< Student Rep. Th. >

$$H^i(-,\mathbb{Q}) : \mathrm{Top} \to \mathrm{Vec}_{\mathbb{Q}} \quad \forall i \geq 0$$

$S_n \curvearrowright X^n$, $\quad X$ fixed top space

Assumptions: ① $\forall n$

$$\begin{tikzcd} X^n \arrow[r] & X^n/S_n & \rightsquigarrow & H^i(X^n,\mathbb{Q}) \arrow[loop below] & H^i(X^n/S_n,\mathbb{Q}) \arrow[l] \arrow[dl, "\cong"] \\ & & & H^i(X^n;\mathbb{Q})^{S_n} & \end{tikzcd} \quad \forall i \geq 0$$

② $\forall n$

$$H^\bullet(X^n;\mathbb{Q}) \cong H^\bullet(X,\mathbb{Q})^{\otimes n}$$

Fact. ① holds when X is HD, locally contractable, and compact (due to Grothendieck)
② holds when X is compact
$\Big\}$ for $H^\bullet = H^\bullet_{sing}$

③ $h^i(X) := \dim_{\mathbb{Q}}(H^i(X,\mathbb{Q})) < +\infty \quad \forall i \geq 0$

④ $h^i(X) = 0 \quad \forall i \gg 0$

Defn $\chi(X, z) = P_X(-z) = \sum_{i=0}^{\infty} (-z)^i h^i(X) \in \mathbb{Z}[[z]]$ with ③

e.g., $\chi(X,1) = \chi(X)$ with ③ + ④

Thm (MD) ① ~ ③ $\Rightarrow \sum_{n \geq 0} \chi(\mathrm{Sym}^n X, z)\, t^n = \prod_{i \geq 0} \left(\frac{1}{1 - z^i t} \right)^{(-1)^i h_i(X)} \in \mathbb{Z}[[z,t]]$

Cor (Rationality of top. zeta) ① ~ ④ $\Rightarrow \sum_{n \geq 0} \chi(\mathrm{Sym}^n X)\, t^n = \left(\frac{1}{1-t} \right)^{\chi(X)}$

§ 1. Quadratic form ass. to a perm.

$\sigma \in S_n \rightsquigarrow Q_\sigma$

Def'n
$$Q_\sigma(z_1,\cdots,z_n) := \sum_{\substack{1\leq i<j\leq n \\ \sigma(i)>\sigma(j)}} z_i z_j \in \mathbb{Z}[z_1,\cdots,z_n]$$

Lemma 1. $\sigma = (r+1 \ \cdots \ r+m)$ m-cycle in S_n

$\Rightarrow Q_\sigma(z_1,\cdots,z_n) = (z_{r+1}+\cdots+z_{r+m-1})z_{r+m}$

pf) $\left.\begin{array}{l} 1\leq i<j\leq n \\ i+1=\sigma(i)>\sigma(j) \end{array}\right\} \Rightarrow j=r+m$ and $\sigma(j)=1$ $\square$

Lemma 2. $\sigma_1,\cdots,\sigma_k$ disjoint cycles in S_n.

$$Q_{\sigma_1\cdots\sigma_k}(\underline{z}) = Q_{\sigma_1}(\underline{z}) + \cdots + Q_{\sigma_k}(\underline{z})$$

pf) $\sigma = \sigma_1\cdots\sigma_k$

$1\leq i<j\leq n$

$\sigma(i)>\sigma(j) \Leftrightarrow \sigma_r(i)>\sigma_r(j)$ for unique $1\leq r\leq n$ $\square$

Def'n $\sigma,\tau \in S_n \rightsquigarrow Q_\sigma^\tau(z_1,\cdots,z_n) := \displaystyle\sum_{\substack{1\leq i<j\leq n \\ (\tau(i)-\tau(j))(\sigma\tau(i)-\sigma\tau(j))<0}} z_i z_j$

Lemma 3. $Q_{\sigma\tau}(\underline{z}) \equiv Q_\sigma^\tau(\underline{z}) + Q_\tau(\underline{z}) \mod 2$.

