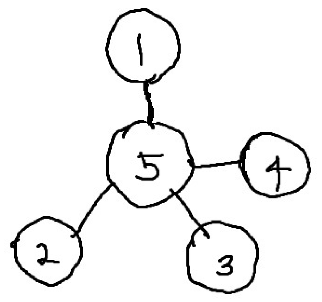


# Pólya enumeration theorems in algebraic geometry

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## Introduction

In combinatorics, the Pólya enumeration theorem concerns how to count colorings on a graph modulo symmetries. My work applies its representation-theoretic philosophy to unify interesting facts in algebraic geometry.

## Macdonald's formula

Let $X$ be a finite CW complex such that $h^i(X) = 0$ for all $i > 2d$ by taking a large integer $d$. Macdonald proved:

$$\sum_{n=0}^{\infty} \chi_u(\mathrm{Sym}^n(X))t^n = \prod_{i=0}^{2d}(1 - u^i t)^{(-1)^{i+1}h^i(X)},$$

where $\chi_u(Y) = h^0(Y) - h^1(Y)u + h^2(Y)u^2 - \cdots$, the signed generating function for singular Betti numbers.

## Grothendieck's formula

Let $X$ be a projective variety of dimension $d$ over a finite field $\mathbb{F}_q$. Consider its zeta series

$$Z_X(t) = \exp\left(\sum_{r=1}^{\infty} \frac{\#X(\mathbb{F}_{q^r})t^r}{r}\right).$$

Grothendieck proved

$$Z_X(t) = \prod_{i=0}^{2d} \det(\mathrm{id}_{H^i(X)} - \mathrm{Fr}_{q,i}^* t)^{(-1)^{i+1}},$$

where $H^i(X)$ is the $l$-adic étale cohmology of $X_{/\overline{\mathbb{F}_q}}$ with a prime $l$ not dividing $q$. Kapranov observed that

$$Z_X(t) = \sum_{n=0}^{\infty} \#(\mathrm{Sym}^n(X)(\mathbb{F}_q))t^n,$$

so rephrasing Grothendieck's formula, we have

$$\sum_{n=0}^{\infty} \#(\mathrm{Sym}^n(X)(\mathbb{F}_q))t^n = \prod_{i=0}^{2d} \det(\mathrm{id}_{H^i(X)} - \mathrm{Fr}_{q,i}^* t)^{(-1)^{i+1}}$$

which looks similar to Macdonald's formula.

## Theorem 1 (C.)

My work provides a way to look at the both formulas as a unified statement. (Vakil had another way.) In either setting, let $F$ be an endomorphism on $X$, which induces a graded linear map $F^*$ on $H^\bullet(X) = H^0(X) \oplus H^1(X) \oplus \cdots$. Consider the signed Lefschetz series:

$$L_u(F^*) = \mathrm{Tr}(F_0^*) - \mathrm{Tr}(F_1^*)u + \mathrm{Tr}(F_2^*)u^2 - \cdots.$$

We have

$$\sum_{n=0}^{\infty} L_u(\mathrm{Sym}^n(F)^*)t^n = \prod_{i=0}^{2d} \det(\mathrm{id}_{H^i(X)} - F_i^* u^i t)^{(-1)^{i+1}}.$$

**Remark**. Taking $F = \mathrm{id}_X$ gives Macdonald in the singular setting. Taking $F = \mathrm{Fr}_q$ and $u = 1$ gives Grothendieck in the $l$-adic setting.

**Technical limitation**. For the $l$-adic setting, where $l$ is a prime not dividing $q$, we have to choose $l > n$ for each $\mathrm{Sym}^n(X)$ appearing in the statement.

## Theorem 2 (C.)

Theorem 1 is a corollary of the following: for any subgroup $G$ of $S_n$, we have

$$L_u(F^*_{X^n/G}) = Z_G(L_u(F^*), L_{u^2}(F^{*2}), \ldots, L_{u^n}(F^{*n})),$$

where

$$Z_G(x_1, \ldots, x_n) = \frac{1}{\#G} \sum_{g \in G} x_1^{m_1(g)} \cdots x_n^{m_n(g)},$$

writing $m_i(g)$ to mean the number of length $i$ cycles in the cycle decomposition of $g$ in $S_n$. The polynomial $Z_G(\boldsymbol{x})$ is called the **Pólya cycle index** of $G$ in $S_n$.

