ORIENTED FLIP GRAPHS, NONCROSSING TREE PARTITIONS, AND REPRESENTATION THEORY OF TILING ALGEBRAS

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Abstract. The purpose of this paper is to understand lattices of certain subcategories in module categories of representation-finite gentle algebras called tiling algebras, as introduced by Coelho Simoes and Parsons. We present combinatorial models for torsion pairs and wide subcategories in the module category of tiling algebras. Our models use the oriented flip graphs and noncrossing tree partitions, developed by the authors, and a description of the extension spaces between indecomposable modules over tiling algebras. In addition, we classify 2-term simple-minded collections in bounded derived categories of tiling algebras. As a consequence, we obtain a characterization of $\mathbf{c}$-matrices for any quiver mutation-equivalent to a type A Dynkin quiver.

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1. Introduction

A **tiling algebra** $\Lambda_T = \Bbbk Q_T/I_T$ is defined by the data of a tree T embedded in a disk. They are a class of representation-finite gentle algebras that were recently introduced in [26]. These algebras also form a subclass of the algebras of partial triangulations introduced in [9]. Examples of tiling algebras include Jacobian algebras [10] of type A and m-cluster-tilted algebras [22] of type A, both of which naturally arise in the study of cluster algebras [11] and in the additive categorification of cluster algebras [4, 5].

The tree T defines two lattices: the **oriented flip graph** of T, denoted $\overrightarrow{FG}(T)$, and the **noncrossing tree partitions** of T, denoted $\text{NCP}(T)$. The former can be regarded as a directed graph whose vertices correspond to partial triangulations of a disk and whose edges correspond to exchanging single arcs in the corresponding partial triangulations. The latter is a generalization of the classical noncrossing set partitions of the set $\{1, 2, \dots, n\}$ where n is the number of nonleaf vertices of T.

We present combinatorial models for the torsion pairs and the wide subcategories in the module category of Λ_T using the oriented flip graph of T and the noncrossing tree partitions of T, respectively. In particular, we prove the following:

- the lattice of torsion-free classes, denoted $\text{torsf}(\Lambda_T)$, (resp., of torsion classes, denoted $\text{tors}(\Lambda_T)$) is isomorphic to $\overrightarrow{FG}(T)$ (resp., $\overrightarrow{FG}(T)^{\text{op}}$) (see Theorem 6.5),

- the lattice of wide subcategories of Λ_T, denoted wide(Λ_T), is isomorphic to NCP(T) (see Theorem 7.1).

We also combinatorially describe all 2-term simple-minded collections in the bounded derived category of Λ_T (see Theorem 8.4). An important application of the latter is a classification of **c**-matrices of quivers that are mutation-equivalent to type A Dynkin quivers (see Theorem 9.1). This classification is similar to one obtained in [27] for acyclic quivers and to the classification found in [13] for type A Dynkin quivers.

1.1. **Organization of the paper.** In Section 2, we review the basics of path algebras, quiver representations, and gentle algebras. In Sections 3.1 and 3.2, we review the notions of the oriented flip graphs and noncrossing tree partitions that were introduced in [15].

In Sections 4 and 5, we define tiling algebras and describe all homomorphisms and extensions between indecomposable modules over tiling algebras.

In Section 6, we show that the lattice of torsion-free classes (resp., torsion classes) of Λ_T ordered by inclusion (resp., reverse inclusion) is isomorphic to $\overrightarrow{FG}(T)$ (see Theorem 6.5). To obtain this result, we make use of the lattice quotient description of $\overrightarrow{FG}(T)$ from [15, Theorem 4.11] and the classification of extensions between indecomposable Λ_T-modules found in Section 4. In Section 7, we show that the lattice of noncrossing tree partitions of T is isomorphic to the lattice of wide subcategories of Λ_T-mod.

In Section 8, we show that the data of a noncrossing tree partition and its Kreweras complement is equivalent to a 2-term simple-minded collection of objects in the bounded derived category of Λ_T (Theorem 8.4). This theorem relies on the description of extensions between indecomposable Λ_T-modules found in Section 4 and on a combinatorial description of the operation of left- and right-mutation on simple-minded collections found in Section 8.1 (see Lemma 8.6).

Acknowledgements. Alexander Garver thanks Kiyoshi Igusa, Gregg Musiker, Ralf Schiffler, Sibylle Schroll, and Hugh Thomas for helpful conversations. The authors thank Emily Barnard for useful discussions and Hugh Thomas for noticing the connection between noncrossing tree partitions and wide subcategories. The authors also thank an anonymous referee for their careful comments. Alexander Garver received support from a Research Training Group, RTG grant DMS-1148634.

2. Preliminaries

2.1. **Path algebras and quiver representations.** Following [1], let Q be a given quiver. We define a **path** of **length** $\ell \geqslant 1$ to be an expression $\alpha_1\alpha_2\cdots\alpha_\ell$ where $\alpha_i \in Q_1$ for all $i \in [\ell] := \{1, \ldots, \ell\}$ and $s(\alpha_i) = t(\alpha_{i+1})$ for all $i \in [\ell - 1]$. We may visualize such a path in the following way

$$\cdot \xleftarrow{\alpha_1} \cdot \xleftarrow{\alpha_2} \cdot \longleftarrow \cdot \quad \cdots \quad \cdot \longleftarrow \cdot \xleftarrow{\alpha_\ell} \cdot\,.$$

The **source** (resp., **target**) of the path $\alpha_1\alpha_2\cdots\alpha_\ell$ is $s(\alpha_\ell)$ (resp., $t(\alpha_1)$). Let Q_ℓ denote the set of all paths in Q of length ℓ. We also associate to each vertex $i \in Q_0$ a path of length $\ell = 0$, denoted ε_i, called the **lazy path** at i.

Definition 2.1. Let Q be a quiver. The **path algebra** of Q, denoted $\Bbbk Q$, is the $\Bbbk$-algebra generated by all paths of length $\ell \geqslant 0$. Throughout this paper, we assume that $\Bbbk$ is algebraically closed. The multiplication of two paths $\alpha_1\cdots\alpha_\ell \in Q_\ell$ and $\beta_1\cdots\beta_k \in Q_k$ is given by the following rule:

$$\alpha_1\cdots\alpha_\ell \cdot \beta_1\cdots\beta_k = \begin{cases} \alpha_1\cdots\alpha_\ell\beta_1\cdots\beta_k \in Q_{\ell+k} & : \quad s(\alpha_\ell) = t(\beta_1) \\ 0 & : \quad s(\alpha_\ell) \neq t(\beta_1). \end{cases}$$

Note that as $\Bbbk$-vector spaces we have

$$\Bbbk Q = \bigoplus_{\ell=0}^{\infty} \Bbbk Q_\ell$$

where $\Bbbk Q_\ell$ is the $\Bbbk$-vector space of all paths of length ℓ.

In this paper, we study certain quivers Q which have **oriented cycles** (i.e., paths $\alpha_1\cdots\alpha_\ell \in Q_\ell$ where $t(\alpha_1) = s(\alpha_\ell)$). If a quiver Q possesses any oriented cycles of length $\ell \geqslant 1$, we see that $\Bbbk Q$ is infinite dimensional. In order to avoid studying infinite dimensional algebras, we will add relations to path algebras whose quivers contain oriented cycles in such a way that we obtain finite dimensional quotients of path algebras. The relations we add are those coming from an **admissible** ideal I of $\Bbbk Q$ meaning that there exists $N \geqslant 2$ such that

$$\bigoplus_{\ell=N}^{\infty} \Bbbk Q_\ell \subseteq I \subseteq \bigoplus_{\ell=2}^{\infty} \Bbbk Q_\ell.$$

If I is an admissible ideal of $\Bbbk Q$, we say that (Q, I) is a **bound quiver** and that $\Bbbk Q/I$ is a **bound quiver algebra**.

In this paper, we study modules over a bound quiver algebra $\Bbbk Q/I$ by studying certain representations of Q that are "compatible" with the relations coming from I. A **representation** $V = ((V_i)_{i\in Q_0}, (\varphi_\alpha)_{\alpha\in Q_1})$ of a quiver Q is an assignment of a $\Bbbk$-vector space V_i to each vertex i and a $\Bbbk$-linear map $\varphi_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$ to each arrow $\alpha \in Q_1$. If $\rho \in \Bbbk Q$, it can be expressed as

$$\rho = \sum_{i=1}^{m} c_i \alpha_1^{(i)} \cdots \alpha_{k_i}^{(i)}$$

for some $c_i \in \Bbbk$ and for some $\alpha_1^{(i)} \cdots \alpha_{k_i}^{(i)} \in Q_{i_k}$ so when considering a representation V of Q, we define

$$\varphi_\rho := \sum_{i=1}^{m} c_i \varphi_{\alpha_1^{(i)}} \cdots \varphi_{\alpha_{k_i}^{(i)}}.$$

If we have a bound quiver (Q, I), we define a representation of Q **bound by** I to be a representation of Q where $\varphi_\rho = 0$ if $\rho \in I$. We say a representation of Q bound by I is **finite dimensional** if $\dim_\Bbbk V_i < \infty$ for all $i \in Q_0$. It turns out that $\Bbbk Q/I$-mod is equivalent to the category of finite dimensional representations of Q bound by I. Additionally, the **dimension vector** of $V \in \Bbbk Q/I$-mod is the vector $\underline{\dim}(V) := (\dim_\Bbbk V_i)_{i\in Q_0}$ and the **dimension** of V is defined as $\dim_\Bbbk(V) := \sum_{i\in Q_0} \dim_\Bbbk V_i$. The **support** of $V \in \Bbbk Q/I$-mod is the set $\mathrm{supp}(V) := \{i \in Q_0 : V_i \neq 0\}$.

In this paper, we focus on bound quiver algebras that are **gentle algebras**. A **gentle algebra** $\Lambda = \Bbbk Q/I$ is a bound quiver algebra that satisfies the following conditions:

i) for each vertex of Q is the starting point of at most two arrows and the ending point of at most two arrows;
ii) for each arrow $\beta \in Q_1$ there is at most one arrow $\alpha \in Q_1$ such that $\beta\alpha \notin I$, and there is at most one arrow $\gamma \in Q_1$ such that $\gamma\beta \notin I$;
iii) for each arrow $\beta \in Q_1$, there is at most one arrow $\delta \in Q_1$ such that $\beta\delta \in I$, and there is at most one arrow $\mu \in Q_1$ such that $\mu\beta \in I$;
iv) the ideal I is generated by paths of length 2.

Gentle algebras have a simple combinatorial parameterization of their indecomposable modules in terms of string modules. A **string** in Λ is a sequence

$$w = x_1 \xleftrightarrow{\alpha_1} x_2 \xleftrightarrow{\alpha_2} \cdots \xleftrightarrow{\alpha_m} x_{m+1}$$

where each $x_i \in Q_0$ and each $\alpha_i \in Q_1$ or $\alpha_i \in Q_1^{-1} := \{\text{formal inverses of arrows of Q}\}$. We require that each α_i **connects** x_i and x_{i+1} (i.e., either $s(\alpha_i) = x_i$ and $t(\alpha_i) = x_{i+1}$ or $s(\alpha_i) = x_{i+1}$ and $t(\alpha_i) = x_i$ where if $\alpha_i \in Q_1^{-1}$ we define $s(\alpha_i) := t(\alpha_i^{-1})$ and $t(\alpha_i) := s(\alpha_i^{-1})$) and that w contains no **substrings** of w of the following forms:

i) $x \xrightarrow{\beta} y \xleftarrow{\beta^{-1}} x$ or $x \xleftarrow{\beta} y \xrightarrow{\beta^{-1}} x$,
ii) $x_{i_1} \xrightarrow{\beta_1} x_{i_2} \cdots x_{i_s} \xrightarrow{\beta_s} x_{i_{s+1}}$ or $x_{i_1} \xleftarrow{\gamma_1} x_{i_2} \cdots x_{i_s} \xleftarrow{\gamma_s} x_{i_{s+1}}$ where $\beta_s \cdots \beta_1, \gamma_1 \cdots \gamma_s \in I$.

In other words, w is an irredundant walk in Q that avoids the relations imposed by I. By convention, we consider w to be a different word in the vertices of Q than $w^{-1} := x_{m+1} \xleftrightarrow{\alpha_m} x_m \xleftrightarrow{\alpha_{m-1}} \cdots \xleftrightarrow{\alpha_1} x_1$. We say the string w is **cyclic** if $x_1 = x_{m+1}$ and we say a cyclic string is a **band** if

$$w^k := \underbrace{x_1 \xleftrightarrow{\alpha_1} x_2 \xleftrightarrow{\alpha_2} \cdots \xleftrightarrow{\alpha_m} x_1 \xleftrightarrow{\alpha_1} x_2 \xleftrightarrow{\alpha_2} \cdots \xleftrightarrow{\alpha_m} x_1 \cdots x_1 \xleftrightarrow{\alpha_1} x_2 \xleftrightarrow{\alpha_2} \cdots \xleftrightarrow{\alpha_m} x_1}_{k \text{ copies of } w}$$

is a string but w is not a proper power of another string u (i.e., there does not exist an integer $s \geqslant 2$ such that $w = u^s$).

Let w be a string in Λ. The **string module** defined by w is the bound quiver representation $M(w) := ((V_i)_{i\in Q_0}, (\varphi_\alpha)_{\alpha\in Q_1})$ where

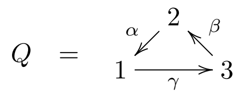
$$V_i \;:=\; \begin{cases} \Bbbk^{s_j} & : \; i = x_j \text{ for some } j \in [m+1] \\ 0 & : \; \text{otherwise} \end{cases}$$

with $s_j := \#\{k \in [m+1] : \; x_k = x_j\}$ and the action of φ_α is induced by the relevant identity morphisms if α lies on w and is zero otherwise. One observes that $M(w) \cong M(w^{-1})$.

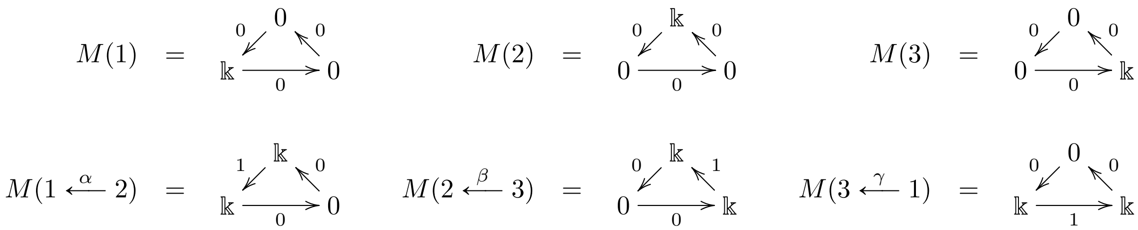
In this paper, we study a family of representation-finite gentle algebras, which we denote by Λ_T. It follows from [28] that the set of indecomposable modules over these algebras, denoted $\mathrm{ind}(\Lambda_T\text{-mod})$, consists of exactly the string modules $M(w)$ where w is a string in Λ_T.

We remark that each band in Λ defines an infinite family of indecomposable modules called **band modules**. However, we omit the definition of band modules since there are no such modules belonging to $\mathrm{ind}(\Lambda_T\text{-mod})$.

Example 2.2. Let Q denote the quiver shown below. Then $\Bbbk Q/I = \Bbbk Q/\langle \beta\alpha, \gamma\beta, \alpha\gamma\rangle$ is a gentle algebra.



The algebra $\Bbbk Q/I$ has the following string modules.



3. Oriented flip graphs and noncrossing tree partitions

3.1. **Oriented flip graphs.** A **tree** is a finite, connected acyclic graph. Any tree may be embedded in a disk D^2 in such a way that a vertex is on the boundary if and only if it is a leaf. We will assume that any tree comes equipped with such an embedding. We refer to non-leaf vertices of a tree as **interior vertices**, and, by convention, any interior vertex has degree at least 3. The embedding that accompanies T also endows each interior vertex with a cyclic ordering. In addition, we say two trees T and T' are **equivalent** if there is an isotopy between the spaces $D^2\backslash T$ and $D^2\backslash T'$.

A tree T embedded in D^2 determines a collection of 2-dimensional regions in D^2 that we will refer to as **faces**. A **corner** of a tree is a pair (v, F) consisting of an interior vertex v and a 2-dimensional face F containing v. We let $\mathrm{Cor}(T)$ denote the set of corners of T.

An **acyclic path** (or **chordless path**) supported by a tree T is a sequence $(v_0, \ldots, v_t)$ of pairwise distinct vertices of T such that v_i and v_j are adjacent if and only if $|i - j| = 1$. We typically identify acyclic paths with their underlying vertex sets; that is, we do not distinguish between acyclic paths of the form $(v_0, \ldots, v_t)$ and $(v_t, \ldots, v_0)$. We will refer to v_0 and v_t as the **endpoints** of the acyclic path $(v_0, \ldots, v_t)$. Note that an acyclic path is determined by its endpoints, and thus we can write $[v_0, v_t] = (v_0, \ldots, v_t)$. As an acyclic path $(v_0, \ldots, v_t)$ defines a subgraph of T (namely, the induced subgraph on the vertices $v_0, \ldots, v_t$), it makes sense to refer to an **edge** of $(v_0, \ldots, v_t)$. Additionally, if $(v_0, \ldots, v_t)$ and $(v_t, \ldots, v_{t+s})$ are acyclic paths that agree only at v_t and where $[v_0, v_{t+s}]$ is an acyclic path, we define their **composition** as $[v_0, v_t] \circ [v_t, v_{t+s}] := [v_0, v_{t+s}]$.