pf) $1\leq i<j\leq n$

$\tau(i) > \tau(j) \Rightarrow z_i z_j$ appears in $Q_\tau(z)$

$\Downarrow$

$\left[\begin{array}{l} z_i z_j \text{ appears in } Q_\sigma^\tau(z) \\ \Leftrightarrow \sigma\tau(i) < \sigma\tau(j) \end{array} \right]$

$\therefore$ Modulo 2 RHS only has $z_i z_j$ with $\sigma\tau(i) > \sigma\tau(j)$ which means LHS. $\square$

§2. Graded vector spaces + power series

K field

$M = \bigoplus_{r \geq 0} M_r$ in $\underset{\text{fin dim}}{Vec_K}$

$b_r(M) = \dim_K(M_r) < +\infty$

$$M \otimes_K N = \bigoplus_{r_1 \geq 0} M_{r_1} \otimes_K \bigoplus_{r_2 \geq 0} N_{r_2}$$

$$\cong \bigoplus_{r_1, r_2 \geq 0} M_{r_1} \otimes_K N_{r_2}$$

$$= \bigoplus_{r \geq 0} \underbrace{\left(\bigoplus_{r_1 + r_2 = r} M_{r_1} \otimes_K N_{r_2} \right)}_{\overset{\shortparallel}{(M \otimes_K N)_r}}$$

Define

$$E(M, z) := \sum_{r \geq 0} (-z)^r b_r(M)$$

$$(\mathrm{Tr}\,\phi)(z) := \sum_{r \geq 0} (-z)^r \mathrm{Tr}\,\phi_r$$

$$\phi = \bigoplus_{r \geq 0} \phi_r : M \longrightarrow M$$

$$\phi_r : M_r \longrightarrow M_r \quad \text{in } Vec_K$$

Lemma 4. $E(M \otimes_K N, z) = E(M, z) E(N, z)$

pf) $b_r(M \otimes_K N) = \sum_{r_1 + r_2 = r} b_{r_1}(M_{r_1}) b_{r_2}(N_{r_2})$

$$\therefore\ E(M\otimes_K N, z) = \sum_{r\geq 0}\sum_{r_1+r_2=r} (-z)^{\overset{r_1+r_2}{r}}\, b_{r_1}(M_{r_1})\, b_{r_2}(N_{r_2})$$

$$= \underbrace{\sum_{r_1\geq 0} (-z)^{r_1} b_{r_1}(M_{r_1})}_{E(M,z)}\ \underbrace{\sum_{r_2\geq 0} (-z)^{r_2} b_{r_2}(N_{r_2})}_{E(N,z)} \qquad \square$$

§3 Decomposition of trace series into E-series

$M = \bigoplus_{r\geq 0} M_r$ in $\overset{\text{fin dim}}{Vec_K}$, $\quad K$ field

$M^{\otimes n}$ graded as before.

$S_n \curvearrowright M^{\otimes n}$

$$\omega(\sigma)\underbrace{(x_1\otimes\cdots\otimes x_n)}_{\in M_{r_1}\otimes\cdots\otimes M_{r_n}} := (-1)^{Q_\sigma(r_1,\dots,r_n)}\ x_{\sigma^{-1}(1)}\otimes\cdots\otimes x_{\sigma^{-1}(n)}$$

Lemma 5. The above is indeed a k-linear S_n-action

pf) k-linearity is evident.

WTS: $\omega(\sigma\tau) = \omega(\sigma)\omega(\tau)$

$$\omega(\sigma\tau)(x_1\otimes\cdots\otimes x_n) = (-1)^{Q_{\sigma\tau}(r_1,\dots,r_n)}\ x_{\tau^{-1}\sigma^{-1}(1)}\otimes\cdots\otimes x_{\tau^{-1}\sigma^{-1}(n)}$$

$$= (-1)^{Q_\sigma^\tau(r_1,\dots,r_n)+Q_\tau(r_1,\dots,r_n)}\ x_{\tau^{-1}\sigma^{-1}(1)}\otimes\cdots\otimes x_{\tau^{-1}\sigma^{-1}(n)}$$

