \( w_1, w_2, \ldots \)

questions : accuracy , stability , ...

2nd order ODE

\[
\frac{d^2x}{dt^2} = f(x) \quad \text{acceleration is a function of position (Newton’s equation)}
\]

\( x(0), \ x'(0) \) : initial position and velocity

\[
\frac{w_{n+1} - 2w_n + w_{n-1}}{(\Delta t)^2} = f(w_n) \quad \Rightarrow \quad w_{n+1} = 2w_n - w_{n-1} + f(w_n)(\Delta t)^2
\]

given \( w_0 \) and \( w_1 \), we can compute \( w_2, w_3, \ldots \)
partial differential equations

heat equation

\( u(x, t) \) : temperature of a metal rod at position \( x \) and time \( t \)

\[
\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}
\]

\( \kappa \) : coefficient of thermal diffusion

The simplest numerical method is a finite-difference scheme.

choose \( \Delta x \) : space step , \( \Delta t \) : time step

define \( w^n_i \) : numerical solution at position \( x_i = i\Delta x \) and time \( t_n = n\Delta t \)

\[
\frac{w^{n+1}_i - w^n_i}{\Delta t} = \kappa \frac{w^n_{i+1} - 2w^n_i + w^n_{i-1}}{(\Delta x)^2}
\]

\[
\Rightarrow w^{n+1}_i = w^n_i + \frac{\kappa \Delta t}{(\Delta x)^2} (w^n_{i+1} - 2w^n_i + w^n_{i-1})
\]

wave equation

\( u(x, t) \) : displacement of a flexible rod at position \( x \) and time \( t \)

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}
\]

\( c \) : wave speed

\[
\frac{w^{n+1}_i - 2w^n_i + w^n_{i-1}}{\Delta t} = c^2 \frac{w^n_{i+1} - 2w^n_i + w^n_{i-1}}{(\Delta x)^2}
\]

\[
\Rightarrow w^{n+1}_i = 2w^n_i - w^{n-1}_i + \frac{c^2 \Delta t}{(\Delta x)^2} (w^n_{i+1} - 2w^n_i + w^n_{i-1})
\]