A **segment** $s = (v_0, \ldots, v_t)$ is an acyclic path consisting of at least two vertices where any two edges (v_{i-1}, v_i) and (v_i, v_{i+1}) are incident to a common face and whose endpoints are *not* leaves. Observe that interior vertices of T are not segments. If the composition $s \circ t$ of two segments s and t is a segment, we say that s and t are **composable**.

Let $\mathrm{Seg}(T)$ be the set of segments supported by a tree T. Given $X \subseteq \mathrm{Seg}(T)$, we say X is **closed** if for any pair of composable segments $s, t \in X$ one has $s \circ t \in X$. If X is any subset of $\mathrm{Seg}(T)$, its **closure**, denoted $\overline{X}$, is the smallest closed set containing X. Say X is **biclosed** if X and $\mathrm{Seg}(T)\backslash X$ are both closed. At times, we will also say X is **co-closed** if $\mathrm{Seg}(T)\backslash X$ is closed.

Define $\mathrm{Bic}(T)$ to be the collection of biclosed subsets of $\mathrm{Seg}(T)$, partially ordered by inclusion. The poset $\mathrm{Bic}(T)$ is a lattice with many nice properties. In particular, for any biclosed sets $B_1, B_2 \in \mathrm{Bic}(T)$, we have that $B_1 \vee B_2 = \overline{B_1 \cup B_2}$. We refer the reader to [15] for more information on the lattice structure of biclosed sets.

Let $s = (v_0, \ldots, v_l)$ be a segment, and orient the segment from v_0 to v_l. Let C_s be the set of segments $(v_i, \ldots, v_j)$ such that

- if $i > 0$ then s turns right at v_i, and
- if $j < l$ then s turns left at v_j.

We note that s is always in C_s since the above conditions are vacuously true. Furthermore if $t \in C_s$, then $C_t \subseteq C_s$. Let $\pi_\downarrow : \mathrm{Bic}(T) \to \mathrm{Bic}(T)$ be the function such that for any $X \in \mathrm{Bic}(T)$,

$$\pi_\downarrow(X) := \{s \in X : \ C_s \subseteq X\}.$$

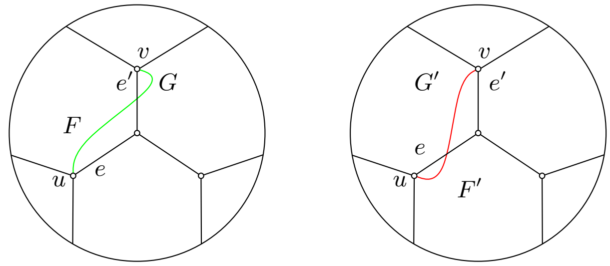


FIGURE 1. A green admissible curve and a red admissible curve for the segment $[u, v]$

It follows from [15, Lemma 4.4] that $\pi_\downarrow(X)$ is actually biclosed. Define the **oriented flip graph** of T, denoted $\overrightarrow{FG}(T)$, to be the partially ordered set $\pi_\downarrow(\text{Bic}(T))$. In [15, Theorem 4.11] we show that the oriented flip graph is a lattice quotient of $\text{Bic}(T)$. As such, it is also a lattice.

In [15], we define the oriented flip graph in terms of a simplicial complex related to T. However, [15, Theorem 4.11] shows that this definition is equivalent to the one we give in this paper.

3.2. **Noncrossing tree partitions.** Let V^o denote the set of interior vertices of T. Fix a small $\epsilon > 0$ such that the ϵ-ball centered at any interior vertex of T is contained in D^2, and no two such ϵ-balls intersect. For each corner (v, F), we fix a point $z(v, F)$ in the interior of F of distance ϵ from v. Let

$$T_\epsilon := T \cup \bigcup_{v \in V^o} \{x \in D^2 : \; |x - v| < \epsilon\}.$$

In words, T_ϵ is the embedded tree T plus the open ϵ-ball around each interior vertex. Similarly, given $[u, v] \in \text{Seg}(T)$, let $[u, v]_\epsilon$ denote the subset of T_ϵ that supports $[u, v]$ and the ϵ balls around u and around v.

It will be convenient to represent segments as certain curves in the disk as follows. A **flag** is a triple (v, e, F) of a vertex v incident to an edge e, which is incident to a face F. Orienting e away from v, we say a flag is **green** if F is left of e. Otherwise, the flag is **red**. Let $(u, e, F),\ (v, e', G)$ be two green flags such that $[u, v]$ is a segment containing the edges e, e' as in Figure 1. A **green admissible curve** $\gamma : [0, 1] \to D^2$ for $[u, v]$ is a simple curve for which $\gamma(0) = z(u, F),\ \gamma(1) = z(v, G)$ and $\gamma([0, 1]) \subseteq D^2 \backslash (T_\epsilon \backslash [u, v]_\epsilon)$. Similarly, if (u, e, F') and (v, e', G') are red flags, then a **red admissible curve** is defined the same way, with $\gamma(0) = z(u, F'),\ \gamma(1) = z(v, G')$. We say a segment is **green** if it is represented by a green admissible curve. Similarly, a segment is **red** if it is represented by a red admissible curve. We may also refer to an **admissible curve** for a segment without specifying a color. Such a curve may be either green or red.

If a colored segment s is represented by a curve with endpoints $z(u, F)$ and $z(v, G)$, we say that (u, F) and (v, G) are the **endpoints** of s. We refer to corners or vertices as the endpoints of a segment at different parts of this paper. The distinction should be clear from context.

Two colored segments are **noncrossing** if they admit admissible curves that do not intersect. Otherwise, they are **crossing**. We remark that if two curves share an endpoint $z(u, F)$ then they are considered to be crossing. To determine whether two colored segments s, t cross, one must check whether the endpoints of t lie in different connected components of $(D^2 \backslash (T_\epsilon \backslash t_\epsilon)) \backslash \gamma$ for some admissible curve γ for s. We will find it convenient to distinguish several cases of crossing as in the following lemma. The three cases correspond to the three columns of Figure 2.

Lemma 3.1. Let γ and γ' be two (green or red) admissible curves corresponding to segments s and s' that meet along a common segment t. Let $t = [a, b]$ and orient γ and γ' from a to b. Assume that γ and γ' do not share an endpoint. Then γ and γ' are noncrossing if and only if one of the following holds:

(1) s (or s') does not share an endpoint with t, and γ turns left (or right) at both endpoints of t;
(2) γ starts at a and turns left (resp., right) at b, and γ' ends at b and turns right (resp., left) at a;
(3) γ and γ' both start at a (resp., both end at b) where γ leaves a (resp., b) to the left, and γ turns left at b (resp., a) or γ' turns right at b (resp., a).

If γ and γ' are both green admissible or both red admissible, then the third case does not occur.

For $B \subseteq V^o$, let $\text{Seg}(B)$ be the set of **inclusion-minimal** segments whose endpoints lie in B. That is, there do not exist distinct segments $s, t \in \text{Seg}(B)$ where every vertex of t appears in s. We say B is **segment-connected** if for any two elements u, v of B, there exists a sequence $u = u_0, \ldots, u_N = v$ of elements of B such

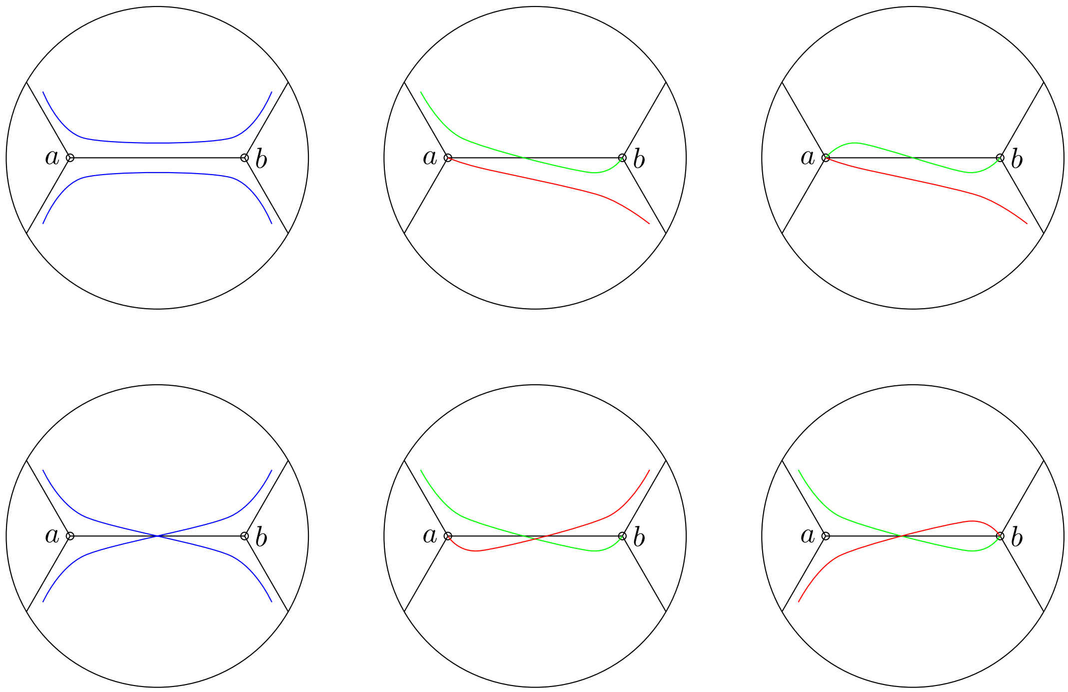


FIGURE 2. Several examples of crossing and noncrossing admissible curves representing segments supported by the tree.

that $[u_{i-1}, u_i] \in \mathrm{Seg}(B)$ for all i. If $\mathbf{B} = \{B_1, \ldots, B_l\}$ is a partition of V^o, we let $\mathrm{Seg}(\mathbf{B}) = \bigcup_{i=1}^{l} \mathrm{Seg}(B_i)$. We let $\mathrm{Seg}_g(\mathbf{B})$ (resp., $\mathrm{Seg}_r(\mathbf{B})$) denote the same set of segments, all colored green (resp., red).

A **noncrossing tree partition B** is a set partition of V^o such that any two segments of $\mathrm{Seg}_r(\mathbf{B})$ are noncrossing and each block of **B** is segment-connected. Note that we intentionally define noncrossing tree partitions using only red segments. Let $\mathrm{NCP}(T)$ be the poset of noncrossing tree partitions of T, ordered by refinement. By [15, Theorem 5.13], the poset $\mathrm{NCP}(T)$ is a lattice. In fact, in [15, Theorem 5.15] we show that $\mathrm{NCP}(T)$ is isomorphic to the "lattice-theoretic" **shard intersection order** (see [24]) of $\overrightarrow{FG}(T)$, denoted $\Psi(\overrightarrow{FG}(T))$, via the isomorphism $\mathbf{B} \mapsto \overline{\mathrm{Seg}(\mathbf{B})}$. Here $\overline{\mathrm{Seg}(\mathbf{B})}$ is the smallest closed set of segments containing $\mathrm{Seg}(\mathbf{B})$.

We give an example of $\mathrm{NCP}(T)$ in Figure 3 where T is the tree appearing in Figure 1. We remark that this lattice of noncrossing tree partitions is not isomorphic to the lattice of noncrossing set partitions of $\{1, 2, 3, 4\}$.

By [15, Corollary 5.12], there is a distinguished bijection $\mathrm{Kr} : \mathrm{NCP}(T) \to \mathrm{NCP}(T)$. We call $\mathrm{Kr}(\mathbf{B})$ the **Kreweras complement** of **B**. The noncrossing tree partition $\mathrm{Kr}(\mathbf{B})$ is characterized by the property that there exist red admissible curves for $\mathrm{Seg}(\mathbf{B})$ and green admissible curves for $\mathrm{Seg}(\mathrm{Kr}(\mathbf{B}))$ such that when one superimposes these curves on T, one obtains a noncrossing tree whose vertex set is V^o. We show an example of a noncrossing tree partition and its Kreweras complement in Figure 5.

4. Trees and their tiling algebras

Let T be a tree embedded in D^2. Then T defines a bound quiver, denoted (Q_T, I_T), as follows. Let Q_T be the quiver whose vertices are in bijection with the edges of T that contain no leaves and whose arrows are exactly those of the form $e_1 \xrightarrow{\alpha} e_2$ satisfying:

i) e_1 and e_2 define a corner of T,

ii) e_2 is counterclockwise from e_1.

The admissible ideal I_T is, by definition, generated by the relations $\alpha\beta$ where $\alpha : e_2 \longrightarrow e_3$ defines the corner (v, F) and $\beta : e_1 \longrightarrow e_2$ defines the corner (v, G). We define the **tiling algebra** of T to be $\Lambda_T := \Bbbk Q_T / I_T$. We remark that the term tiling algebra first appeared in [26] where a tiling algebra is defined by a partial triangulation of a polygon.

Example 4.1. In Figure 4, we show three trees. The tree T_1 determines the quiver $Q_{T_1} = 1 \xrightarrow{\beta} 2 \xrightarrow{\alpha} 3$. The tiling algebra of T_1 is $\Lambda_{T_1} = \Bbbk Q_{T_1}/I_{T_1}$ where $I_{T_1} = \langle \alpha\beta \rangle$. Also note that $Q_{T_2} \cong Q_{T_3} \cong Q$ and $\Lambda_{T_2} \cong \Lambda_{T_3} \cong \Lambda$ where Q is the quiver from Example 2.2 and Λ is the algebra from Example 2.2.

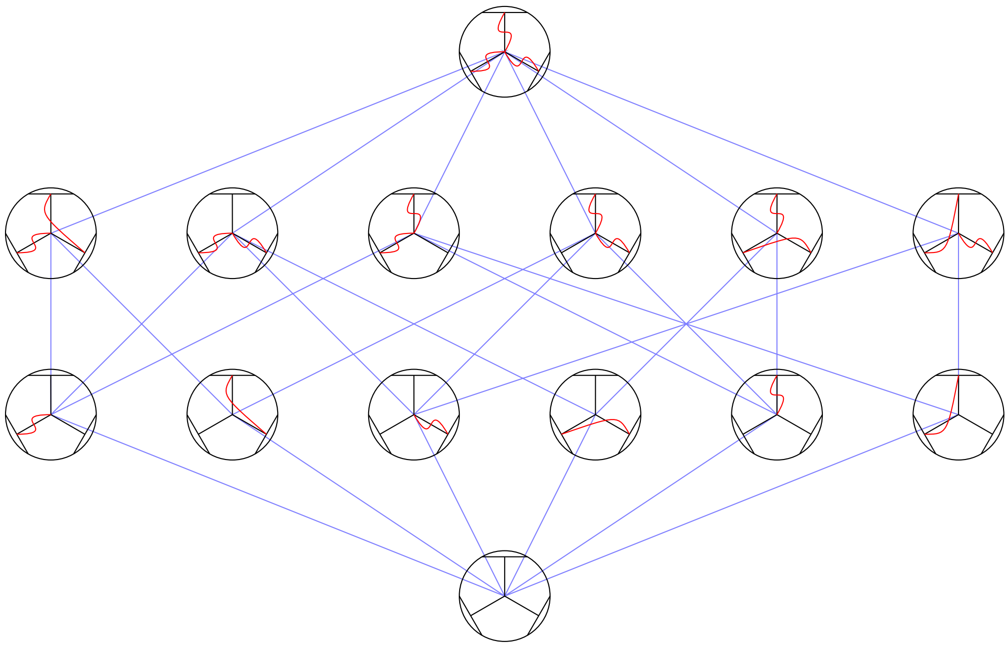


FIGURE 3. A lattice of noncrossing tree partitions

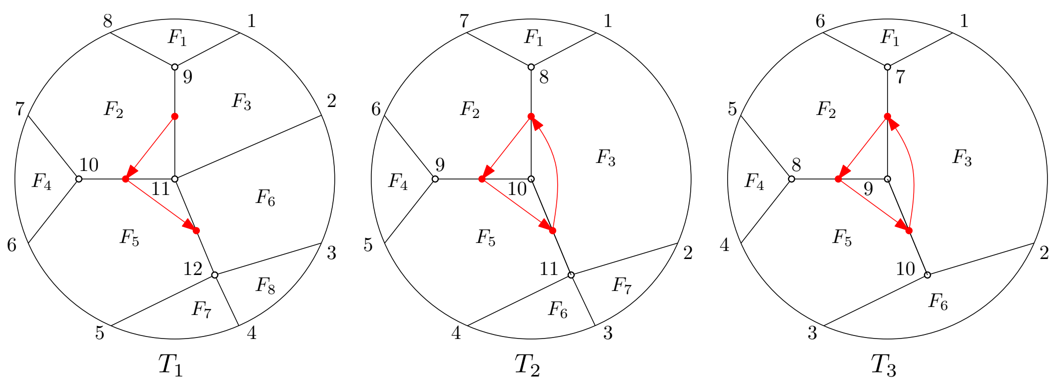


FIGURE 4

Proposition 4.2. The algebra Λ_T is a gentle algebra. Furthermore, the algebra Λ_T is representation-finite and its indecomposables are exactly the string modules.

Proof. The first assertion follows from [26, Proposition 3.2]. To prove the second assertion, it is enough to observe that any string w in Λ_T can be regarded as a full, connected subquiver of Q_T with at most one arrow from any cycle in Q_T. This implies that there are no cyclic strings in Λ_T and therefore no bands in Λ_T. □

Corollary 4.3. The following hold for the tiling algebra Λ_T.