$\uparrow$ Lemma 3

$$\omega(\sigma)\omega(\tau)(x_1\otimes\cdots\otimes x_n) = \omega(\sigma)\left((-1)^{Q_\tau(r_1,\dots,r_n)}\ x_{\tau^{-1}(1)}\otimes\cdots\otimes x_{\tau^{-1}(n)}\right)$$

$$= (-1)^{Q_\tau(r_1,\dots,r_n)}\ \omega(\sigma)(x_{\tau^{-1}(1)}\otimes\cdots\otimes x_{\tau^{-1}(n)})$$

k-linearity

$$= (-1)^{Q_\tau(r_1,\dots,r_n)}(-1)^{Q_\sigma(r_{\tau^{-1}(1)},\dots,r_{\tau^{-1}(n)})}$$

$$\cdot\ x_{\tau^{-1}\sigma^{-1}(1)}\otimes\cdots\otimes x_{\tau^{-1}\sigma^{-1}(n)}$$

$\therefore$ Enough to show: $Q_\sigma^\tau(z_1, \cdots, z_n) = Q_\sigma(z_{\tau^{-1}(1)}, \cdots, z_{\tau^{-1}(n)})$

Check:
$$Q_\sigma(z_{\tau^{-1}(1)}, \cdots, z_{\tau^{-1}(n)}) = \sum_{\substack{1 \le i < j \le n \\ \sigma(i) > \sigma(j)}} z_{\tau^{-1}(i)} z_{\tau^{-1}(j)}$$

$$= \sum_{\substack{1 \le \tau\tau^{-1}(i) < \tau\tau^{-1}(j) \le n \\ \sigma(i) > \sigma(j)}} z_{\tau^{-1}(i)} z_{\tau^{-1}(j)}$$

$\tau(i) \longleftrightarrow i$

$$= \sum_{\substack{1 \le \tau(i) < \tau(j) \le n \\ \sigma\tau(i) > \sigma\tau(j)}} z_i z_j$$

$$= Q_\sigma^\tau(z_1, \cdots, z_n) \qquad \square$$

$\therefore$ get a rep $\omega : S_n \longrightarrow GL_K(M^{\otimes n})$

Note: this $\rightsquigarrow$ $S_n \xrightarrow{\omega_r} GL_K((M^{\otimes n})_r) \quad \forall r \geq 0$

$\because$ $(M^{\otimes n})_r = \bigoplus_{r_1 + \cdots + r_n = r} M_{r_1} \underset{K}{\otimes} \cdots \underset{K}{\otimes} M_{r_n} \circlearrowleft \omega(\sigma) \quad \forall \sigma \in S_n$

Def'n deg of a rep := dim of target v.s.

$\deg \omega_r = \dim_K (M^{\otimes n}_r)$

$E(M^{\otimes n}; z) = E(M, z)^n$

The coefficient of $(-z)^r$ here.

Lemma 6. $\sigma \in S_n$ cycle type $= [n_1, \cdots, n_k]$

$\Rightarrow (\operatorname{Tr} \omega(\sigma))(z) = E(M, z^{n_1}) \cdots E(M, z^{n_k})$

pf) $(\mathrm{Tr}\,\omega(\sigma))(z) = \sum_{r \geq 0} (-z)^r \mathrm{Tr}(\omega_r(\sigma))$

Recall: $\sigma, \tau \in S_n$ have the same cycle type

$\Leftrightarrow \sigma \sim \tau$ conjugate in S_n

$\sigma = \gamma\tau\gamma^{-1}$ $\longrightarrow$ $\omega_r(\sigma) = \omega_r(\gamma)\omega_r(\tau)\omega_r(\gamma)^{-1} \in GL_K((M^{\otimes n})_r)$

$\Rightarrow \mathrm{Tr}(\omega_r(\sigma)) = \mathrm{Tr}(\omega_r(\tau))$

$\therefore (\mathrm{Tr}\,\omega(\sigma))(z) = (\mathrm{Tr}\,\omega(\tau))(z)$.