## Example: Unweighted Pólya

Let $X$ be a finite set with the discrete topology so that $\#X = \chi(X) = \chi_1(X)$. In this case, writing $m(g) = m_1(g) + \cdots + m_n(g)$, Theorem 2 implies

$$\#(X^n/G) = \frac{1}{\#G} \sum_{g \in G} (\#X)^{m(g)}.$$

## Example of cycle index

If $G$ is the group of rotational symmetries for the graph in the left-top corner, we have

$$Z_G(x_1, x_2, x_3, x_4, x_5) = \frac{x_1}{12}(x_1^4 + 8x_1x_3 + 3x_2^2).$$

## Weighted Pólya

We discuss the weighted Pólya enumeration with the example above. The main combinatorial objective here is to answer the following question:

**Question**. Given a two-element set $X = \{C, H\}$, how many ways to use one $C$ and four $H$'s to fill up the vertices of the structure given in the left-top corner?

**Remark**. We are ignoring some rules of chemistry if we are considering $C$ as carbon and $H$ hydrogen.

**Answer**. Is it 5? No, I think the natural answer is 2, because we need to count up to rotational symmetries of the figure. This can be encoded as the coefficient of $CH^4$ for the following **weight polynomial**:

$$C^5 + 2C^4H + 2C^3H^2 + 2C^2H^3 + 2CH^4 + H^5.$$

Now, the **weighted Pólya enumeration theorem** states that: the above weight polynomial is equal to

$$Z_G(C + H, C^2 + H^2, C^3 + H^3, C^4 + H^4, C^5 + H^5).$$

John Stembridge told me that I should look at this with the following perspective:

$$C^r + H^r = \mathrm{Tr}\begin{bmatrix} C & 0 \\ 0 & H \end{bmatrix}^r,$$

by letting the matrix $\mathrm{diag}(C, H)$ act on $\mathbb{C}e_C \oplus \mathbb{C}e_H$. More generally, if $\phi$ is a linear endomorphism that acts on this 2-dimensional vector space, then by density argument, one can prove that

$$\mathrm{Tr}(\phi_{X^5/G}) = Z_G(\mathrm{Tr}(\phi), \mathrm{Tr}(\phi^2), \mathrm{Tr}(\phi^3), \mathrm{Tr}(\phi^4), \mathrm{Tr}(\phi^5)).$$

## Theorem 3 (C.)

In the situation of Theorem 2, consider any graded linear enormophism $\phi$ on $H^\bullet(X)$. It turns out that

$$H^\bullet(X^n/G) \simeq H^\bullet(X^n)^G,$$

and we may explicitly understand how $G$ acts on $H^\bullet(X^n) \simeq H^\bullet(X)^{\otimes n}$ using Macdonald's work. With this, we see $\phi$ induces a linear endomorphism $\phi_{X^n/G}$ on $H^\bullet(X^n/G)$. Carefully applying density argument in this more complicated setting, my generalization goes:

$$L_u(\phi_{X^n/G}) = Z_G(L_u(\phi), L_{u^2}(\phi^2), \ldots, L_{u^n}(\phi^n)).$$

## Corollary (Cheah's formula)

Let $X$ be a smooth projective complex variety of (complex) dimension $d$. Then the $i$-th singular cohomology over $\mathbb{C}$ has the Hodge decomposition:

$$H^i(X) = \bigoplus_{p+q=i} H^{p,q}(X).$$

Taking

$$\phi = \bigoplus_{i=0}^{2d} \bigoplus_{p+q=i} x^p y^q \mathrm{id}_{H^{p,q}(X)}$$

in Theorem 3 gives a formula due to Cheah:

$$\chi_{u,x,y}(X^n/G) = Z_G(\chi_{u,x,y}(X), \cdots, \chi_{u^n,x^n,y^n}(X)),$$

where for any complex variety $Y$, we write

$$\chi_{u,x,y}(Y) = \sum_{i=0}^{\infty} \sum_{p+q=i} h^{p,q}(H^i(Y))x^p y^q(-u)^i.$$

## Follow-up research (in progress)

My ongoing philosophy is that the weights in the Pólya enumeration theorem in combinatorics are tied to the weights in Deligne's Hodge theory. More concrete follow-ups to understand this philosophy:

(1) Replace $X^n$ with $X^n \setminus \text{big diagonal}$ (with Yifeng Huang).

(2) Extend Cheah's result more intrinsically by using Dolbeault cohomology (with Y. Nancy Wang).

## Reference (containing a complete reference)

G. Cheong, *Pólya enumeration theorems in algebraic geometry*, https://arxiv.org/abs/2003.04825