1. Assume $M(w) := ((V_i)_{i\in Q_0}, (\varphi_\alpha)_{\alpha\in Q_1})$ is a string module of Λ_T. Then $\dim_{\Bbbk}(V_i) = 1$ if $i \in \mathrm{supp}(M(w))$ and $\dim_{\Bbbk}(V_i) = 0$ otherwise.
2. The map $\mathrm{ind}(\Lambda_T\text{-mod}) \longrightarrow \mathrm{Seg}(T)$ defined by
$$M(w) \longmapsto s_w := (v_0, \ldots, v_t)$$
where each v_i is a vertex of T belonging to some $e_j \in \mathrm{supp}(M(w))$ and where each pair v_i and v_{i+1} belongs to a common $e_j \in \mathrm{supp}(M(w))$ is a bijection.

Proof. Assertion 1. follows from the proof of Proposition 4.2.

To prove assertion 2., note that any string module $M(w) \in \mathrm{ind}(\Lambda_T\text{-mod})$ can be regarded as a full, connected subquiver of Q_T with at most one arrow from any cycle in Q_T. With this identification, we observe that $M(w)$

is equivalent to a sequence of interior vertices $(v_0, \dots, v_t)$ of Q_T with the property that any two edges (v_{i-1}, v_i) and (v_i, v_{i+1}) are contained in a common face of T. Thus the given map is a bijection. □

We now present a description of the spaces of extensions between indecomposable Λ_T-modules. These results generalize, in the finite representation type case, the description of extensions between indecomposables found in [6]. The proofs of the following results depend on some technical lemmas presented in Section 5.

The following results (Propositions 4.4 and 4.5 and Theorems 4.6 and 4.7) were new when we originally announced them in [16]. Since then, the space of extensions between indecomposable modules over a general gentle algebra has been described in [7]. Although the results therein imply ours, we include the proofs of our results as they appeared in [16].

Proposition 4.4. Let $0 \to M(u) \xrightarrow{f} X \xrightarrow{g} M(v) \to 0$ be an extension where $\text{supp}(M(u)) \cap \text{supp}(M(v)) = \varnothing$ and where s_u and s_v do not have any common vertices. Then the given extension is split so $\text{Ext}^1_{\Lambda_T}(M(v), M(u)) = 0$.

Proof. Since s_u and s_v have no common vertices, there is no arrow $\alpha \in (Q_T)_1$ such that $u \xleftarrow{\alpha} v$ is a string in Λ_T. By exactness of the given sequence and by Lemma 5.6, it is clear that $X = M(u) \oplus M(v)$. Thus the given sequence is split. □

Proposition 4.5. Let $M(u), M(v) \in \text{ind}(\Lambda_T\text{-mod})$ where s_u and s_v either share an endpoint and agree along a segment or they have a common vertex that is an endpoint of at most one of s_u and s_v. Then $\text{Ext}^1_{\Lambda_T}(M(v), M(u)) = 0$.

Proof. Let $0 \to M(u) \xrightarrow{f} X \xrightarrow{g} M(v) \to 0$ be an extension. By Lemma 5.9 *i*), X has at least two summands $M(y)$ and $M(z)$ for some nonempty strings y and z in Λ_T. By Lemma 5.9 *ii*), without loss of generality, we have that $M(y) = M(u)$ and $M(z) = M(v)$ so the given sequence is split. □

Theorem 4.6. Let $M(u), M(v) \in \text{ind}(\Lambda_T\text{-mod})$ where s_u and s_v agree only at an endpoint. Then there is a nonsplit extension $\xi = 0 \to M(u) \to M(u \xleftarrow{\alpha} v) \to M(v) \to 0$ if and only if there exists an arrow $\alpha \in (Q_T)_1$ such that $u \xleftarrow{\alpha} v$ is a string in Λ_T. In this case, ξ is the unique nonsplit extension of $M(v)$ by $M(u)$.

Proof. Assume that there exists an arrow $\alpha \in (Q_T)_1$ such that $u \xleftarrow{\alpha} v$ is a string in Λ_T. Thus $M(u \xleftarrow{\alpha} v)$ is a string module and so ξ is a nonsplit extension.

Assume that there does not exist an arrow $\alpha \in (Q_T)_1$ such that $u \xleftarrow{\alpha} v$ is a string in Λ_T. Let $0 \to M(u) \xrightarrow{f} X \xrightarrow{g} M(v) \to 0$ be an extension. Lemma 5.6 implies that $X = M(u) \oplus M(v)$ so all such extensions are split.

The last assertion follows from the fact that $\dim_{\Bbbk} \text{Ext}^1_{\Lambda_T}(M(v), M(u)) = 1$ by Lemma 5.5. □

Theorem 4.7. Let $M(u), M(v) \in \text{ind}(\Lambda_T\text{-mod})$ and suppose that $\text{supp}(M(u)) \cap \text{supp}(M(v)) \neq \varnothing$. Now let w denote the unique maximal string supported on $\text{supp}(M(u)) \cap \text{supp}(M(v))$. Furthermore, assume that the segments s_u and s_v do not have any common endpoints. Write $u = u^{(1)} \leftrightarrow w \leftrightarrow u^{(2)}$ and $v = v^{(1)} \leftrightarrow w \leftrightarrow v^{(2)}$ for some strings $u^{(1)}, u^{(2)}, v^{(1)}$, and $v^{(2)}$ in Λ_T, some of which may be empty. Then $\text{Ext}^1_{\Lambda_T}(M(v), M(u)) \neq 0$ if and only if $u = u^{(1)} \leftarrow w \to u^{(2)}$ and $v = v^{(1)} \to w \leftarrow v^{(2)}$. Additionally, in this case,

$$0 \to M(u) \to M(u^{(1)} \leftarrow w \leftarrow v^{(2)}) \oplus M(v^{(1)} \to w \to u^{(2)}) \to M(v) \to 0$$

is the unique nonsplit extension of $M(v)$ by $M(u)$.

Proof. Assume that $u = u^{(1)} \leftarrow w \to u^{(2)}$ and $v = v^{(1)} \to w \leftarrow v^{(2)}$ for some strings $u^{(1)}, u^{(2)}, v^{(1)}$, and $v^{(2)}$ in Λ_T, not all of which are empty. Note that the segments s_u and s_v have no common endpoints. This means that $M(u^{(1)} \leftarrow w \leftarrow v^{(2)})$ is not isomorphic to $M(u)$ or $M(v)$ and the same is true for $M(v^{(1)} \to w \to u^{(2)})$. Thus

$$0 \to M(u) \to M(u^{(1)} \leftarrow w \leftarrow v^{(2)}) \oplus M(v^{(1)} \to w \to u^{(2)}) \to M(v) \to 0$$

is a nonsplit extension. This implies that $\text{Ext}^1_{\Lambda_T}(M(v), M(u)) \neq 0$.

Conversely, assume that $\text{Ext}^1_{\Lambda_T}(M(v), M(u)) \neq 0$. Let $0 \to M(u) \xrightarrow{f} X \xrightarrow{g} M(v) \to 0$ be a nonsplit extension and let $X = \oplus_{i=1}^k X_i$ be a direct sum decomposition of X into indecomposables. By Corollary 5.8, we have that $X = M(u^{(1)} \leftrightarrow w \leftrightarrow v^{(2)}) \oplus M(v^{(1)} \leftrightarrow w \leftrightarrow u^{(2)})$. Since the given sequence is exact, we must have that $(u^{(1)} \leftrightarrow w \leftrightarrow v^{(2)}) = (u^{(1)} \leftarrow w \leftarrow v^{(2)})$ and $(v^{(1)} \leftrightarrow w \leftrightarrow u^{(2)}) = (v^{(1)} \to w \to u^{(2)})$. Thus $u = u^{(1)} \leftarrow w \to u^{(2)}$ and $v = v^{(1)} \to w \leftarrow v^{(2)}$.

The last assertion follows from the fact that $\dim_{\Bbbk} \text{Ext}^1_{\Lambda_T}(M(v), M(u)) = 1$ by Lemma 5.5. □

5. Homomorphisms and extensions between string modules

In this section, we present the technical facts required to prove Propositions 4.4 and 4.5 and Theorems 4.6 and 4.7. We prove Lemma 5.1, which is used in the statement of Theorem 4.7, Lemma 5.7, and Corollary 5.8. We omit the proofs of Lemma 5.2, 5.3, and 5.5 as they are nearly identical to that of [14, Lemma 9.2], [14, Lemma 9.3], and [14, Lemma 9.4], respectively.

Lemma 5.1. Let $M(u), M(v) \in \text{ind}(\Lambda_T\text{-mod})$ with $\text{supp}(M(u)) \cap \text{supp}(M(v)) \neq \varnothing$. Then $w = x_1 \leftrightarrow x_2 \cdots x_{k-1} \leftrightarrow x_k$ where $\text{supp}(M(u)) \cap \text{supp}(M(v)) = \{x_i\}_{i\in[k]}$ is a string in Λ_T. Furthermore, w is the unique maximal string along which u and v agree.

Proof. Any string in Λ_T includes at most two vertices from any oriented cycle in Q_T. Thus a string $u = y_1 \leftrightarrow y_2 \cdots y_{s-1} \leftrightarrow y_s$ is the shortest path connecting y_1 and y_s in the underlying graph of Q_T. This implies that for any y_i and y_j appearing in u, the string $y_i \leftrightarrow y_{i+1} \cdots y_{j-1} \leftrightarrow y_j$ is the shortest path connecting y_i and y_j in the underlying graph of Q_T. Therefore if $\text{supp}(M(u)) \cap \text{supp}(M(v)) \neq \varnothing$, then $w = x_1 \leftrightarrow x_2 \cdots x_{k-1} \leftrightarrow x_k$ where $\text{supp}(M(u)) \cap \text{supp}(M(v)) = \{x_i\}_{i\in[k]}$ is a string in Λ_T. Clearly, w is the unique maximal string along which u and v agree. □

Lemma 5.2. Let $M(u), M(v) \in \text{ind}(\Lambda_T\text{-mod})$. If $M(u) \hookrightarrow M(v)$ or $M(u) \twoheadrightarrow M(v)$, then

$$\dim_{\Bbbk} \text{Hom}_{\Lambda_T}(M(u), M(v)) = 1.$$

Lemma 5.3. Let $M(u), M(v) \in \text{ind}(\Lambda_T\text{-mod})$. Then $\dim_{\Bbbk} \text{Hom}_{\Lambda_T}(M(u), M(v)) \leqslant 1$. Additionally, assume $\text{Hom}_{\Lambda_T}(M(u), M(v)) \neq 0$, but $M(u)$ is not a submodule of $M(v)$ and $M(u)$ does not surject onto $M(v)$. Then there exists a string w in Λ_T distinct from both u and v such that $M(u) \twoheadrightarrow M(w) \hookrightarrow M(v)$.

Lemma 5.4. Assume s_u and s_v agree at an endpoint. Then s_u and s_v agree along a segment s_w if and only if either $\text{Hom}_{\Lambda_T}(M(u), M(v)) \neq 0$ or $\text{Hom}_{\Lambda_T}(M(v), M(u)) \neq 0$.

Proof. Assume s_u and s_v agree along a segment s_w. By Lemma 5.1, assume that s_w is the unique largest segment along which s_u and s_v agree. We have that either $u = u^{(1)} \leftarrow w$ and $v = v^{(1)} \rightarrow w$ or $u = u^{(1)} \rightarrow w$ and $v = v^{(1)} \leftarrow w$. In the former case, $\text{Hom}_{\Lambda_T}(M(u), M(v)) \neq 0$. In the latter case, $\text{Hom}_{\Lambda_T}(M(v), M(u)) \neq 0$.

The converse statement is obvious. □

Lemma 5.5. Let $M(u), M(v) \in \text{ind}(\Lambda_T\text{-mod})$. Then $\dim_{\Bbbk} \text{Ext}^1_{\Lambda_T}(M(u), M(v)) \leqslant 1$.

Next, we present four results, each of which is crucial to classifying extensions between indecomposable Λ_T-modules. Lemma 5.6 is used in the proof of Proposition 4.4 and Theorem 4.6. Corollary 5.8, which is used in the proof Theorem 4.7, follows from Lemma 5.7. Lemma 5.7 establishes several restrictions on which indecomposable Λ_T-modules can appear as middle terms of a nonsplit extension between two indecomposables whose corresponding segments agree along a segment, but have no shared endpoints. Lastly, Lemma 5.9 is used in the proof of Proposition 4.5.

Lemma 5.6. Let $0 \to M(u) \xrightarrow{f} X \xrightarrow{g} M(v) \to 0$ be an extension where $\text{supp}(M(u)) \cap \text{supp}(M(v)) = \varnothing$. Assume that there does not exist an arrow $\alpha \in (Q_T)_1$ such that $u \xleftarrow{\alpha} v$ is a string in Λ_T and let $X = \oplus_{i=1}^k X_i$ be a direct sum decomposition of X in to indecomposables (i.e., $X_i \in \text{ind}(\Lambda_T\text{-mod})$ for each $i \in [k]$). Then none of the modules X_i have any of the following properties

i) $\text{supp}(X_i) \cap \text{supp}(M(u)) \neq \varnothing$ and $\text{supp}(X_i) \cap \text{supp}(M(v)) \neq \varnothing$

ii) $\text{supp}(X_i) \subsetneq \text{supp}(M(u))$

iii) $\text{supp}(X_i) \subsetneq \text{supp}(M(v))$.

Proof. Suppose some X_i satisfies *i)*. Then we can write $X_i = M(w)$, $u = u' \leftrightarrow w'$, and $v = w'' \leftrightarrow v''$ where $w = w' \leftrightarrow w''$ is a string in Λ_T. By assumption, $w = w' \leftrightarrow w'' = w' \xrightarrow{\beta} w''$. Observe that the direction of β implies that $\text{Hom}_{\Lambda_T}(M(u), M(w)) = 0$ and $\text{Hom}_{\Lambda_T}(M(w), M(v)) = 0$. Since $\text{supp}(M(u)) \cap \text{supp}(M(v)) = \varnothing$, $\{\text{supp}(X_i)\}_{i=1}^k$ is a set partition of the set $\text{supp}(X)$. Thus we have that $M(w) \cap \text{im}(f) = 0$, but $M(w) \subseteq \ker(g)$. This contradicts that the given sequence is exact.

As none of the X_i satisfy *i)*, we can separate these modules into those supported on $M(u)$ and those supported on $M(v)$. We denote the former modules by $\{M(u^{(j)})\}_{j=1}^s$ and the latter by $\{M(v^{(\ell)})\}_{\ell=1}^t$.

Suppose $M(u^{(j)})$ satisfies *ii)*. Then there exist $M(u^{(j')})$ for some $j' \neq j$ such that $u^{(j)} \xleftrightarrow{\beta} u^{(j')}$ is a string in Λ_T supported on u. Thus if $u^{(j)} \xleftarrow{\beta} u^{(j')}$ (resp., $u^{(j)} \xrightarrow{\beta} u^{(j')}$) is a string in Λ_T, we have that $\text{Hom}_{\Lambda_T}(M(u), M(u^{(j)})) = 0$ (resp., $\text{Hom}_{\Lambda_T}(M(u), M(u^{(j')})) = 0$). This implies that there exists a summand

$M(u^{(j'')})$ of X such that $M(u^{(j'')}) \cap \mathrm{im}(f) = 0$. However, $M(u^{(j'')}) \subseteq \ker(g)$ since $\mathrm{supp}(M(u^{(j'')})) \cap \mathrm{supp}(M(v)) = \varnothing$. This contradicts that the given sequence is exact. The proof that there are no summands $M(v^{(\ell)})$ of X that satisfy *iii*) is similar so we omit it. □

Lemma 5.7. Let $M(u), M(v) \in \mathrm{ind}(\Lambda_T\text{-mod})$ where s_u and s_v have no common endpoints. Let $0 \to M(u) \xrightarrow{f} X \xrightarrow{g} M(v) \to 0$ be a nonsplit extension where $\mathrm{supp}(M(u)) \cap \mathrm{supp}(M(v)) \neq \varnothing$, and let w denote the unique maximal string supported on $\mathrm{supp}(M(u)) \cap \mathrm{supp}(M(v))$. Let $X = \oplus_{i=1}^k X_i$ be a direct sum decomposition of X into indecomposables and write $u = u^{(1)} \leftrightarrow w \leftrightarrow u^{(2)}$ and $v = v^{(1)} \leftrightarrow w \leftrightarrow v^{(2)}$ for some strings $u^{(1)}, u^{(2)}, v^{(1)}$, and $v^{(2)}$ in Λ_T, some of which may be empty. Then the following hold.

i) X is not indecomposable.

ii) There is no X_i such that $\mathrm{supp}(X_i) \cap \mathrm{supp}(M(x)) \neq \varnothing$ for any $x \in \{w, u^{(1)}, u^{(2)}\}$, assuming that both $u^{(1)}$ and $u^{(2)}$ are nonempty strings.

iii) There is no X_i such that $\mathrm{supp}(X_i) \cap \mathrm{supp}(M(x)) \neq \varnothing$ for any $x \in \{w, v^{(1)}, v^{(2)}\}$, assuming that both $v^{(1)}$ and $v^{(2)}$ are nonempty strings.

iv) There is no X_i such that $\mathrm{supp}(X_i) \subsetneq \mathrm{supp}(M(x))$ where $x \in \{u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}\}$. Thus each X_i satisfies $\mathrm{supp}(X_i) \cap \mathrm{supp}(M(w)) \neq \varnothing$.

v) If X_i and $x \in \{w, u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}\}$ satisfy $\mathrm{supp}(X_i) \cap \mathrm{supp}(M(x)) \neq \varnothing$, then $\mathrm{supp}(M(x)) \subseteq \mathrm{supp}(X_i)$.