$\hookrightarrow$ Upshot: We can pick <u>any</u> $\sigma \in S_n$ with cycle type $[n_1, \dots, n_k]$ to prove the result

$\hookrightarrow$ Reduces to: $\sigma = \underbrace{(1\ 2 \cdots n_1)}_{\sigma_1}\underbrace{(n_1+1 \cdots n_1+n_2)}_{\sigma_2} \cdots \underbrace{(n_1+\cdots+n_{k-1}+1 \cdots n_1+\cdots+n_k)}_{\sigma_k}$

May assume: $n_1 \geq \cdots \geq n_k \ (\geq 1)$

Goal: to compute $\mathrm{Tr}\,\omega_r(\sigma)$

$$\omega_r(\sigma) : (M^{\otimes n})_r \xrightarrow{\sim} (M^{\otimes n})_r \quad \text{in } \mathrm{Vec}_K.$$

$$(M^{\otimes n})_r = \bigoplus_{\sum_{i=1}^n r_i = r} M_{r_1} \otimes_K \cdots \otimes_K M_{r_n}$$

$$\omega_r(\sigma) : M_{r_1} \otimes_K \cdots \otimes_K M_{r_n} \xrightarrow{\sim} M_{r_{\sigma^{-1}(1)}} \otimes \cdots \otimes M_{r_{\sigma^{-1}(n)}}$$

$$x_1 \otimes \cdots \otimes x_n \longmapsto (-1)^{Q_\sigma(r_1, \dots, r_n)} x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}$$

$Tr(\omega_r(\sigma)) \rightsquigarrow$ only care about diagonals!

$\rightsquigarrow$ Focus: $x_1 \otimes \cdots \otimes x_n = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}$

$x_i \in M_{r_i}$ a basis elt $\forall i$.

$$= x_2 \otimes \cdots \otimes x_{n_1} \otimes x_1 \otimes x_{n_1+2} \otimes \cdots \otimes x_{n_1+n_2} \otimes x_{n_1+1} \otimes \cdots x_{n_1+\cdots+n_{k-1}+2} \otimes \cdots \otimes x_{n_1+\cdots+n_k} \otimes x_{n_1+\cdots+n_{k-1}+1}$$

$V_1 := x_1 = x_2 = x_3 = \cdots = x_{n_1} \in M_{r_1}$

$V_2 := x_{n_1+1} = \cdots = x_{n_1+n_2} \in M_{r_2}$

$\vdots \qquad \vdots$

$V_k := x_{n_1+\cdots+n_{k-1}+1} = \cdots = x_{n_1+\cdots+n_k} \in M_{r_n}$

Note: $r = n_1 r_1 + \cdots + n_k r_k$
$= r_1 + \cdots + r_n$

$x_1 \otimes \cdots \otimes x_n = V_1^{\otimes n_1} \otimes \cdots \otimes V_k^{\otimes n_k}$

$\therefore \omega_r(\sigma)(x_1 \otimes \cdots \otimes x_n) = (-1)^{Q_\sigma(r_1,\ldots,r_1,\ldots,r_k,\ldots,r_k)} x_1 \otimes \cdots \otimes x_n$

Q. $Q_\sigma(\overbrace{r_1,\cdots,r_1}^{n_1 \text{ times}},\cdots,\overbrace{r_k,\cdots,r_k}^{n_k \text{ times}}) \equiv ?$ mod 2.

$\sigma = \sigma_1 \cdots \sigma_k$ $\qquad$ Lemma 1

$Q_{\sigma_i}(z_1,\ldots,z_n) = (z_{n_1+\cdots+n_{i-1}+1} + \cdots + z_{n_1+\cdots+n_{i-1}-1}) z_{n_1+\cdots+n_{i-1}+n_i}$

$Q_\sigma(\underline{z}) = Q_{\sigma_1}(\underline{z}) + \cdots + Q_{\sigma_k}(\underline{z})$

Lemma 2

$Q_{\sigma_i}(r_1,\cdots,r_1,\cdots,r_k,\cdots,r_k)$
$= (\underbrace{r_i + \cdots + r_i}_{n_i-1 \text{ times}}) r_i = (n_i-1) r_i^2$

