**Polya Enumeration Theorems in Algebraic Geometry**

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**Introduction**

In combinatorics, the Polya enumeration theorem concerns how to count colorings on a graph modulo symmetries. My work applies its representation-theoretic philosophy to unify interesting facts in algebraic geometry.

**Macdonald's Formula**

Let $X$ be a finite CW complex such that $b_i(X) = 0$ for all $i > 2d$ by taking a large integer $d$. Macdonald proved:

$$\sum_{n=0}^\infty \chi_n(X)|t^n = \prod_{i=0}^d (1 - u^i)(1 - v^i)^{b_i(X)},$$

where $\chi_n(X) = h^n(Y) - h^n(Y)(Y) + h^n(Y)(Y)^2$.

**Grothendieck's Formula**

Let $X$ be a projective variety of dimension $d$ over a finite field $F$. Consider its zeta series

$$Z_X(t) = \exp \left( \sum_{n=1}^\infty \frac{|X(F)\backslash r|}{r} t^n \right).$$

Grothendieck proved

$$Z_X(t) = \prod_{i=0}^d \det(\id_{H^i(X)} - F_q^*, t)^{b_i(X)},$$

where $H^i(X)$ is the $i$-adic étale cohomology of $X_{\overline{\mathbb{F}}}$ with a prime $l$ not dividing $q$. Kapranov observed that

$$Z_X(t) = \sum_{n=0}^\infty \chi_n(X)(t)^n,$$

so rephrasing Grothendieck's formula, we have

$$\sum_{n=0}^\infty \chi_n(X)(t)^n = \prod_{i=0}^d (1 - \chi_i(X) t)^{b_i(X)},$$

which looks similar to Macdonald's formula.

**Theorem 1 (C.)**

My work provides a way to look at the both formulas as a unified statement. (Vakil had another way.) In this setting, let $F$ be an endomorphism on $X$, which induces a graded linear map $F^*$ on $H^*(X) = H^0(X) \oplus H^1(X) \oplus \cdots$. Consider the signed Lefschetz series:

$$L_n(F^*) = \Tr(F^n_0) - \Tr(F^n_1) + \Tr(F^n_2) u^2 - \cdots.$$

We have

$$\sum_{n=0}^\infty L_n(Sym^d(F^*)^r) = \prod_{i=0}^d \det(\id_{H^i(X)} - F_q^*, u)(t)^{1-\chi_i(X)}.$$

**Remark.** Taking $F = \id_X$ gives Macdonald in the singular setting. Taking $F = F_q^*$ and $u = 1$ gives Grothendieck in the $l$-adic setting.

**Technical limitation.** For the $l$-adic setting, where $l$ is a prime not dividing $q$, we have to choose $l > n$ for each $Sym^d(X)$ appearing in the statement.

**Theorem 2 (C.)**

Theorem 1 is a corollary of the following: for any subgroup $G$ of $S_n$, we have

$$L_n(F_{X_{G}}(X)) = Z_G(L_n(F^*), L_n(F^2), \ldots, L_n(F^{m(n)})),$$

where

$$Z_G(x_1, \ldots, x_n) = \frac{1}{|G|} \sum_{g \in G} \prod_{i=0}^d z_{m_i(g)}(x_1) \cdots z_{m_i(g)}(x_n),$$

writing $m_i(g)$ to mean the number of length $i$ cycles in the cycle decomposition of $g$ in $S_n$. The polynomial $Z_G(x)$ is called the Polya cycle index of $G$ in $S_n$.

**Example: Unweighted Polya**

Let $X$ be a finite set with the discrete topology so that $\#X = \chi(X) = \chi_1(X)$. In this case, writing $m(g) = m_1(g) + \cdots + m_n(g)$, Theorem 2 implies

$$\#(X^n/G) = \frac{1}{|G|} \sum_{g \in G} \chi(m(g)).$$

**Example of Cycle Index**

If $G$ is the group of rotational symmetries for the graph in the left-top corner, we have

$$Z_G(x_1, x_2, x_3, x_4, x_5) = \frac{x_1^3}{12}(x_1^4 + 8x_1 x_3 + 3x_2^2).$$

**Weighted Polya**

We discuss the weighted Polya enumeration with the example above. The main combinatorial objective here is to answer the following question:

**Question.** Given a two-element set $X = \{C, H\}$, how many ways to use one $C$ and four $H$'s to fill the vertices of the cycle given in the left-top corner?

**Remark.** We are ignoring some rules of chemistry if we are considering $C$ as carbon and $H$ hydrogen.

**Answer.** It is 57? No, I think the natural answer is 2, because we need to count up to rotational symmetries of the figure. This can be encoded as the coefficient of $CH^4$ for the following weight polynomial:

$$C^5 + 4C^2H + 2C^3H^2 + 2C^2H^3 + 2CH^4 + H^5.$$

Now, the weighted Polya enumeration theorem states that: the above weight polynomial is equal to

$$Z_G(C + H, C^2 + H^2, C^3 + H^3, C^4 + H^4, C^5 + H^5).$$

John Stembridge told me that I should look at this with the following perspective:

$$C^* + H^* = \Tr\begin{pmatrix} C & 0 \\ 0 & H \end{pmatrix},$$

by letting the matrix $\text{diag}(C, H)$ act on $C_C \oplus C_H$. More generally, if $\phi$ is a linear endomorphism that acts on this 2-dimensional vector space, then by density argument, one can prove that

$$\Tr(\phi_{X_{G}}) = Z_G(\Tr(\phi), \Tr(\phi^2), \ldots, \Tr(\phi^n)).$$

**Theorem 3 (C.)**

In the situation of Theorem 2, consider any graded linear endomorphism $\phi$ on $H^*(X)$. It turns out that

$$H^*(X^n / G) \cong H^*(X)^G,$$

and we may explicitly understand how $G$ acts on $H^*(X^n) \cong H^*(X)^G$ using Macdonald's work. With this, see we $\phi$ induces a linear endomorphism $\phi_{\infty}(X/G)$ on $H^*(X^n / G)$. Carefully applying density argument in this more complicated setting, my generalization goes:

$$L_n(\phi_{X_{G}}) = Z_G(\phi_{X}(\phi), \phi_{X}(\phi^2), \ldots, \phi_{X}(\phi^n)).$$

**Corollary (Cheah's Formula)**

Let $X$ be a smooth projective complex variety of (complex) dimension $d$. Then the $i$-th singular cohomology over $C$ has the Hodge decomposition:

$$H^i(X) = \bigoplus_{p+q=i} H^{p,q}(X).$$

Taking

$$\phi = \bigoplus_{i=0}^d \bigoplus_{p+q=i} x^p y^q \id_{H^r(X)}$$

in Theorem 3 gives a formula due to Cheah:

$$\chi_{\mu, \nu}(X/G) = Z_G(\chi_{\mu, \nu}(X), \cdots, \chi_{\mu, \nu}(x^n / G)(X)),$$

where for any complex variety $Y$, we write

$$\chi_{\mu, \nu}(Y) = \sum_{i=0}^\infty \sum_{p+q=i} H^p,q(H^r(Y))x^p y^q(-u)^q.$$

**Follow-up Research (in Progress)**

My ongoing philosophy is that the weights in the Polya enumeration theorem in combinatorics are tied to the weights in Deligne’s Hodge theory. More concrete follow-ups to understand this philosophy:

1. Replace $X^n$ with $X^n \\setminus \text{big diagonal}$ (with Yifeng Huang).
2. Extend Cheah's result more intrinsically by using Dolbeault cohomology (with Y. Nancy Wang).

**Reference (Containing a Complete Reference)**