Proof. We first show that each X_i satisfies $X_i \not\cong M(u)$ and $X_i \not\cong M(v)$. Without loss of generality, suppose a summand X_i of X satisfies $X_i \cong M(u)$. Since s_u and s_v have no common endpoints, $\mathrm{im}(f) = X_i$. By dimension considerations and the fact that g is surjective, $M(v)$ is also a summand of X. Thus the given sequence is split, a contradiction.

i) We observe that by exactness, $\underline{\dim}_{\Bbbk}(X) = \underline{\dim}_{\Bbbk}(M(u)) + \underline{\dim}_{\Bbbk}(M(v))$. Since $\mathrm{supp}(M(u)) \cap \mathrm{supp}(M(v)) \neq \varnothing$, Lemma 4.3 1. implies that X is not a string module and therefore not indecomposable.

ii) Suppose that such an X_i exists. Then $\mathrm{supp}(M(w)) \subseteq \mathrm{supp}(X_i)$. Now note that since $X_i \not\cong M(u)$ and $X_i \not\cong M(v)$ we can assume without loss of generality, that $\mathrm{supp}(X_i) \cap \mathrm{supp}(M(u^{(1)})) \subsetneq \mathrm{supp}(M(u^{(1)}))$ and $\mathrm{supp}(X_i) \cap \mathrm{supp}(M(u^{(1)})) \neq \varnothing$. This implies that we can write $u^{(1)} = x^{(1)} \leftrightarrow x^{(2)}$ for some nonempty strings $x^{(1)}$ and $x^{(2)}$ in Λ_T where $\mathrm{supp}(M(x^{(2)})) = \mathrm{supp}(X_i) \cap \mathrm{supp}(M(u^{(1)}))$ and $u = x^{(1)} \leftrightarrow x^{(2)} \leftrightarrow w \leftrightarrow u^{(2)}$.

Suppose $u^{(1)} = x^{(1)} \leftarrow x^{(2)}$. Now write $x^{(1)} = x_1^{(1)} \leftrightarrow x_2^{(1)} \leftrightarrow \cdots \leftrightarrow x_\ell^{(1)}$ so that

$$u^{(1)} = x^{(1)} \leftarrow x^{(2)} = (x_1^{(1)} \leftrightarrow x_2^{(1)} \leftrightarrow \cdots \leftrightarrow x_\ell^{(1)}) \leftarrow x^{(2)}.$$

In this case, $\mathrm{Hom}_{\Lambda_T}(M(u), X_j) = 0$ if X_j is any summand of X where $\mathrm{supp}(X_j) \subseteq \mathrm{supp}(M(x^{(1)}))$ and $x_\ell^{(1)} \in \mathrm{supp}(X_j)$. Thus any such X_j satisfies $X_j \cap \mathrm{im}(f) = 0$. One also observes that $\mathrm{supp}(M(x^{(1)})) \cap \mathrm{supp}(M(v)) = \varnothing$ so $\mathrm{Hom}_{\Lambda_T}(X_j, M(v)) = 0$. Therefore, any such $X_j \subseteq \ker(g)$. This means that if such a summand X_j exists, then the given sequence is not exact.

We show that there must be a summand X_j of X satisfying $\mathrm{supp}(X_j) \subseteq \mathrm{supp}(M(x^{(1)}))$ and whose string contains $x_\ell^{(1)}$. First note that by the exactness of the given sequence, there must exist a summand X_j of X whose support contains $x_\ell^{(1)}$ and thus intersects $\mathrm{supp}(M(x^{(1)}))$. It is enough to show that, without loss of generality, there is no string y in Λ_T such that $\mathrm{supp}(M(y)) \cap \mathrm{supp}(M(x^{(1)})) \neq \varnothing$ and $\mathrm{supp}(M(y)) \cap \mathrm{supp}(M(v^{(1)})) \neq \varnothing$. To show this, it is enough to observe that the segments $s_{x^{(1)}}$ and $s_{v^{(1)}}$ have no common vertices. The latter follows from the fact that $x^{(2)}$ is a nonempty string. We obtain a contradiction.

We now have that $u^{(1)} = x^{(1)} \to x^{(2)}$. This implies that $\mathrm{Hom}_{\Lambda_T}(M(u), X_i) = 0$. Let us express X_i as $X_i = ((V_i)_{i \in Q_0}, (\varphi_\alpha)_{\alpha \in Q_1})$. By exactness and dimension considerations, the module X_i is the only summand of X satisfying $\mathrm{supp}(X_i) \cap \mathrm{supp}(M(x^{(2)})) \neq \varnothing$. Thus there exists $\lambda \in V_i$ that is nonzero with $i \in \mathrm{supp}(X_i) \cap \mathrm{supp}(M(x^{(2)}))$ and $\lambda \notin \mathrm{im}(f)$. However, λ is also a nonzero element of $M(u)$, and this contradicts that f is injective.

iii) The proof of this assertion is similar to the proof of assertion *ii*) so we omit it.

iv) It suffices to show that there does not exist a summand X_i of X such that $\mathrm{supp}(X_i) \subsetneq \mathrm{supp}(M(v^{(1)}))$. Suppose there exists such a summand X_i. Then there exist summands $M(x)$ and $M(y)$ of X such that $x \leftrightarrow y$ is a string in Λ_T where $\mathrm{supp}(M(x)) \subsetneq \mathrm{supp}(M(v^{(1)}))$ and $\mathrm{supp}(M(y)) \cap \mathrm{supp}(M(v^{(1)})) \neq \varnothing$. If $(x \leftrightarrow y) = (x \leftarrow y)$, then $\mathrm{Hom}_{\Lambda_T}(M(y), M(v)) = 0$. Let us express $M(y)$ as $M(y) = ((V_i)_{i \in Q_0}, (\varphi_\alpha)_{\alpha \in Q_1})$. Then any nonzero $\lambda \in V_i$ where $i \in \mathrm{supp}(M(y)) \cap \mathrm{supp}(M(v^{(1)}))$ satisfies $\lambda \in \ker(g)$. Since λ does not belong to any summand besides $M(y)$, we have that g is not surjective, a contradiction. If instead $(x \leftrightarrow y) = (x \to y)$, then $\mathrm{Hom}_{\Lambda_T}(M(x), M(v)) = 0$. Similarly, this implies that $M(x) \subseteq \ker(g)$, which contradicts that g is surjective.

v) We first prove the assertion for any $x \in \{u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}\}$. It suffices to prove this for $x = v^{(1)}$. Suppose that there exists X_i such that $\text{supp}(X_i) \cap \text{supp}(M(v^{(1)})) \neq \varnothing$ and $\text{supp}(M(v^{(1)})) \nsubseteq \text{supp}(X_i)$. By iv), we have that $\text{supp}(X_i) \cap \text{supp}(M(w)) \neq \varnothing$. Now by exactness of the given sequence, there exists another summand X_j of X such that $\text{supp}(X_j) \subseteq \text{supp}(M(v^{(1)}))\backslash\text{supp}(X_i) \subseteq \text{supp}(M(v^{(1)}))$. This contradicts iv).

Next, we suppose $x = w$. By assertion iv), each summand X_i satisfies $\text{supp}(X_i) \cap \text{supp}(M(w)) \neq \varnothing$. Thus it is enough to show that there are no summands X_i such that $\text{supp}(X_i) \subsetneq \text{supp}(M(w))$. Suppose there exists such a summand $X_i = M(y^{(2)})$. We can assume, without loss of generality, that there is another summand $X_j = M(y^{(1)})$ of X such that

- $y^{(1)} \leftrightarrow y^{(2)}$ is a string in Λ_T,
- $\text{supp}(M(y^{(1)})) \cap \text{supp}(M(v^{(1)})) \neq \varnothing$,
- $\text{supp}(M(y^{(1)})) \cap \text{supp}(M(w)) \neq \varnothing$.

Suppose that $(y^{(1)} \leftrightarrow y^{(2)}) = (y^{(1)} \rightarrow y^{(2)})$. Then $\text{Hom}_{\Lambda_T}(M(y^{(1)}), M(v)) = 0$. Let us express $M(y^{(1)})$ as $M(y^{(1)}) = ((V_i)_{i\in Q_0}, (\varphi_\alpha)_{\alpha\in Q_1})$. The for any nonzero $\lambda \in V_i$ where $i \in \text{supp}(M(y^{(1)})) \cap \text{supp}(M(v^{(1)}))$ satisfies $\lambda \in \ker(g)$. Since $M(y^{(1)})$, is the only summand containing λ, this contradicts that g is surjective.

Now suppose $(y^{(1)} \leftrightarrow y^{(2)}) = (y^{(1)} \leftarrow y^{(2)})$ and write $y^{(2)} = y_1^{(2)} \leftrightarrow \cdots \leftrightarrow y_\ell^{(2)}$. Then $\text{Hom}_{\Lambda_T}(M(y^{(2)}), M(v)) = 0$. This means that any other summand $M(y^{(3)})$ of X where $(y^{(1)} \leftarrow y^{(3)})$ is a string in Λ_T and $y_1^{(2)} \in \text{supp}(M(y^{(3)}))$ has the property that $\text{Hom}_{\Lambda_T}(M(y^{(3)}), M(v)) = 0$. Since $M(y^{(1)})$ is the only summand of X whose support intersects $\text{supp}(M(y^{(1)})) \cap \text{supp}(M(v^{(1)}))$ and since $\text{supp}(M(y^{(1)})) \subseteq \text{supp}(M(v))$, we have that there is an inclusion $M(y^{(1)}) \hookrightarrow M(v)$. Since the given sequence is exact, there must exist a summand $M(z) = ((V_i)_{i\in Q_0}, (\varphi_\alpha)_{\alpha\in Q_1})$ of X where z satisfies

- $\text{supp}(M(z)) \cap \text{supp}(M(y^{(1)})) \neq \varnothing$ where any nonzero $\lambda \in V_i$ with $i \in \text{supp}(M(z)) \cap \text{supp}(M(y^{(1)}))$ satisfies $\lambda \notin \ker(g) = \text{im}(f)$, and
- $\text{supp}(M(z)) \cap \text{supp}(M(y^{(2)})) \neq \varnothing$ where any nonzero $\lambda \in V_i$ with $i \in \text{supp}(M(z)) \cap \text{supp}(M(y^{(2)}))$ satisfies $\lambda \in \text{im}(f)$.

However, since $(y^{(1)} \leftrightarrow y^{(2)}) = (y^{(1)} \leftarrow y^{(2)})$ there are no homomorphisms from $M(u)$ to $M(z)$ satisfying these properties. Thus there are no summands X_i of X such that $\text{supp}(X_i) \subsetneq \text{supp}(M(w))$. ☐

Corollary 5.8. Let $M(u), M(v) \in \text{ind}(\Lambda_T\text{-mod})$ where s_u and s_v have no common endpoints. Let $0 \to M(u) \xrightarrow{f} X \xrightarrow{g} M(v) \to 0$ be a nonsplit extension where $\text{supp}(M(u)) \cap \text{supp}(M(v)) \neq \varnothing$, and let w denote the unique maximal string supported on $\text{supp}(M(u)) \cap \text{supp}(M(v))$. Let $X = \bigoplus_{i=1}^k X_i$ be a direct sum decomposition of X into indecomposables and write $u = u^{(1)} \leftrightarrow w \leftrightarrow u^{(2)}$ and $v = v^{(1)} \leftrightarrow w \leftrightarrow v^{(2)}$ for some strings $u^{(1)}, u^{(2)}, v^{(1)}$, and $v^{(2)}$ in Λ_T some of which may be empty. Then $X = M(u^{(1)} \leftrightarrow w \leftrightarrow v^{(2)}) \oplus M(v^{(1)} \leftrightarrow w \leftrightarrow u^{(2)})$.

Proof. By Lemma 5.7 i), X has at least two indecomposable summands. By Lemma 5.7 iv) and v), X has exactly two summands, $M(y)$ and $M(z)$, where $\text{supp}(M(w)) \subseteq \text{supp}(M(y))$ and $\text{supp}(M(w)) \subseteq \text{supp}(M(z))$. By exactness of the given sequence and by Lemma 5.7 v), for any $x \in \{u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}\}$ we have that $\text{supp}(M(x))$ is contained in $\text{supp}(M(y))$ or $\text{supp}(M(z))$. By combining Lemma 5.7 ii) and iii), we have that $M(y) = M(u^{(1)} \leftrightarrow w \leftrightarrow v^{(2)})$ and $M(z) = M(v^{(1)} \leftrightarrow w \leftrightarrow u^{(2)})$. ☐

Lemma 5.9. Let $M(u), M(v) \in \text{ind}(\Lambda_T\text{-mod})$ where s_u and s_v either share an endpoint and agree along a segment or they have a common vertex that is an endpoint of at most one of s_u and s_v. If $0 \to M(u) \xrightarrow{f} X \xrightarrow{g} M(v) \to 0$ is an extension and $X = \bigoplus_{i=1}^k X_i$ is a direct sum decomposition into indecomposables, then the following hold.

i) X is not indecomposable.

ii) There is no X_i such that $\text{supp}(X_i) \subsetneq \text{supp}(M(x))$ where $x \in \{u, v\}$.

Proof. Only Lemma 5.7 ii) and iii) relied on the assumption that the given extension was nonsplit. Thus one proves these assertions by adapting the proofs of Lemmas 5.7 i), iv), and v), since these did not depend on Lemma 5.7 ii) and iii). ☐

6. Oriented flip graphs and torsion-free classes

In this section, we recall the definition of torsion-free classes and their lattice structure. After that, we show that the oriented flip graph of T is isomorphic as a poset to the lattice of torsion-free classes of Λ_T ordered by inclusion and torsion classes of Λ_T ordered by reverse inclusion.

Let Λ be a finite dimensional $\Bbbk$-algebra. A full, additive subcategory $\mathcal{C} \subseteq \Lambda\text{-mod}$ is **extension closed** if for any objects $X, Y \in \mathcal{C}$ satisfying $0 \to X \to Z \to Y \to 0$ one has $Z \in \mathcal{C}$. We say $\mathcal{C}$ is **quotient closed** (resp.,

submodule closed) if for any $X \in \mathcal{C}$ satisfying $X \xrightarrow{\alpha} Z$ where α is a surjection (resp., $Z \xrightarrow{\beta} X$ where β is an injection), then $Z \in \mathcal{C}$. A full, additive subcategory $\mathcal{T} \subseteq \Lambda$-mod is called a **torsion class** if $\mathcal{T}$ is quotient closed and extension closed. Dually, a full, additive subcategory $\mathcal{F} \subseteq \Lambda$-mod is called a **torsion-free class** if $\mathcal{F}$ is extension closed and submodule closed.

Let $\text{tors}(\Lambda)$ (resp., $\text{torsf}(\Lambda)$) denote the lattice of torsion classes (resp., of torsion-free classes) of Λ, ordered by inclusion. We have the following proposition, which shows that a torsion class of Λ uniquely determines a torsion-free class of Λ and vice versa. Given $\mathcal{T}$ a torsion class and $\mathcal{F}$ its corresponding torsion-free class, we say that the data $(\mathcal{T}, \mathcal{F})$ is a **torsion pair**.

Proposition 6.1. [18, Prop. 1.1 a)] The maps

$$\begin{array}{rcl} \text{tors}(\Lambda) & \xrightarrow{(-)^\perp} & \text{torsf}(\Lambda) \\ \mathcal{T} & \longmapsto & \mathcal{T}^\perp := \{X \in \Lambda\text{-mod} : \ \text{Hom}_\Lambda(\mathcal{T}, X) = 0\} \end{array}$$

and

$$\begin{array}{rcl} \text{torsf}(\Lambda) & \xrightarrow{{}^\perp(-)} & \text{tors}(\Lambda) \\ \mathcal{F} & \longmapsto & {}^\perp\mathcal{F} := \{X \in \Lambda\text{-mod} : \ \text{Hom}_\Lambda(X, \mathcal{F}) = 0\} \end{array}$$

are inverse bijections.

Proposition 6.2. [18, Prop. 1.3] Let Λ be a finite dimensional algebra. Then $\text{tors}(\Lambda)$ and $\text{torsf}(\Lambda)$ are complete lattices. The join and meet operations are described as follows.

a) Let $\{\mathcal{T}_i\}_{i \in I} \subseteq \text{tors}(\Lambda)$ be a collection of torsion classes. Then we have $\bigwedge_{i\in I} \mathcal{T}_i = \bigcap_{i \in I} \mathcal{T}_i$ and $\bigvee_{i \in I} \mathcal{T}_i = {}^\perp\left(\bigcap_{i\in I} \mathcal{T}_i^\perp\right)$.

b) Let $\{\mathcal{F}_i\}_{i \in I} \subseteq \text{torsf}(\Lambda)$ be a collection of torsion-free classes. Then we have $\bigwedge_{i\in I} \mathcal{F}_i = \bigcap_{i\in I} \mathcal{F}_i$ and $\bigvee_{i\in I} \mathcal{F}_i = \left(\bigcap_{i \in I} {}^\perp\mathcal{F}_i\right)^\perp$.