$\therefore Q_\sigma(r_1,\cdots,r_1,\cdots,r_k,\cdots,r_k) = (n_1-1)r_1^2 + \cdots + (n_k-1)r_k^2$
$\equiv (n_1+1)r_1 + \cdots + (n_k+1)r_k$
$= r + r_1 + \cdots + r_k$

$\Rightarrow (-1)^{Q_\sigma(r_1,\ldots,r_1,\ldots,r_k,\ldots,r_k)} = (-1)^r (-1)^{r_1+\cdots+r_k}$

$\exists\ b_{r_1}(M)\cdots b_{r_k}(M)$ many elts of the form $V_1^{\otimes n_1}\otimes\cdots\otimes V_k^{\otimes n_k}$ in $M^{\otimes n}$

$$\therefore\ Tr(\omega_r(\sigma)) = \sum_{n_1r_1+\cdots+n_kr_k=r} (-1)^r(-1)^{r_1+\cdots+r_k}\, b_{r_1}(M)\cdots b_{r_k}(M)$$

$$\Downarrow$$

$$\begin{aligned}
(Tr(\omega(\sigma)))(z) &= \sum_{r\geq 0} Tr(\omega_r(\sigma))\,(-1)^r z^r \\
&= \sum_{r\geq 0}\ \sum_{n_1r_1+\cdots+n_kr_k=r} (-1)^{r_1} b_{r_1}(M)\, z^{n_1r_1}\cdots(-1)^{r_k} b_{r_k}(M)\, z^{n_kr_k} \\
&= \left(\sum_{r_1\geq 0}(-1)^{r_1} b_{r_1}(M)\, z^{n_1r_1}\right)\cdots\left(\sum_{r_k\geq 0}(-1)^{r_k} b_{r_k}(M)\, z^{n_kr_k}\right) \\
&= E(M,z^{n_1})\cdots E(M,z^{n_k})
\end{aligned}$$

$\square$ Lemma 6.

§4. S_n-rep. th. / char 0.

Notations as in §3 + char 0 for K + alg closed

ψ_r = char. of rep ω_r

$\lambda\vdash n$ feed any $\sigma\in S_n$ with cycle type λ.

$$\psi(\lambda)(t) := \sum_{r\geq 0}(-z)^r\psi_r(\lambda)t^r$$

$\psi_r(\lambda)=\psi_r(\sigma)$
$= Tr\,\omega_r(\sigma)$

$$\underset{\text{Lemma 6}}{=} E(M,z^{\lambda_1})\cdots E(M,z^{\lambda_k})$$

where $\lambda=[\lambda_1,\cdots,\lambda_k]$

f.d. $\because \forall v \in V$ irr $\quad K[S_n] \twoheadrightarrow K[S_n]v = V$

Review 1 Recall : $\{\vdash n\} \longleftrightarrow \{$irred. S_n-reps / $K\}$ ☆☆ char $K = 0$

S_n-rep. $\lambda \longmapsto S^\lambda$ (Specht-modules) ↳ some positive definiteness

alg closed → Not 100% sure...

Defn $\lambda = [\lambda_1, \cdots, \lambda_k]$, $\lambda_1 \geq \cdots \geq \lambda_k$

$S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_k}$

$\star$ $S^\lambda = K[S_n]$-submodule of $K[S_n/S_\lambda]$ generated by polytabloids.

What is a polytabloid?

A λ-tableau :

$$\begin{array}{|c|c|c|c|} \hline 2 & 5 & \cdots & 10 \\ \hline 11 & 1 & \cdots & 8 \\ \hline \end{array} \quad \begin{array}{l} \lambda_1 \text{ boxes} \\ \lambda_2 \text{ boxes} \end{array}$$

$$\vdots$$

$$\begin{array}{|c|c|c|c|} \hline 13 & 4 & \cdots & 9 \\ \hline \end{array} \quad \lambda_k \text{ boxes}$$