Lemma 6.3. [18, Prop. 1.4 a), c)] The maps

$$\begin{array}{rcl} \text{tors}(\Lambda) & \xrightarrow{D(-)} & \text{torsf}(\Lambda^{\text{op}}) \cong \text{torsf}(\Lambda)^{\text{op}} \\ \mathcal{T} & \longmapsto & D\mathcal{T} \end{array}$$

and

$$\begin{array}{rcl} \text{torsf}(\Lambda) & \xrightarrow{D(-)} & \text{tors}(\Lambda^{\text{op}}) \cong \text{tors}(\Lambda)^{\text{op}} \\ \mathcal{F} & \longmapsto & D\mathcal{F} \end{array}$$

are isomorphisms of lattices where $D(-) := \text{Hom}_\Lambda(-, \Bbbk)$ is the **standard duality**. Furthermore, the functor $D((-)^\perp) : \text{tors}(\Lambda) \to \text{tors}(\Lambda^{\text{op}})$ is an anti-isomorphism of posets.

For the proofs of Theorems 6.5 and 7.1, we will use the following lemma.

Lemma 6.4. Let $\mathcal{C} \subset \Lambda$-mod be a full, additive subcategory with the property that for any indecomposables $X, Y \in \text{ind}(\Lambda\text{-mod})$ and any extension $0 \to X \to Z \to Y \to 0$, the module Z belongs $\mathcal{C}$. Then $\mathcal{C}$ is extension closed.

Proof. Let $0 \to X \to Z \to Y \to 0$ be a an extension where $X, Y \in \mathcal{C}$, and let $\xi \in \text{Ext}^1_\Lambda(Y, X)$ denote the corresponding extension.

Now, write $X \cong \bigoplus_{i=1}^k X_i$ and $Y \cong \bigoplus_{j=1}^\ell Y_j$ for some indecomposables $X_i, Y_j \in \Lambda_T$-mod where $i = 1, \ldots, k$ and $j = 1, \ldots, \ell$. Recall that we have an isomorphism

$$\text{Ext}^1_\Lambda(Y, X) \cong \bigoplus_{i,j} \text{Ext}^1_\Lambda(Y_j, X_i).$$

We let $\sum_{i,j} \xi_{i,j} \in \bigoplus_{i,j} \text{Ext}^1_{\Lambda_T}(Y_j, X_i)$ denote the image of ξ under this isomorphism, and we let

$$0 \longrightarrow X_i \xrightarrow{f_{i,j}} Z_{i,j} \xrightarrow{g_{i,j}} Y_j \longrightarrow 0$$

denote the extension corresponding to $\xi_{i,j}$. This implies that ξ is equivalent to the extension

$$0 \longrightarrow \bigoplus_{i=1}^k X_i \xrightarrow{[f_{i,j}]} \bigoplus_{i,j} Z_{i,j} \xrightarrow{[g_{i,j}]} \bigoplus_{j=1}^\ell Y_j \longrightarrow 0$$

By assumption, $Z_{i,j} \in \mathcal{C}$ for all i and j. Since $\mathcal{C}$ is additive, we have that $\bigoplus_{i,j} Z_{i,j} \in \mathcal{C}$. Thus $\mathcal{C}$ is extension closed. $\square$

Theorem 6.5. For any tree T, we have that $\overrightarrow{FG}(T) \cong \text{torsf}(\Lambda_T)$ and $\overrightarrow{FG}(T) \cong \text{tors}(\Lambda_T)^{\text{op}}$ where $\text{tors}(\Lambda_T)^{\text{op}}$ denotes the lattice of torsion classes ordered by reverse inclusion.

Proof. We claim that the map

$$\begin{array}{rcl} \overrightarrow{FG}(T) = \pi_\downarrow(\text{Bic}(T)) & \xrightarrow{\zeta} & \text{torsf}(\Lambda_T) \\ \pi_\downarrow(X) & \longmapsto & \mathcal{F} := \text{add}(\oplus_{s_u} M(u) : \ s_u \in \pi_\downarrow(X)) \end{array}$$

is an isomorphism of posets where $\text{add}(\oplus_{i=1}^k X_i)$ denotes the smallest full, additive subcategory of Λ_T-mod closed under taking summands of $\oplus_{i=1}^k X_i$. Furthermore, we claim that the inverse of this map is given by

$$\begin{array}{rcl} \text{torsf}(\Lambda_T) & \xrightarrow{\delta} & \pi_\downarrow(\text{Bic}(T)) \\ \mathcal{F} = \text{add}(\oplus_{i\in[k]} M(w^{(i)})) & \longmapsto & \pi_\downarrow(\{s_{w^{(1)}}, \ldots, s_{w^{(k)}}\}). \end{array}$$

We can see that these maps are order-preserving, since $\pi_\downarrow$ is order-preserving by [15, Lemma 4.5 (7)]. Assuming that $\zeta(\pi_\downarrow(X))$ is a torsion-free class and that $\delta(\mathcal{F}) \in \pi_\downarrow(\text{Bic}(T))$, we have that $\delta = \zeta^{-1}$ as $\pi_\downarrow$ is an idempotent map (see [15, Lemma 4.5 (5)]).

To show that $\delta(\mathcal{F}) \in \pi_\downarrow(\text{Bic}(T))$ where $\mathcal{F} = \text{add}(\oplus_{i\in[k]} M(w^{(i)}))$, we must show that $\{s_{w^{(1)}}, \ldots, s_{w^{(k)}}\}$ is biclosed. Let $s_u, s_v \in \{s_{w^{(1)}}, \ldots, s_{w^{(k)}}\}$ and assume $s_u \circ s_v \in \text{Seg}(T)$. Then, up to reversing the roles of u and v, $u \leftarrow v$ is a string in Λ_T so there is an extension $0 \to M(u) \to M(u \leftarrow v) \to M(v) \to 0$. Since $\mathcal{F}$ is extension closed, $M(u \leftarrow v) \in \mathcal{F}$ so $s_u \circ s_v = s_{(u\leftarrow v)} \in \{s_{w^{(1)}}, \ldots, s_{w^{(k)}}\}$. Thus $\{s_{w^{(1)}}, \ldots, s_{w^{(k)}}\}$ is closed. Since $\mathcal{F}$ is submodule closed, there are no extensions of the form $0 \to M(u) \to M(u \leftarrow v) \to M(v) \to 0$ where $s_u, s_v \notin \{s_{w^{(1)}}, \ldots, s_{w^{(k)}}\}$, but $s_{(u\leftarrow v)} \in \{s_{w^{(1)}}, \ldots, s_{w^{(k)}}\}$. Thus $\{s_{w^{(1)}}, \ldots, s_{w^{(k)}}\}$ is co-closed.

Next, we show that $\mathcal{F} := \text{add}(\oplus_{s_u} M(u) : \ s_u \in \pi_\downarrow(X))$ is a torsion-free class given $X \in \text{Bic}(T)$. We begin by showing that it is submodule closed. Assume that there is an inclusion $M(v) \hookrightarrow M(u)$ where $M(u) \in \mathcal{F}$. Write $s_u = (x_0, \ldots, x_\ell)$ and orient this segment from x_0 to x_ℓ. Let $s_v = (x_i, \ldots, x_j)$ where we can assume that $0 < i$ and $j < \ell$. The inclusion $M(v) \hookrightarrow M(u)$ implies that $u = u^{(1)} \to v \leftarrow u^{(2)}$ for some nonempty strings $u^{(1)}$ and $u^{(2)}$ in Λ_T. Now we have that s_v turns right (resp., left) at x_i (resp., at x_j). Thus $s_v \in C_{s_u} \subseteq X$. This implies that $C_{s_v} \subseteq C_{s_u} \subseteq X$ so $s_v \in \pi_\downarrow(X)$. We obtain that $M(v) \in \mathcal{F}$.

Now suppose $f : M(v) \hookrightarrow X = \oplus_{i\in[k]} M(w^{(i)})^{a_i}$ for some $a_i \geqslant 0$ and $M(v)$ does not include into any summand of X. Furthermore, suppose any indecomposable $M(u)$ with $\dim_{\Bbbk}(M(u)) < \dim_{\Bbbk}(M(v))$ that includes into an object of $\mathcal{F}$ belongs to $\mathcal{F}$. Let $M(w^{(i)})$ be a summand of X where the component map $g : M(v) \to M(w^{(i)})$ of f is nonzero. By Lemma 5.3, we can assume that there exists a nonempty string w in Λ_T not equal to u or $w^{(i)}$ such that $M(v) \twoheadrightarrow M(w) \hookrightarrow M(w^{(i)})$. By the previous paragraph, $M(w) \in \mathcal{F}$. Now express v as $v = v^{(1)} \leftarrow w \to v^{(2)}$ where, without loss of generality, both $v^{(1)}$ and $v^{(2)}$ are nonempty. This implies that $M(v^{(i)}) \hookrightarrow X$ so $M(v^{(i)}) \in \mathcal{F}$ for $i = 1, 2$ since $\dim_{\Bbbk}(M(v^{(i)})) < \dim_{\Bbbk}(M(v))$. Observe that we have an extension $0 \to M(v^{(2)}) \to M(w \to v^{(2)}) \to M(w) \to 0$, which shows that $M(w \to v^{(2)}) \in \mathcal{F}$ since $s_{(w\to v^{(2)})} = s_w \circ s_{v^{(2)}} \in \pi_\downarrow(X)$. This implies that we have an extension $0 \to M(v^{(1)}) \to M(v) \to M(w \to v^{(2)}) \to 0$, which shows that $M(v) \in \mathcal{F}$ since $s_v = s_{v^{(1)}} \circ s_w \circ s_{v^{(2)}} \in \pi_\downarrow(X)$. We conclude that $\mathcal{F}$ is submodule closed.

Lastly, we show that $\mathcal{F}$ is extension closed. By Lemma 6.4, it is enough to show that $\mathcal{F}$ is extension closed with respect to extensions $0 \to X \to Z \to Y \to 0$ where $X, Y \in \text{ind}(\mathcal{F})$. Since $\pi_\downarrow(X)$ is closed, it is easy to see that $\mathcal{F}$ is extension closed with respect to extensions whose nonzero terms are indecomposable. By our description of nonsplit extensions in Λ_T-mod (see Section 4), it suffices to show that if $M(u), M(v) \in \mathcal{F}$ where $u = u^{(1)} \leftarrow w \to u^{(2)}$ and $v = v^{(1)} \to w \leftarrow v^{(2)}$ and

$$0 \to M(u) \to M(u^{(1)} \leftarrow w \leftarrow v^{(2)}) \oplus M(v^{(1)} \to w \to u^{(2)}) \to M(v) \to 0$$

is the nonsplit extension defined by these modules, then $M(u^{(1)} \leftarrow w \leftarrow v^{(2)}), M(v^{(1)} \to w \to u^{(2)}) \in \mathcal{F}$. We show $M(u^{(1)} \leftarrow w \leftarrow v^{(2)}) \in \mathcal{F}$ and the proof that $M(v^{(1)} \to w \to u^{(2)}) \in \mathcal{F}$ is very similar. Notice that $M(u^{(1)}) \hookrightarrow M(u)$ and $M(w \leftarrow v^{(2)}) \hookrightarrow M(v)$ so $M(u^{(1)}), M(w \leftarrow v^{(2)}) \in \mathcal{F}$. Thus we obtain a nonsplit extension $0 \to M(u^{(1)}) \to M(u^{(1)} \leftarrow w \leftarrow v^{(2)}) \to M(w \leftarrow v^{(2)}) \to 0$, which shows that $M(u^{(1)} \leftarrow w \leftarrow v^{(2)}) \in \mathcal{F}$. □

7. Noncrossing tree partitions and wide subcategories

In this section, we show that noncrossing tree partitions of a tree T provide a combinatorial model for the wide subcategories of Λ_T-mod.

If Λ is a finite dimensional $\Bbbk$-algebra, we say that a full, additive subcategory $\mathcal{W} \subseteq \Lambda$-mod is a **wide subcategory** if it is abelian and extension closed. We let $\text{wide}(\Lambda)$ denote the poset of wide subcategories of Λ-mod, partially ordered by inclusion. It is easy to see that the intersection of two wide subcategories is also a wide subcategory, and the zero subcategory (resp., Λ-mod) is the unique minimal (resp., unique maximal) element of wide(Λ). Thus if Λ is representation-finite, the poset wide(Λ) is a lattice.

Theorem 7.1. For any tree T, we have the following isomorphisms of posets:

$$\begin{array}{rcl} \mathrm{NCP}(T) & \longrightarrow & \mathrm{wide}(\Lambda_T) \\ \mathbf{B} & \longmapsto & \mathrm{add}\left(\oplus_{s_u} M(u) : s_u \in \overline{\mathrm{Seg}(\mathbf{B})}\right). \end{array}$$

Proof. The map is order-preserving because $\mathbf{B}_1 \leqslant \mathbf{B}_2$ in NCP(T) if and only if $\overline{\mathrm{Seg}(\mathbf{B}_1)} \subseteq \overline{\mathrm{Seg}(\mathbf{B}_2)}$ in $\Psi(\overrightarrow{FG}(T))$. Thus it is enough to show that this map defines a wide subcategories and has an order-preserving inverse.

We now show that $\mathcal{W} := \mathrm{add}\left(\oplus_{s_u} M(u) : s_u \in \overline{\mathrm{Seg}(\mathbf{B})}\right) \in \mathrm{wide}(\Lambda_T)$. By Lemma 7.2, we know that $\mathcal{W}$ is closed under taking kernels and cokernels of maps between modules $M(u), M(v) \in \mathcal{W}$ where $s_u, s_v \in \mathrm{Seg}(\mathbf{B})$. Now suppose that $f \in \mathrm{Hom}_{\Lambda_T}(M(u), M(v))$ is nonzero where $s_u \in \overline{\mathrm{Seg}(B)}$, $s_v \in \overline{\mathrm{Seg}(B')}$, and where B and B' are blocks of $\mathbf{B}$. We can further assume that f is neither injective nor surjective. We write $s_u = s_{u^{(1)}} \circ \cdots \circ s_{u^{(k)}}$ and $s_v = s_{v^{(1)}} \circ \cdots \circ s_{v^{(\ell)}}$ where $s_{u^{(1)}}, \ldots, s_{u^{(k)}} \in \mathrm{Seg}(B)$ and $s_{v^{(1)}}, \ldots, s_{v^{(\ell)}} \in \mathrm{Seg}(B')$.

Assume $B \neq B'$. Since f is nonzero, then $s_u \notin \mathrm{Seg}(B)$ or $s_v \notin \mathrm{Seg}(B')$. Without loss of generality, we assume that $s_u \notin \mathrm{Seg}(B)$. Thus we have an inclusion $M(u^{(t)}) \hookrightarrow M(u)$ for some $t = 1, \ldots, k$. This implies that $\mathrm{Hom}_{\Lambda_T}(M(u^{(t)}), M(v)) \neq 0$, which contradicts Lemma 7.2.

Now assume $B = B'$. Suppose that any $g \in \mathrm{Hom}_{\Lambda_T}(X, Y)$ has $\ker(g), \mathrm{coker}(g) \in \mathcal{W}$ for any $X, Y \in \mathcal{W}$ with $\dim_{\Bbbk}(X) + \dim_{\Bbbk}(Y) < \dim_{\Bbbk}(M(u)) + \dim_{\Bbbk}(M(v))$. Define $s_w := s_{w^{(1)}} \circ \cdots \circ s_{w^{(t)}}$ where $\{s_{w^{(1)}}, \ldots, s_{w^{(t)}}\} = \{s_{u^{(1)}}, \ldots, s_{u^{(k)}}\} \cap \{s_{v^{(1)}}, \ldots, s_{v^{(\ell)}}\}$. Now we have that f factors as $M(u) \overset{\alpha}{\twoheadrightarrow} M(w) \overset{\beta}{\hookrightarrow} M(v)$. This implies that $\dim_{\Bbbk}(M(u)) + \dim_{\Bbbk}(M(w))$ and $\dim_{\Bbbk}(M(w)) + \dim_{\Bbbk}(M(v))$ are both less than $\dim_{\Bbbk}(M(u)) + \dim_{\Bbbk}(M(v))$. We thus obtain that $\ker(f) = \ker(\alpha) \in \mathcal{W}$ and $\mathrm{coker}(f) = \mathrm{coker}(\beta) \in \mathcal{W}$. We conclude that $\mathcal{W}$ is abelian.

Next, we show that $\mathcal{W}$ is extension closed. By Lemma 6.4, it is enough to show that $\mathcal{W}$ is extension closed with respect to extensions $0 \to X \to Z \to Y \to 0$ where $X, Y \in \mathrm{ind}(\mathcal{W})$. It is clear that if $M(u), M(v) \in \mathcal{W}$ appear in an extension of the form $0 \to M(u) \to M(u \leftarrow v) \to M(v) \to 0$, then $M(u \leftarrow v) \in \mathcal{W}$. Thus, it is enough to show that if $M(u), M(v) \in \mathcal{W}$ where $u = u^{(-)} \leftarrow w \to u^{(+)}$, $v = v^{(-)} \to w \leftarrow v^{(+)}$, and

$$0 \to M(u) \to M(u^{(-)} \leftarrow w \leftarrow v^{(+)}) \oplus M(v^{(-)} \to w \to u^{(+)}) \to M(v) \to 0$$

is the nonsplit extension defined by these modules, then $M(u^{(-)} \leftarrow w \leftarrow v^{(+)}), M(v^{(-)} \to w \to u^{(+)}) \in \mathcal{W}$.

Let $M(u), M(v) \in \mathcal{W}$ be two such indecomposables. We have that if $f : M(u) \to M(v)$ is a nonzero map, then $\ker(f) = M(u^{(-)}) \oplus M(u^{(+)})$ and $\mathrm{coker}(f) = M(v^{(-)}) \oplus M(v^{(+)})$. Since $\mathcal{W}$ is abelian, we conclude that $M(w), M(u^{(-)}), M(u^{(+)}), M(v^{(-)}), M(v^{(+)}) \in \mathcal{W}$. We have already noticed that $\mathcal{W}$ is extension closed with respect to taking extensions involving only indecomposables. One can thus construct such extensions showing that $M(u^{(-)} \leftarrow w \leftarrow v^{(+)}), M(v^{(-)} \to w \to u^{(+)}) \in \mathcal{W}$. We conclude that $\mathcal{W} \in \mathrm{wide}(\Lambda_T)$.

We now claim that the map $\omega : \mathrm{wide}(\Lambda_T) \longrightarrow \Psi(\overrightarrow{FG}(T))$ defined by

$$\mathcal{W} \mapsto \mathcal{S} := \overline{\{s_u : \ M(u) \text{ is a simple object of } \mathcal{W}\}}$$

is an order-preserving inverse to the map $\Psi(\overrightarrow{FG}(T)) \longrightarrow \mathrm{wide}(\Lambda_T)$. Assuming that $\omega(\mathcal{W}) \in \Psi(\overrightarrow{FG}(T))$, it is clear that ω is an inverse as a map of sets.

Our earlier argument shows that the elements $M(u^{(i)})$ are the simple objects of $\mathcal{W}$ where $s_{u^{(i)}} \in \mathrm{Seg}(\mathbf{B})$. Thus to prove that $\mathcal{S} = \overline{\mathrm{Seg}(\mathbf{B})}$ for some $\mathbf{B} \in \mathrm{NCP}(T)$, it is enough to show that any two distinct simple objects $M(u), M(v) \in \mathcal{W}$ have the property that s_u and s_v are noncrossing. Note that $\mathrm{Hom}_{\Lambda_T}(M(u), M(v)) = \mathrm{Hom}_{\Lambda_T}(M(v), M(u)) = 0$, since $M(u)$ and $M(v)$ are simple objects and $\mathcal{W}$ is a wide subcategory.

If s_u and s_v share an endpoint, then Lemma 5.4 implies that s_u and s_v do not agree along a segment. Thus they are noncrossing in this case.

If s_u and s_v are crossing, then, up to reversing the roles of u and v, Theorem 4.7 implies that they define a unique nonsplit extension

$$0 \to M(u) \to M(u^{(-)} \leftarrow w \leftarrow v^{(+)}) \oplus M(v^{(-)} \to w \to u^{(+)}) \to M(v) \to 0$$

where $u = u^{(-)} \leftarrow w \to u^{(+)}$, $v = v^{(-)} \to w \leftarrow v^{(+)}$. Using this description of the strings u and v, we notice that there is map $f \in \mathrm{Hom}_{\Lambda_T}(M(u), M(v))$ where $\mathrm{im}(f) = M(w)$, a contradiction. Thus s_u and s_v are noncrossing.

Next, we show that ω is order-preserving. Since any two simple objects of $\mathcal{W} \in \mathrm{wide}(\Lambda_T)$ correspond to noncrossing segments, the segment defined by any indecomposable object of $\mathcal{W}$ can be expressed as a concatenation of segments corresponding to simple objects of $\mathcal{W}$. That is, the segments of $\omega(\mathcal{W})$ are in bijection with the indecomposable objects of $\mathcal{W}$. Thus if $\mathcal{W}_1 \subseteq \mathcal{W}_2$, one has $\omega(\mathcal{W}_1) \subseteq \omega(\mathcal{W}_2)$. ☐

Lemma 7.2. Let $\mathbf{B} \in \mathrm{NCP}(T)$ and let $M(u), M(v)$ be two distinct indecomposable Λ_T-modules whose corresponding segments appear in $\mathrm{Seg}(B)$ and $\mathrm{Seg}(B')$, respectively, for some blocks B and B' of $\mathbf{B}$. Then one has $\mathrm{Hom}_{\Lambda_T}(M(u), M(v)) = 0$ and $\mathrm{Hom}_{\Lambda_T}(M(v), M(u)) = 0$.

Proof. First assume $B = B'$. Since $M(u)$ and $M(v)$ are distinct, the corresponding segments s_u and s_v share at most one vertex of T. This means u and v are supported on disjoint sets of vertices of Q_T so the statement holds. Thus we can assume that $s_u \in \mathrm{Seg}(B)$ and $s_v \in \mathrm{Seg}(B')$ where B and B' are distinct blocks of $\mathbf{B}$. Since $\mathbf{B} \in \mathrm{NCP}(T)$, this implies that s_u and s_v have no common endpoints.

Let γ_u and γ_v be left admissible curves for s_u and s_v, respectively, witnessing that s_u and s_v are noncrossing. Write $s_w = [a, b]$ for the unique maximal segment along which s_u and s_v agree, if it exists, and orient γ_u and γ_v from a to b. Without loss of generality, we have two cases:

i) $\mathrm{supp}(M(u)) \subsetneq \mathrm{supp}(M(v))$,
ii) $\mathrm{supp}(M(v))\backslash\mathrm{supp}(M(u)) \neq \varnothing$ and $\mathrm{supp}(M(u))\backslash\mathrm{supp}(M(v)) \neq \varnothing$.

Suppose $\mathrm{supp}(M(u)) \subsetneq \mathrm{supp}(M(v))$. Here $s_w = s_u$. By Lemma 3.1 (1), with s_u playing the role of t, we have that γ_v either turns left at both a and b or it turns right at both a and b. This means that either $v = v^{(1)} \leftarrow u \leftarrow v^{(2)}$ or $v = v^{(1)} \rightarrow u \rightarrow v^{(2)}$ for some nonempty strings $v^{(1)}$ and $v^{(2)}$ in Λ_T. Thus $\mathrm{Hom}_{\Lambda_T}(M(u), M(v)) = 0$ and $\mathrm{Hom}_{\Lambda_T}(M(v), M(u)) = 0$.

Now suppose that $\mathrm{supp}(M(v))\backslash\mathrm{supp}(M(u)) \neq \varnothing$ and $\mathrm{supp}(M(u))\backslash\mathrm{supp}(M(v)) \neq \varnothing$. We can assume that a (resp., b) is an endpoint of s_v (resp., s_u). Thus we can write $s_v = [a, b] \circ s_{v'}$ and $s_u = s_{u'} \circ [a, b]$ for some nonempty segments $s_{v'}, s_{u'} \in \mathrm{Seg}(T)$. By Lemma 3.1 (2), with $[a, b]$ playing the role of t, we have that either γ_v turns right at b and γ_u turns left at a or γ_v turns left at b and γ_u turns right at a. Thus either $v = w \rightarrow v'$ and $u = u' \leftarrow w$ or $v = w \leftarrow v'$ and $u = u' \rightarrow w$. We conclude that $\mathrm{Hom}_{\Lambda_T}(M(u), M(v)) = 0$ and $\mathrm{Hom}_{\Lambda_T}(M(v), M(u)) = 0$. □

8. Simple-minded collections

In this section, we interpret noncrossing tree partitions in terms of the representation theory of Λ_T using simple-minded collections in the bounded derived category of Λ_T, denoted $\mathcal{D}^b(\Lambda_T)$. We show that the data of a noncrossing tree partition and its Kreweras complement is equivalent to a certain type of simple-minded collection.

Simple-minded collections were originally used by Rickard [25] in the construction of derived equivalences of symmetric algebras from stable equivalences. A standard example of a simple-minded collection in representation theory is a complete set of non-isomorphic simple Λ-modules regarded as elements of $\mathcal{D}^b(\Lambda)$. Note that any Λ-module X becomes an element of $\mathcal{D}^b(\Lambda)$ by mapping it to the stalk complex concentrated in degree 0 whose degree 0 term is X. Additionally, in [20], simple-minded collections were useful in computing spaces of Bridgeland stability conditions [2].

Here we recall some of the definitions we will need in order to study simple-minded collections. For a more complete presentation of the notions of derived categories and triangulated categories, we refer the reader to Chapter 1 of [19].

Let Λ be a finite dimensional $\Bbbk$-algebra (or, more generally, a ring). By a **complex**, we mean a diagram of finitely generated Λ-modules

$$X \quad = \quad \cdots \xrightarrow{d_X^{-2}} X^{-1} \xrightarrow{d_X^{-1}} X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} X^2 \xrightarrow{d_X^2} \cdots$$

that satisfies $d_X^{i+1} \circ d_X^i = 0$ for each $i \in \mathbb{Z}$. We say that the Λ-module X^i in the complex X is in **degree** i. We refer to the Λ-module homomorphisms $d_X^i : X^i \rightarrow X^{i-1}$ as **differentials**. If the only nonzero module of a complex X is in degree i, we say that X is a **stalk complex** concentrated in degree i. Given a complex X, it is natural to define the **shift** of X, denoted $X[1]$, where

$$X[1] \quad = \quad \cdots \xrightarrow{-d_X^{-1}} X^0 \xrightarrow{-d_X^0} X^1 \xrightarrow{-d_X^1} X^2 \xrightarrow{-d_X^2} X^3 \xrightarrow{-d_X^3} \cdots$$

and where in $X[1]$ the module in degree i is X^{i+1}. Now let $f : X \rightarrow Y$ be a morphism of complexes. We define the **mapping cone** or **cone** of f, denoted $\mathrm{Cone}(f)$, to be the componentwise direct sum of complexes

$$X[1] \oplus Y \quad = \quad \cdots \xrightarrow{d_{\mathrm{Cone}(f)}^{-2}} X^0 \oplus Y^{-1} \xrightarrow{d_{\mathrm{Cone}(f)}^{-1}} X^1 \oplus Y^0 \xrightarrow{d_{\mathrm{Cone}(f)}^{0}} X^2 \oplus Y^1 \xrightarrow{d_{\mathrm{Cone}(f)}^{1}} X^3 \oplus Y^2 \xrightarrow{d_{\mathrm{Cone}(f)}^{2}} \cdots$$

with differential given by

$$d_{\mathrm{Cone}(f)}^i = \begin{bmatrix} -d_X^{i+1} & 0 \\ f^{i+1} & d_Y^i \end{bmatrix}.$$

Dually, one defines the **cocone** of f, denoted $\mathrm{Cocone}(f)$.

The bounded derived category of Λ has objects given by complexes X of Λ-modules with $X^i = 0$ when $|i|$ is sufficiently large. We say that two objects $X, Y \in \mathcal{D}^b(\Lambda)$ are **quasi-isomorphic** if there exists a morphism of complexes $\varphi : X \to Y$ that induces an isomorphism $H^k(X) \to H^k(Y)$ for all k. Two objects $X, Y \in \mathcal{D}^b(\Lambda)$ are isomorphic if and only if there exists a sequence of quasi-isomorphisms

$$X = X_1 \xrightarrow{\varphi_1} X_2 \xleftarrow{\varphi_2} X_3 \xrightarrow{\varphi_3} \cdots \xleftarrow{\varphi_{\ell-1}} X_\ell = Y$$

for some $\ell \geqslant 1$.

The category $\mathcal{D}^b(\Lambda)$, which is a triangulated category, also has the property that any triangle is isomorphic to a triangle of the form

$$X \xrightarrow{f} Y \longrightarrow \mathrm{Cone}(f) \longrightarrow X[1].$$

In addition, one shows that any triangle in $\mathcal{D}^b(\Lambda)$ is isomorphic to one of the form

$$X[-1] \longrightarrow \mathrm{Cocone}(f) \longrightarrow X \xrightarrow{f} Y.$$

Morphism spaces between objects in derived categories can be very complicated. However, the objects in the collections we will study turn out to be stalk complexes. In this situation, the problem of understanding morphisms between such objects in $\mathcal{D}^b(\Lambda)$ is more tractable, as the following well-known proposition shows.

Proposition 8.1. Let $X, Y \in \mathcal{D}^b(\Lambda)$ be stalk complexes concentrated in degree 0. Then

$$\mathrm{Hom}_{\mathcal{D}^b(\Lambda)}(X[i], Y[j]) = \mathrm{Ext}_\Lambda^{j-i}(X, Y).$$

We now give the main definition of this section.

Definition 8.2. Let $\mathcal{C}$ be a triangulated category. A collection $\{X_1, \ldots, X_n\}$ of objects of $\mathcal{C}$ is said to be **simple-minded** if the following hold for any $i, j \in [n]$:

i) $\mathrm{Hom}_\mathcal{C}(X_i, X_j[k]) = 0$ for any $k < 0$,

ii) $\mathrm{Hom}_\mathcal{C}(X_i, X_j) = \begin{cases} \Bbbk & : \text{ if } i = j \\ 0 & : \text{ otherwise,} \end{cases}$

iii) $\mathcal{C} = \mathrm{thick}\langle X_1, \ldots, X_n\rangle$ (i.e., the smallest triangulated category containing $X_1, \ldots, X_n$ and closed under taking summands of objects is $\mathcal{C}$). One says that the objects $\{X_1, \ldots, X_n\}$ form a **thick subcategory** of $\mathcal{C}$.

Now let Λ be a finite dimensional $\Bbbk$-algebra and consider a simple-minded collection $\{X_1, \ldots, X_n\}$ in $\mathcal{D}^b(\Lambda)$. If for each $i \in [n]$ one has $H^k(X_i) = 0$ for any $k \neq 0, -1$, we say the collection is **2-term**. We let 2-smc(Λ) denote the set of isomorphism classes of 2-term simple-minded collections of $\mathcal{D}^b(\Lambda)$.

It turns out that, as the following lemma shows, it is easy to say what objects can appear in a 2-term simple minded collection in $\mathcal{D}^b(\Lambda_T)$.

Lemma 8.3. Let $\mathcal{X} = \{X_1, \ldots, X_n\} \in$ 2-smc(Λ_T). Each $X_i \in \mathcal{X}$ is isomorphic to a stalk complex of an indecomposable Λ_T-module concentrated in degree 0 or -1.

Proof. By [3, Remark 4.11], each $X \in \mathcal{X}$ is isomorphic to a stalk complex of a Λ_T-module concentrated in degree 0 or -1. Suppose $X \in \mathcal{X}$ is of the form $X \cong M[1]$ where $M \in \Lambda_T$-mod. Now we have that

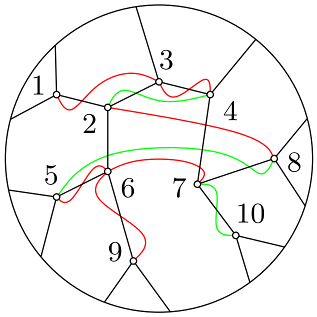
$$\mathrm{End}_{\Lambda_T}(M) = \mathrm{Hom}_{\Lambda_T}(M, M) = \mathrm{Hom}_{\mathcal{D}^b(\Lambda_T)}(M, M) = \mathrm{Hom}_{\mathcal{D}^b(\Lambda_T)}(M[1], M[1]) = \Bbbk$$

where the last equality follows from the fact that $\mathcal{X} \in$ 2-smc(Λ_T). Since $\mathrm{End}_{\Lambda_T}(M)$ is a local ring, M is indecomposable. The proof is similar when $X \cong M$ for some $M \in \Lambda_T$-mod. □

From Lemma 8.3, we have that any 2-term simple-minded collection $\mathcal{X} = \{X_1, \ldots, X_n\}$ in $\mathcal{D}^b(\Lambda_T)$ can be regarded as a collection of segments of T. We define $\mathrm{Seg}(\mathcal{X}) = \{s_1, \ldots, s_n\}$ to be this collection where $s_i \in \mathrm{Seg}(\mathcal{X})$ corresponds to $X_i \in \mathcal{X}$. Moreover, we can write $\mathrm{Seg}(\mathcal{X}) = \mathrm{Seg}^0(\mathcal{X}) \sqcup \mathrm{Seg}^{-1}(\mathcal{X})$ where

$$\mathrm{Seg}^i(\mathcal{X}) := \{s_j \in \mathrm{Seg}(\mathcal{X}) : \ X_j \text{ is concentrated in degree } i\}.$$

The simple-minded collection $\mathcal{X}$ also naturally defines a graph lying on D^2 as follows. Let $\mathcal{SEG}(\mathcal{X})$ be the graph whose vertices are the internal vertices of T and whose edges are admissible curves γ_i defined by the segments $s_i \in \mathrm{Seg}(\mathcal{X})$ up to isotopy fixing the endpoints of γ_i where if $s_i \in \mathrm{Seg}^0(\mathcal{X})$ (resp., $s_i \in \mathrm{Seg}^{-1}(\mathcal{X})$) then γ_i is a green (resp., red) admissible curve. By abuse of notation, we will write $\mathcal{SEG}(\mathcal{X}) = \{\gamma_1, \ldots, \gamma_n\}$. It will also be useful to define $\mathcal{SEG}^0(\mathcal{X})$ (resp., $\mathcal{SEG}^{-1}(\mathcal{X})$) to be the subgraph of $\mathcal{SEG}(\mathcal{X})$ consisting of green (resp., red) admissible curves from $\mathcal{SEG}(\mathcal{X})$.



$$\longleftrightarrow \quad \begin{matrix} \{M(w_{(1,3)})[1], & M(w_{(3,4)})[1], & M(w_{(2,8)})[1], \\ M(w_{(5,6)})[1], & M(w_{(6,7)})[1], & M(w_{(6,9)})[1], \\ M(w_{(2,4)}), & M(w_{(5,8)}), & M(w_{(7,10)})\} \end{matrix}$$

FIGURE 5. The noncrossing tree partition $\mathbf{B} = (\{1,3,4\}, \{2,8\}, \{5,6,7,9\}, \{10\})$ with its Kreweras complement $\mathrm{Kr}(\mathbf{B}) = (\{1\}, \{2,4\}, \{3\}, \{5,8\}, \{6\}, \{7,10\}, \{9\})$ and its corresponding simple-minded collection via the map θ in Theorem 8.4. Here $w_{(i,j)}$ denotes the string corresponding to the segment of T connecting i and j.