S_n-action on λ-tableaux $\rightsquigarrow K[S_n]$

A λ-tabloid :

$$\begin{array}{c} \hline 2 \;\; 5 \;\; \cdots \;\; 10 \\ \hline 11 \;\; 1 \;\; \cdots \;\; 8 \\ \hline \vdots \\ \hline 13 \;\; 4 \;\; \cdots \;\; 9 \\ \hline \end{array}$$

S_n-action on λ-tabloids $\rightsquigarrow K[S_n/S_\lambda]$

$S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_k}$

$(\sigma_1, \cdots, \sigma_k)$

$\downarrow \qquad \updownarrow$

$S_n \qquad \sigma_1 * \cdots * \sigma_k$

concatenation

Given a λ-tableau t

$C_t :=$ the column stabilizer

$\wedge$ subgroup of S_n given by t

S_n (i.e., permutations that act on columns)

$e_t := \sum_{\sigma \in C_t} (\operatorname{sgn} \sigma) \{t\} \in K[S_n/S_\lambda]$.

↑ tabloid given by t

Back to discussion

class functions $f : S_n \to K$ invariant under conjugacy

Hence : $c(S_n, K) = \bigoplus_{\mu \vdash n} K \chi_{S^\mu}$

using that K alg closed char 0 $\Rightarrow c(S_n, K)$ has basis given by χ_V

$[V]$ isom classes of irr reps of S_n / K

Write $\chi_\mu = \chi_{S^\mu}$

See Serre's book Ch 2.

Have $\psi_r = \sum_{\mu \vdash n} a_r^\mu \chi^\mu$, $\quad a_r^\mu \in \mathbb{Z}_{\geq 0}$

can replace this with $\overline{\psi(\sigma)}$ if $K = \mathbb{C}$. and ψ is a character

$\boxed{\text{Review 2}}$ $\langle \varphi, \psi \rangle := \frac{1}{n!} \sum_{\sigma \in S_n} \varphi(\sigma) \psi(\sigma^{-1})$ $\qquad \varphi, \psi : S_n \to K$ set maps.

$\langle \chi_W, \chi_V \rangle = \dim_K (\mathrm{Hom}_{S_n}(V, W))$ $\qquad \forall$ fin. dim. $K[S_n]$-modules V, W.

$= \begin{cases} 1 & V = W \\ 0 & V \neq W \end{cases}$ $\qquad$ "Orthogonality"

if V, W are irred.

$\varphi : S_n \to K$ class ftn $\qquad \lambda \vdash n$

$\overline{\varphi}(\lambda) := \varphi(\sigma^{-1})$ where σ has cycle type λ.

$\therefore \; a_r^\mu = \langle \psi_r, \chi^\mu \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \psi_r(\sigma) \overline{\chi^\mu}(\sigma)$

$= \frac{1}{n!} \sum_{\lambda \vdash n} c_\lambda \psi_r(\lambda) \overline{\chi^\mu}(\lambda)$

c_λ: # elts in S_n w/ cycle type λ

* The only time we use defn of Specht module

$S^{[n]} \subset k[S_n/S_n] = k$ $\; \left.\right\} \Rightarrow \chi^{[n]}(\lambda) = 1 \;\; \forall \lambda \vdash n$

$\mathcal{L}$ must be trivial rep k. $\qquad \chi^{[n]} = \chi_{triv}$

$a_r^{[n]} = \langle \psi_r, \chi_{triv} \rangle$

= # of times trivial reps occur in decomp of ω_r.

$\because$ every f.d. S_n-rep $/K$ is semi-simple $= \bigoplus_{\text{finite}}$ irreducibles.

$$\therefore\ a_r^{[n]} = \dim_K \{\eta \in (M^{\otimes n})_r : \omega_r(\sigma)(\eta) = \eta \quad \forall \sigma \in S_n\}$$

$$\because\ (M^{\otimes n})_r = \text{nontriv} \oplus \cdots \oplus \text{nontriv} \oplus \underbrace{\text{triv}}_{k} \oplus \underbrace{\text{triv}}_{k} \oplus \cdots \oplus \underbrace{\text{triv}}_{k}$$

Call this V_r

$$\therefore\ \dim_K(V_r) = a_r^{[n]} = \frac{1}{n!}\sum_{\lambda \vdash n} c_\lambda \psi_r(\lambda)$$

Defn $V := \bigoplus_{r\geq 0} V_r$

Lemma 7. $E(V,z) = \dfrac{1}{n!}\displaystyle\sum_{[n_1,\cdots,n_k]\vdash n} c_{[n_1,\cdots,n_k]} E(M : z^{n_1}) \cdots E(M : z^{n_k})$

pf)
$$E(V,z) \overset{\text{def}}{=} \sum_{r\geq 0} \dim_K(V_r)(-z)^r$$

$$= \sum_{r\geq 0} a_r^{[n]} (-z)^r$$

$$= \frac{1}{n!}\sum_{r\geq 0}\sum_{\lambda\vdash n} c_\lambda \psi_r(\lambda)(-z)^r$$

$$= \frac{1}{n!}\sum_{\lambda\vdash n} c_\lambda \sum_{r\geq 0}\psi_r(\lambda)(-z)^r$$

$\lambda = [n_1,\cdots,n_k]$

$$\overset{\text{Lemma 6}}{=} \frac{1}{n!}\sum_{[n_1,\cdots,n_k]\vdash n} c_\lambda E(M,z^{n_1})\cdots E(M,z^{n_k})$$

$\square$

§5. Product expansion

We work in $R = \mathbb{Z}[[t,z,u_1,\cdots,u_n]] \big/ \left(u_1^r + \cdots + u_n^r - E(M;z^r)\right)_{r\geq 0}$

Def'n. $P(z,t) := \prod_{r\geq 0} (1-z^r t)^{(-1)^{r+1} b_r(M)}$, $b_r(M) = \dim_K(M_r)$

$$\frac{d}{dt}\log P(z,t) = \frac{d}{dt}\sum_{r\geq 0} \log(1-z^r t)^{(-1)^{r+1} b_r(M)}$$

$$= \frac{d}{dt}\sum_{r\geq 0} (-1)^{r+1} b_r(M) \log(1-z^r t)$$

$$= \sum_{r\geq 0} (-1)^r b_r(M) \cdot \frac{z^r}{1-z^r t}$$

$$= \sum_{r\geq 0} (-1)^r b_r(M) \sum_{i\geq 0} z^r \cdot (z^r t)^i$$

$$= \sum_{i\geq 0} \Big(\underbrace{\sum_{r\geq 0} (-1)^r b_r(M) (z^{i+1})^r}_{E(M,z^{i+1})} \Big) t^i$$

$$\therefore \frac{d}{dt}\log P(z,t) = \sum_{i\geq 0} E(M,z^{i+1}) t^i$$

$$\equiv E(M,z) + E(M,z^2)t + \cdots + E(M,z^n)t^{n-1} \mod t^n$$

$$= u_1 + \cdots + u_n + (u_1^2 + \cdots + u_n^2)t + \cdots + (u_1^n + \cdots + u_n^n)t^{n-1}$$

$$= \sum_{j=1}^n u_j + u_j^2 t + u_j^3 t^3 + \cdots$$

$$= \sum_{j=1}^n \frac{u_j}{1-u_j t}$$

$$= \sum_{j=1}^n \frac{d}{dt}\log(1-u_j t)^{-1}$$

$$= \frac{d}{dt}\sum_{j=1}^n \log(1-u_j t)^{-1}$$

$$= \frac{d}{dt}\log\Big(\prod_{j=1}^n (1-u_j t)^{-1}\Big)$$

$$\therefore \frac{d}{dt}\log P(z,t) = \frac{d}{dt}\left(\log \prod_{j=1}^{n}(1-u_j t)^{-1}\right) + t^n f(z,u_1,\cdots,u_n,t)$$

in R

Hence mod t^{n+1}, have

$$P(z,t) \equiv \prod_{j=1}^{n}(1-u_j t)^{-1} \quad \text{mod } t^{n+1}$$

$$= \prod_{j=1}^{n}\sum_{i\geq 0}(u_i t)^i$$

$$= \underbrace{(u_0 + u_1 t + u_2 t^2 + \cdots)\cdots(u_0 + u_1 t + u_2 t^2 + \cdots)}_{n \text{ times}}$$

$$= 1 + S_1(\underline{u})t + S_2(\underline{u})t^2 + \cdots$$

$$\equiv 1 + S_1(\underline{u})t + \cdots + S_n(\underline{u})t^n \quad \text{mod } t^{n+1}.$$

$$S_r(\underline{u}) = \sum_{\substack{i_1+\cdots+i_n=r \\ \text{allow } 0}} u_1^{i_1}\cdots u_n^{i_n}$$