Our next theorem gives a combinatorial classification of the 2-term simple-minded collections for the algebras Λ_T. This theorem implies that the data of a noncrossing tree partition paired with its Kreweras complement is equivalent to that of $\mathcal{SEG}(\mathcal{X})$ for a unique $\mathcal{X} \in$ 2-smc(Λ_T).

Theorem 8.4. There is a bijection $\theta : \{(\mathbf{B}, \mathrm{Kr}(\mathbf{B}))\}_{\mathbf{B}\in \mathrm{NCP}(T)} \longrightarrow$ 2-smc(Λ_T) given by

$$(\mathbf{B}, \mathrm{Kr}(\mathbf{B})) \stackrel{\theta}{\longmapsto} \{M(u)[1] : s_u \in \mathrm{Seg}(B) \text{ where } B \in \mathbf{B}\} \sqcup \{M(v) : s_v \in \mathrm{Seg}(B') \text{ where } B' \in \mathrm{Kr}(\mathbf{B})\}.$$

Proof. The image of θ lies in 2-smc(Λ_T) by Lemma 7.2, Lemma 8.10, and Lemma 8.11.

Next, decompose $\mathrm{Seg}^0(\mathcal{X})$ and $\mathrm{Seg}^{-1}(\mathcal{X})$ into segment-connected subsets of maximal size as follows:

$$\mathrm{Seg}^0(\mathcal{X}) = \bigsqcup_{i=1}^{\ell} \mathrm{Seg}^0_i(\mathcal{X}) \quad \text{and} \quad \mathrm{Seg}^{-1}(\mathcal{X}) = \bigsqcup_{i=1}^{k} \mathrm{Seg}^{-1}_i(\mathcal{X}).$$

In Section 8.2, we construct a map $\epsilon :$ 2-smc$(\Lambda_T) \longrightarrow \{(\mathbf{B}, \mathrm{Kr}(\mathbf{B}))\}_{\mathbf{B}\in \mathrm{NCP}(T)}$ defined by

$$\mathcal{X} \stackrel{\epsilon}{\longmapsto} (\mathbf{B}_{\mathcal{X}}, \mathrm{Kr}(\mathbf{B}_{\mathcal{X}}))$$

where $\mathbf{B}_{\mathcal{X}} := \{B_1, \ldots, B_k\}$ and where $B_i := \{$vertices of T that are endpoints of segments in $\mathrm{Seg}^{-1}_i(\mathcal{X})\}$. It follows from Proposition 8.9 that $\mathbf{B}_{\mathcal{X}} \in \mathrm{NCP}(T)$ and that any block B'_i in $\mathrm{Kr}(\mathbf{B}_{\mathcal{X}}) = \{B'_1, \ldots, B'_\ell\}$ satisfies $B'_i = \{$vertices of T that are endpoints of segments in $\mathrm{Seg}^0_i(\mathcal{X})\}$.

The map θ takes a collection of red or green segments to the corresponding stalk complexes, whereas ϵ takes a collection of stalk complexes to the corresponding red or green segments. Hence, to show that they are inverse bijections, it is enough to check that they are well-defined as functions between sets of segments of the form $(\mathbf{B}, \mathrm{Kr}(\mathbf{B}))$ and 2-term simple-minded collections. For ϵ, this is done in Proposition 8.9. □

8.1. **Mutation of simple-minded collections.** Here we recall the notion of **mutation** of simple-minded collections and interpret this as a combinatorial operation on configurations of admissible curves. Our interpretation of mutation will be a key ingredient in the following results. Mutation was first introduced in [21, Section 8.1] for spherical collections and generalized in [20] to Hom-finite, Krull-Schmidt triangulated categories. This notion is defined using the language of **approximations**, which we now briefly review.

Let $\mathcal{C}$ be an arbitrary category (not necessarily triangulated), and let $\mathcal{A}$ be any subcategory of $\mathcal{C}$. We say that a morphism $f : C \to A$ where $C \in \mathcal{C}$ and $A \in \mathcal{A}$ is a **left $\mathcal{A}$-approximation** of C if for any morphism $g : C \to A'$ where $A' \in \mathcal{A}$ one has $g = g'f$ for some morphism $g' : A \to A'$. Dually, one defines the notion of a **right $\mathcal{A}$-approximation** of C. Additionally, we say that $f : C \to A$ where $C \in \mathcal{C}$ and $A \in \mathcal{A}$ is **left minimal** morphism if for every morphism $g : A \to A$ that satisfies $gf = f$ one has that g is an isomorphism. Dually, one defines **right minimal morphisms**. A morphism $f : C \to A$ (resp., $f : A \to C$) is a **left minimal $\mathcal{A}$-approximation** (resp., **right minimal $\mathcal{A}$-approximation**) if f is left minimal and is a left $\mathcal{A}$-approximation (resp., right minimal and is a right $\mathcal{A}$-approximation).

Let $\mathcal{X} = \{X_1, \ldots, X_n\}$ be a simple-minded collection in $\mathcal{D}^b(\Lambda)$ where Λ is any finite dimensional $\Bbbk$-algebra. Let $\mathrm{ext}(X_k)$ denote the **extension closure** of X_k in $\mathcal{D}^b(\Lambda)$ (i.e., the smallest subcategory of $\mathcal{D}^b(\Lambda)$ that contains

X_k and is closed under extensions). We define the **left mutation** of $\mathcal{X}$ to be $\mu_k^+(\mathcal{X}) := \{X_1^+, \ldots, X_n^+\}$ where

$$X_i^+ \quad := \quad \begin{cases} X_k[1] & : \text{ if } i = k \\ \text{Cone}(g_i^+ : X_i[-1] \to X_{k,i}) & : \text{ if } i \neq k \end{cases}$$

where g_i^+ is a left minimal $\text{ext}(X_k)$-approximation. It is known that such approximations exist and that $\mu_k^+(\mathcal{X})$ is a simple-minded collection in $\mathcal{D}^b(\Lambda)$ (see [20, Section 7.2]). Dually, one defines the **right mutation** of $\mathcal{X}$, denoted $\mu_k^-(\mathcal{X})$. The resulting collection $\mu_k^-(\mathcal{X}) := \{X_1^-, \ldots, X_n^-\}$ has objects given by

$$X_i^- \quad := \quad \begin{cases} X_k[-1] & : \text{ if } i = k \\ \text{Cocone}(g_i^- : X_{k,i} \to X_i[1]) & : \text{ if } i \neq k \end{cases}$$

where g_i^- is a right minimal $\text{ext}(X_k)$-approximation. By [20, Proposition 7.6 (a)], we have $\mu_k^-\mu_k^+(\mathcal{X}) = \mathcal{X}$ and $\mu_k^+\mu_k^-(\mathcal{X}) = \mathcal{X}$.

Remark 8.5. Let $\mathcal{X} = \{X_1, \ldots, X_n\} \in$ 2-smc(Λ_T). By Lemma 8.3, we have that $\mu_k^+(\mathcal{X}) \in$ 2-smc(Λ_T) (resp., $\mu_k^-(\mathcal{X}) \in$ 2-smc(Λ_T)) if and only if X_k is a stalk complex of an indecomposable concentrated in degree 0 (resp., -1). Using Proposition 4.5, we have that, when performing the mutation μ_k^+ (resp., μ_k^-) on $\mathcal{X}$, $\text{ext}(X_k) = \text{add}(X_k)$ (resp., $\text{ext}(X_k) = \text{add}(X_k[1])$).

Lemma 8.6. Let $\mathcal{X} = \{X_1, \ldots, X_n\} = \{M(u^{(1)})[1], \ldots, M(u^{(n_1)})[1]\} \sqcup \{M(v^{(1)}), \ldots, M(v^{(n_2)})\} \in$ 2-smc(Λ_T) and let $g_i^+ : X_i[-1] \to X_{k,i}$ and $g_i^- : X_{k,i} \to X_i[1]$ be approximations used in the mutations $\mu_k^+(\mathcal{X})$ and $\mu_k^-(\mathcal{X})$. Then if $X_k = M(v^{(j)})$, we have

$$X_i^+ \quad = \quad \text{Cone}(g_i^+) \quad \cong \quad \begin{cases} M(v^{(j)} \leftarrow v^{(j')}) & : \text{ Ext}^1_{\Lambda_T}(X_i, X_k) \neq 0 \text{ and } X_i = M(v^{(j')}) \text{ where} \\ & \quad \text{supp}(M(v^{(j)})) \cap \text{supp}(M(v^{(j')})) = \varnothing, \\ M(w) & : \text{ Hom}_{\Lambda_T}(X_i[-1], X_k) \neq 0 \text{ and } X_i = M(u^{(j')})[1] \text{ where} \\ & \quad \text{supp}(M(w)) = \text{supp}(M(v^{(j)})) \backslash \text{supp}(M(u^{(j')})) \text{ and} \\ & \quad \text{supp}(M(u^{(j')})) \subseteq \text{supp}(M(v^{(j)})), \\ M(w)[1] & : \text{ Hom}_{\Lambda_T}(X_i[-1], X_k) \neq 0 \text{ and } X_i = M(u^{(j')})[1] \text{ where} \\ & \quad \text{supp}(M(w)) = \text{supp}(M(u^{(j')})) \backslash \text{supp}(M(v^{(j)})) \text{ and} \\ & \quad \text{supp}(M(u^{(j)})) \subseteq \text{supp}(M(v^{(j')})), \\ X_i & : \text{ otherwise.} \end{cases}$$

If $X_k = M(u^{(j)})[1]$, we have

$$X_i^- \quad = \quad \text{Cocone}(g_i^-) \quad \cong \quad \begin{cases} M(u^{(j')} \leftarrow u^{(j)})[1] & : \text{ Ext}^1_{\Lambda_T}(X_k, X_i) \neq 0 \text{ and } X_i = M(u^{(j')})[1] \text{ where} \\ & \quad \text{supp}(M(u^{(j)})) \cap \text{supp}(M(u^{(j')})) = \varnothing, \\ M(w)[1] & : \text{ Hom}_{\Lambda_T}(X_k, X_i[1]) \neq 0 \text{ and } X_i = M(v^{(j')}) \text{ where} \\ & \quad \text{supp}(M(w)) = \text{supp}(M(u^{(j)})) \backslash \text{supp}(M(v^{(j')})) \text{ and} \\ & \quad \text{supp}(M(v^{(j')})) \subseteq \text{supp}(M(u^{(j)})), \\ M(w) & : \text{ Hom}_{\Lambda_T}(X_k, X_i[1]) \neq 0 \text{ and } X_i = M(v^{(j')}) \text{ where} \\ & \quad \text{supp}(M(w)) = \text{supp}(M(v^{(j')})) \backslash \text{supp}(M(u^{(j)})) \text{ and} \\ & \quad \text{supp}(M(u^{(j)})) \subseteq \text{supp}(M(v^{(j')})), \\ X_i & : \text{ otherwise.} \end{cases}$$

Lemma 8.6 shows how mutation of a 2-term simple-minded collection $\mathcal{X}$ of $\mathcal{D}^b(\Lambda_T)$ can be understood combinatorially as an operation on admissible curves in $\mathcal{SEG}(\mathcal{X})$. In Figure 6, we illustrate the possible ways that mutation can effect $\mathcal{SEG}(\mathcal{X})$. Lemma 8.6 also shows that $\mu_k^+(\mathcal{X})$ differs from $\mathcal{X}$ by at most three objects.

Proof of Lemma 8.6. It is easy to see that $X_{k,i}$ is isomorphic to X_k or 0, since g_i^+ is a left minimal $\text{add}(X_k)$-approximation. Note that the map g_i^+ defines the triangle $X_i[-1] \xrightarrow{g_i^+} X_{k,i} \longrightarrow \text{Cone}(g_i^+) \longrightarrow X_i$ in $\mathcal{D}^b(\Lambda_T)$. This triangle gives rise to the long exact sequence

$$0 \longrightarrow H^{-1}(\text{Cone}(g_i^+)) \longrightarrow H^0(X_i[-1]) \xrightarrow{(g_i^+)^*} H^0(X_{k,i}) \longrightarrow H^0(\text{Cone}(g_i^+)) \longrightarrow H^1(X_i[-1]) \longrightarrow 0,$$

which, by Lemma 8.3, vanishes outside of the terms shown. This sequence becomes

$$0 \longrightarrow H^{-1}(\text{Cone}(g_i^+)) \longrightarrow H^{-1}(X_i) \xrightarrow{(g_i^+)^*} H^0(X_{k,i}) \longrightarrow H^0(\text{Cone}(g_i^+)) \longrightarrow H^0(X_i) \longrightarrow 0.$$

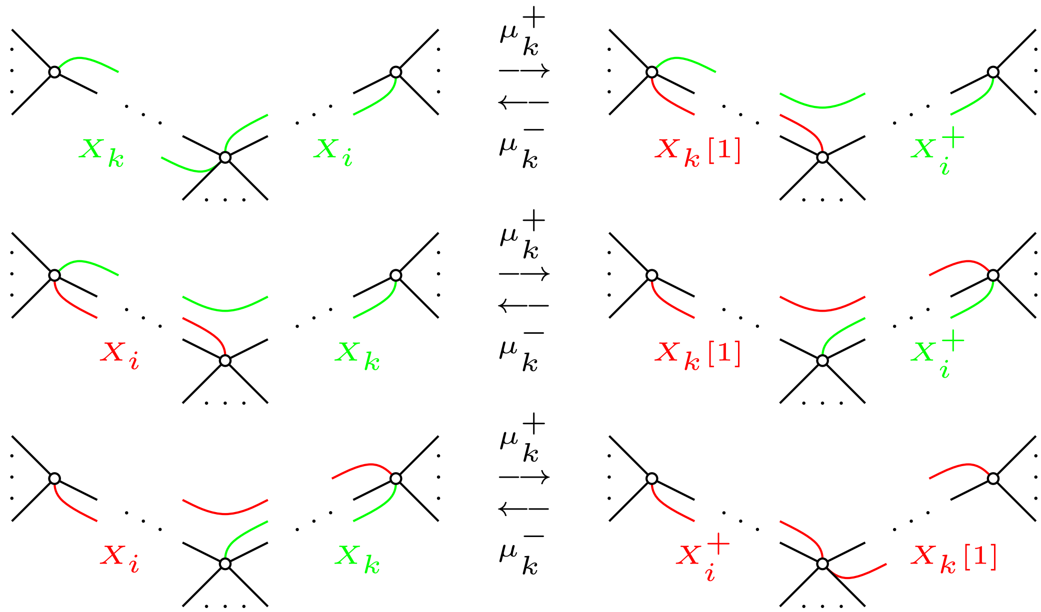


FIGURE 6. The three types of nontrivial transformations.

Now note that since X_i is a stalk complex concentrated in degree 0 or -1, we have the following two cases

$$\mathrm{Hom}_{\mathcal{D}^b(\Lambda_T)}(X_i[-1], X_{k,i}) = \begin{cases} \mathrm{Ext}^1_{\Lambda_T}(X_i, X_{k,i}) & : \text{ if } H^0(X_i) = X_i, \\ \mathrm{Hom}_{\Lambda_T}(X_i[-1], X_{k,i}) & : \text{ if } H^{-1}(X_i) = X_i. \end{cases}$$

We first consider the case when $H^0(X_i) = X_i$. By Lemma 5.5, $\dim_{\Bbbk}\mathrm{Ext}^1_{\Lambda_T}(X_i, X_{k,i}) \leqslant 1$. Suppose that $\dim_{\Bbbk}\mathrm{Ext}^1_{\Lambda_T}(X_i, X_{k,i}) = 0$. This means that $g_i^+ : X_i[-1] \to X_{k,i}$ is the zero map. Since g_i^+ is a left minimal morphism, this implies that $X_{k,i} = 0$. Then the long exact sequence implies that $H^{-1}(\mathrm{Cone}(g_i^+)) \cong H^{-1}(X_i) = 0$ and $H^0(\mathrm{Cone}(g_i^+)) \cong H^0(X_i) = X_i$. Thus $\mathrm{Cone}(g_i^+) \cong X_i$.

Next, suppose that $\dim_{\Bbbk}\mathrm{Ext}^1_{\Lambda_T}(X_i, X_{k,i}) = 1$. Since g_i^+ is a left minimal morphism, we know that g_i^+ is nonzero and thus $X_{k,i} = X_k$. Assume X_i is concentrated in degree 0 and write $X_k = M(v^{(j)})$, $X_i = M(v^{(j')})$. Since $\mathcal{X}$ is a simple-minded collection, $\mathrm{Hom}_{\Lambda_T}(M(v^{(j)}), M(v^{(j')})) = 0$ and $\mathrm{Hom}_{\Lambda_T}(M(v^{(j')}), M(v^{(j)})) = 0$. Thus Theorems 4.6 and 4.7 imply that $0 \to M(v^{(j)}) \to M(v^{(j)} \leftarrow v^{(j')}) \to M(v^{(j')}) \to 0$ is the unique nonsplit extension of $M(v^{(j')})$ by $M(v^{(j)})$ up to equivalence of extensions.