$E(V, z^{\lambda_1})$ $E(V, z^{\lambda_k})$

Lemma 8 $S_n(\underline{u}) = \frac{1}{n!}\sum_{\lambda\vdash n} C_\lambda J_{\lambda_1}(\underline{u})\cdots J_{\lambda_k}(\underline{u})$ $\qquad \lambda = [\lambda_1,\cdots,\lambda_k]$

$J_r(\underline{u}) = u_1^r + \cdots + u_n^r$

pf) $RHS = \frac{1}{n!}\sum_{\lambda\vdash n} C_\lambda (u_1^{\lambda_1}+\cdots+u_n^{\lambda_1})\cdots(u_1^{\lambda_k}+\cdots+u_n^{\lambda_k})$

↑ a term here is $u_{i_1}^{\lambda_1}\cdots u_{i_k}^{\lambda_k}$ $\quad 1\leq i_1,\cdots,i_k\leq n$

$$= \frac{1}{n!}\sum_{\lambda\vdash n} C_\lambda \sum_{1\leq j_1<\cdots<j_k\leq n} u_{j_1}^{\lambda_1}\cdots u_{j_k}^{\lambda_k}$$

$\updownarrow$

$$= \frac{1}{n!}\sum_{\sigma\in S_n}\sum_{\substack{i_1+\cdots+i_n=n \\ \text{allow } 0}} u_1^{i_1}\cdots u_n^{i_n} \qquad (\underbrace{i_1,\cdots,i_n}_{\text{allowing } 0}) = \lambda$$

$$= S_n(\underline{u}) \quad \square$$

Lemma 9. $E(V,z) = S_n(u)$.

pf) Lemma 7 + Lemma 8. $\square$

Lemma 10. $E(V,z)$ is the coefficient of t^n in the expansion of

$$\prod_{r\geq 0}\left(\frac{1}{1-z^r t}\right)^{(-1)^r b_r(M)} \qquad \left(= P(z,t)\right)$$

pf) Know: $P(z,t) \equiv 1 + S_1(u)t + \cdots + \underbrace{S_n(u)}_{E(V,z)}t^n \mod t^{n+1}$

$\uparrow$ Lemma 9.

$E(V,z)$. $\square$

§6. Proof of Thm

$K = \mathbb{C}$

$M = H^\circ(X, \mathbb{Q})_{\mathbb{C}}$ $\qquad H^\circ(X,\mathbb{Q})_{\mathbb{C}} = \bigoplus_{r\geq 0} H^r(X,\mathbb{Q})_{\mathbb{C}}$

$M_r = H^r(X, \mathbb{Q})_{\mathbb{C}}$

$M^{\otimes n} \cong H^\circ(X^n, \mathbb{Q})_{\mathbb{C}}$ as S_n-reps.

$$\begin{aligned} V^{\otimes n} = (M^{\otimes n})^{S_n} &\cong H^\circ(X^n, \mathbb{Q})_{\mathbb{C}}^{S_n} \\ &\cong H^\circ(X^n/S_n, \mathbb{Q})_{\mathbb{C}} \\ &= H^\circ(\mathrm{Sym}^n X, \mathbb{Q})_{\mathbb{C}} \end{aligned}$$

Apply Lemma 10. $\square$

Rmk: 1) $p \nmid n$ (prime char.) $\Rightarrow$ The arguments seem to work (possibly except Grothendieck)

2) We can generalize this to trigraded M (Cheah's observation)

3) Rationality of Hasse-Weil zeta function

End of Talk