Let $M(v^{(j)}) \longrightarrow M(v^{(j)} \leftarrow v^{(j')}) \longrightarrow M(v^{(j')}) \xrightarrow{\xi} M(v^{(j)})[1]$ be the triangle in $\mathcal{D}^b(\Lambda_T)$ defined by this nonsplit extension where ξ is the class of this extension in $\mathrm{Ext}^1_{\Lambda_T}(M(v^{(j')}), M(v^{(j)}))$. As $\dim_{\Bbbk}\mathrm{Ext}^1_{\Lambda_T}(M(v^{(j')}), M(v^{(j)})) = 1$, we know that $\xi \neq 0$. Furthermore, we have that $g_i^+ = c \cdot \xi$ for some $c \in \Bbbk\backslash\{0\}$. Thus we have the following isomorphism of triangles in $\mathcal{D}^b(\Lambda_T)$

$$\begin{array}{ccccccc} M(v^{(j')})[-1] & \xrightarrow{-\xi} & M(v^{(j)}) & \longrightarrow & M(v^{(j)} \leftarrow v^{(j')}) & \longrightarrow & M(v^{(j')}) \\ \| & & \downarrow{\scriptstyle (-c)\cdot 1} & & \downarrow{\scriptstyle \cong} & & \| \\ M(v^{(j')})[-1] & \xrightarrow{c\cdot\xi} & M(v^{(j)}) & \longrightarrow & \mathrm{Cone}(c\cdot\xi) & \longrightarrow & M(v^{(j')}) \end{array}$$

This implies that $\mathrm{Cone}(g_i^+) \cong M(v^{(j)} \leftarrow v^{(j')})$.

Next, we consider the case when $H^{-1}(X_i) = X_i$. By Lemma 5.3, $\dim_{\Bbbk}\mathrm{Hom}_{\Lambda_T}(X_i[-1], X_{k,i}) \leqslant 1$. Suppose that $\dim_{\Bbbk}\mathrm{Hom}_{\Lambda_T}(X_i[-1], X_{k,i}) = 0$. This means that $g_i^+ : X_i[-1] \to X_{k,i}$ is the zero map. Since g_i^+ is a left minimal morphism, this implies that $X_{k,i} = 0$. Then the long exact sequence implies that $H^{-1}(\mathrm{Cone}(g_i^+)) \cong H^{-1}(X_i) = X_i$ and $H^0(\mathrm{Cone}(g_i^+)) \cong H^0(X_i) = 0$. Thus $\mathrm{Cone}(g_i^+) \cong X_i$.

Now suppose that $\dim_{\Bbbk}\mathrm{Hom}_{\Lambda_T}(X_i[-1], X_{k,i}) = 1$. Since g_i^+ is a left minimal morphism, we know that g_i^+ is nonzero and thus $X_{k,i} = X_k$. Thus if we write $X_i[-1] = M(u^{(j')})$ and $X_k = M(v^{(j)})$, we have that

$\text{supp}(M(u^{(j')})) \cap \text{supp}(M(v^{(j)})) \neq \varnothing$. Furthermore, since $\mathcal{X}$ is a simple-minded collection, we have that

$$\text{Ext}^1_{\Lambda_T}(M(u^{(j')}), M(v^{(j)})) = \text{Ext}^1_{\Lambda_T}(X_i[-1], X_k) = \text{Hom}_{\mathcal{D}^b(\Lambda_T)}(X_i, X_k) = 0.$$

Thus Theorem 4.7 implies that the segments $s_{u^{(j)}}$ and $s_{v^{(j')}}$ must share an endpoint. As $\text{Hom}_{\Lambda_T}(M(u^{(j')}), M(v^{(j)})) \neq 0$, the two segments must agree along a segment.

We know from Lemma 8.3 that $\text{Cone}(g_i^+)$ must be isomorphic in $\mathcal{D}^b(\Lambda_T)$ to either $M(w)$ or $M(w)[1]$ for some $M(w) \in \text{ind}(\Lambda_T\text{-mod})$ in order to have $\mu_k^+(\mathcal{X}) \in 2\text{-smc}(\Lambda_T)$. This implies that either $\ker((g_i^+)^*) = 0$ or $\text{coker}((g_i^+)^*) = 0$. In the former case $\text{Cone}(g_i^+) \cong M(w)$ where $\text{supp}(M(w)) = \text{supp}(M(v^{(j)}))\backslash\text{supp}(M(u^{(j')}))$. In the latter case $\text{Cone}(g_i^+) \cong M(w)[1]$ where $\text{supp}(M(w)) = \text{supp}(M(u^{(j')}))\backslash\text{supp}(M(v^{(j)}))$.

The computation of $\text{Cocone}(g_i^-)$ is similar so we omit it. □

8.2. **From simple-minded collections to noncrossing tree partitions.** In this section, we show how any 2-term simple-minded collection gives rise to a noncrossing tree partition paired with its Kreweras complement.

Using left mutation, we can endow $2\text{-smc}(\Lambda_T)$ with a poset structure by regarding it as the transitive closure of the relation $\mathcal{X}_1 \lessdot \mathcal{X}_2$ if and only if $\mathcal{X}_2 = \mu_k^+(\mathcal{X}_1)$ for some $k \in [n]$. Perhaps surprisingly, this poset can be understood more globally. In [20, Proposition 7.9] it is shown that the partial order on $(2\text{-smc}(\Lambda_T), <)$ can be described as follows. If $\mathcal{X}_1 = \{X_1^{(1)}, \ldots, X_n^{(1)}\}, \mathcal{X}_2 = \{X_1^{(2)}, \ldots, X_n^{(2)}\} \in 2\text{-smc}(\Lambda_T)$, then

$$\mathcal{X}_1 \leqslant \mathcal{X}_2 \text{ if and only if } \text{Hom}_{\mathcal{D}^b(\Lambda_T)}(X_i^{(1)}, X_j^{(2)}[m]) = 0$$

for any $m < 0$ and any $i, j \in [n]$. The next proposition shows that the poset $(2\text{-smc}(\Lambda_T), <)$ has an even richer structure.

Proposition 8.7. The poset $(2\text{-smc}(\Lambda_T), <)$ is a finite lattice whose unique minimal (resp., maximal) element is $\{M(i) : i \in (Q_T)_0\}$ (resp., $\{M(i)[1] : i \in (Q_T)_0\}$).

Proof. We will show that $(2\text{-smc}(\Lambda_T), <)$ is isomorphic to the lattice of torsion-free classes $\text{torsf}(\Lambda_T)$. The lattice $\text{torsf}(\Lambda_T)$ is finite since Λ_T is representation-finite.

By [17, Theorem 3.1] and [29, Proposition 2.3], the poset $\text{torsf}(\Lambda_T)$ is isomorphic to the poset of **bounded t-structures** $(\mathcal{C}_1^{\leqslant 0}, \mathcal{C}_1^{\geqslant 0})$ on $\mathcal{D}^b(\Lambda_T)$ that satisfy $\mathcal{C}^{\leqslant 0}[1] \subseteq \mathcal{C}_1^{\leqslant 0} \subseteq \mathcal{C}^{\leqslant 0}$ or equivalently, $\mathcal{C}^{\geqslant 0}[1] \subseteq \mathcal{C}_1^{\geqslant 0} \subseteq \mathcal{C}^{\geqslant 0}$ where

$$\mathcal{C}^{\leqslant 0} := \{X \in \mathcal{D}^b(\Lambda_T) : \ H^i(X) = 0 \text{ for } i > 0\} \quad \text{and} \quad \mathcal{C}^{\geqslant 0} := \{X \in \mathcal{D}^b(\Lambda_T) : \ H^i(X) = 0 \text{ for } i \leqslant -1\}.$$

In the latter poset, bounded t-structures are partially ordered by inclusion:

$$(\mathcal{C}_1^{\leqslant 0}, \mathcal{C}_1^{\geqslant 0}) \leqslant (\mathcal{C}_2^{\leqslant 0}, \mathcal{C}_2^{\geqslant 0}) \text{ if and only if } \mathcal{C}_1^{\geqslant 0} \subseteq \mathcal{C}_2^{\geqslant 0}, \text{ or equivalently, } \mathcal{C}_1^{\leqslant 0} \supset \mathcal{C}_2^{\leqslant 0}$$

The isomorphism sends a torsion-free class $\mathcal{F}$ and its corresponding torsion class $\mathcal{T}$ to the bounded t-structure $(\mathcal{C}'^{\leqslant 0}, \mathcal{C}'^{\geqslant 0})$ where

$$\mathcal{C}_1^{\leqslant 0} := \{X \in \mathcal{D}^b(\Lambda_T) : \ H^i(X) = 0 \text{ for } i > 0, \ H^0(X) \in \mathcal{T}\}$$

and

$$\mathcal{C}_1^{\geqslant 0} := \{X \in \mathcal{D}^b(\Lambda_T) : \ H^i(X) = 0 \text{ for } i < -1, \ H^{-1}(X) \in \mathcal{F}\}.$$

Now, by [3, Corollary 4.3] and the remarks following its proof, this poset of bounded t-structures is isomorphic to $(2\text{-smc}(\Lambda_T), <)$.

Remark 8.5 shows that the unique minimal (resp., maximal) element of $(2\text{-smc}(\Lambda_T), <)$ is $\{M(i) : i \in (Q_T)_0\}$ (resp., $\{M(i)[1] : i \in (Q_T)_0\}$). □

Proposition 8.8. Let $\mathcal{X} \in 2\text{-smc}(\Lambda_T)$. The graph $\mathcal{SEG}(\mathcal{X})$ is a **noncrossing** tree (i.e., any two admissible curves in $\mathcal{SEG}(\mathcal{X})$ are noncrossing in the sense of Lemma 3.1).

Proof. It is clear that $\mathcal{SEG}(\{M(i) : i \in (Q_T)_0\})$ is a noncrossing tree. By Proposition 8.7, for any $\mathcal{X} \in 2\text{-smc}(\Lambda_T)$ there exists a sequence of left mutations such that $\mathcal{X} = \mu_{i_k}^+ \circ \cdots \circ \mu_{i_1}^+(\{M(i) : i \in (Q_T)_0\})$. By Lemma 8.6, we have that if $\mathcal{X}_2 = \mu_k^+(\mathcal{X}_1)$ and $\mathcal{SEG}(\mathcal{X}_1)$ is a tree, then $\mathcal{SEG}(\mathcal{X}_2)$ is a tree. Thus $\mathcal{SEG}(\mathcal{X})$ is a tree.

It remains to prove that if $\mathcal{X}_2 = \mu_k^+(\mathcal{X}_1)$ where $\mathcal{X}_1, \mathcal{X}_2 \in 2\text{-smc}(\Lambda_T)$ and $\mathcal{SEG}(\mathcal{X}_1) = \{\gamma_1, \ldots, \gamma_n\}$ is noncrossing, then $\mathcal{SEG}(\mathcal{X}_2) = \{\gamma_1^+, \ldots, \gamma_n^+\}$ is noncrossing. It is clear that the admissible curves in $\mathcal{SEG}(\mathcal{X}_1)\backslash\mathcal{SEG}(\mathcal{X}_2)$ are noncrossing. Write $\mathcal{X}_1 = \{X_1, \ldots, X_n\}$, $\text{Seg}(\mathcal{X}_1) = \{s_1, \ldots, s_n\}$, and $\text{Seg}(\mathcal{X}_2) = \{s_1^+, \ldots, s_n^+\}$. Without loss of generality, we can assume $k = 1$ and then $\mathcal{X}_2 = \{X_1[1], X_2^+, \ldots, X_n^+\}$. By Lemma 8.6, $\mathcal{X}_2$ differs from $\mathcal{X}_1$ in at most three objects. This implies that, without loss of generality, $X_i^+ = X_i$ if $i \notin \{1, 2, 3\}$. Furthermore, the description of mutation in Lemma 8.6 shows that the admissible curves in $\mathcal{SEG}(\mathcal{X}_2)\backslash\mathcal{SEG}(\mathcal{X}_1)$ are noncrossing. Thus it suffices to show any admissible curve from $\mathcal{SEG}(\mathcal{X}_2)\backslash\mathcal{SEG}(\mathcal{X}_1)$ and any admissible curve from $\mathcal{SEG}(\mathcal{X}_1)\backslash\mathcal{SEG}(\mathcal{X}_2)$ are noncrossing. Note that from our interpretation of mutation in terms of admissible curves (see Figure 6), we see that there is no curve in $\mathcal{SEG}(\mathcal{X}_2)\backslash\mathcal{SEG}(\mathcal{X}_1)$ that crosses one from $\mathcal{SEG}(\mathcal{X}_1)\backslash\mathcal{SEG}(\mathcal{X}_2)$ in the sense that the two have a common endpoint $z(w,F)$ for some corner (w,F) of T.

Next, we show that if $\gamma_\ell^+ \in \mathcal{SEG}(\mathcal{X}_2)\backslash\mathcal{SEG}(\mathcal{X}_1)$, then γ_ℓ^+ and any $\gamma_i^+ = \gamma_i \in \mathcal{SEG}(\mathcal{X}_1)\backslash\mathcal{SEG}(\mathcal{X}_2)$ are noncrossing. Write $s_\ell^+ = s_{w^{(\ell,+)}}$ and $s_i = s_{w^{(i)}}$ for some strings $w^{(\ell,+)}$ and $w^{(i)}$ in Λ_T. Let $s_w = [a,b]$ be the unique maximal string along which $s_{w^{(\ell,+)}}$ and $s_{w^{(i)}}$ agree and orient γ_ℓ^+ and γ_i from a to b.

Assume $s_{w^{(\ell,+)}}$ and $s_{w^{(i)}}$ share an endpoint and that a is the shared endpoint. In this situation, one of γ_ℓ^+ and γ_i is red admissible and the other is green admissible. We assume γ_ℓ^+ is green admissible and γ_i is red admissible, and the following argument can be adapted to the case where γ_ℓ^+ is red admissible and γ_i is green admissible. Since $\mathcal{X}_2$ is a simple-minded collection, Definition 8.2 i) implies that

$$\mathrm{Hom}_{\Lambda_T}(M(w^{(\ell,+)}), M(w^{(i)})) = \mathrm{Hom}_{\mathcal{D}^b(\Lambda_T)}(M(w^{(\ell,+)}), M(w^{(i)})[1][-1]) = 0$$

and so by Lemma 5.4, there is a nonzero morphism $f : M(w^{(i)}) \to M(w^{(\ell,+)})$. Thus $w^{(i)} = w \to u^{(i)}$ and $w^{(\ell,+)} = w \leftarrow u^{(\ell)}$ for some strings $u^{(i)}$ and $u^{(\ell)}$ in Λ_T, one of which may be empty. This implies that γ_ℓ^+ turns left at b or γ_i turns right at b. By Lemma 3.1 (c) (with γ_ℓ^+ playing the role of γ), we have that γ_ℓ^+ and γ_i are noncrossing.

Now suppose that $s_{w^{(\ell,+)}}$ and $s_{w^{(i)}}$ do not share an endpoint. Assume that γ_ℓ^+ is green admissible and γ_i is red admissible. The following argument can be adapted to the case when γ_ℓ^+ is red admissible and γ_i is green admissible. Then since $\mathcal{X}_2$ is a simple-minded collection, Definition 8.2 ii) implies that

$$\mathrm{Ext}^1_{\Lambda_T}(M(w^{(\ell,+)}), M(w^{(i)})) = \mathrm{Hom}_{\mathcal{D}^b(\Lambda_T)}(M(w^{(\ell,+)}), M(w^{(i)})[1]) = 0.$$

By Theorem 4.7 and the structure of Q_T, we have that one the following holds:

a) $w^{(i)} = u^{(i,1)} \leftarrow w \leftarrow u^{(i,2)}$ and $w^{(\ell,+)} = v^{(\ell,1)} \xrightarrow{\alpha} w \xrightarrow{\alpha} v^{(\ell,2)}$ where $u^{(i,1)}$ and $u^{(i,2)}$ are nonempty strings and $v^{(\ell,1)}$ and $v^{(\ell,2)}$ may be empty strings,

b) $w^{(i)} = u^{(i,1)} \leftarrow w$ and $w^{(\ell,+)} = v^{(\ell,1)} \xrightarrow{\alpha} w \xrightarrow{\beta} v^{(\ell,2)}$ where $u^{(i,1)}$ and $v^{(\ell,2)}$ are nonempty strings and $v^{(\ell,1)}$ may be an empty string,

c) $w^{(i)} = w \leftarrow u^{(i,2)}$ and $w^{(\ell,+)} = v^{(\ell,1)} \xrightarrow{\beta} w \xrightarrow{\alpha} v^{(\ell,2)}$ where $u^{(i,1)}$ and $v^{(\ell,1)}$ are nonempty strings and $v^{(\ell,2)}$ may be an empty string,

a′) $w^{(i)} = u^{(i,1)} \to w \to u^{(i,2)}$ and $w^{(\ell,+)} = v^{(\ell,1)} \xleftarrow{\alpha} w \xleftarrow{\alpha} v^{(\ell,2)}$ where $u^{(i,1)}$ and $u^{(i,2)}$ are nonempty strings and $v^{(\ell,1)}$ and $v^{(\ell,2)}$ may be empty strings,

b′) $w^{(i)} = u^{(i,1)} \to w$ and $w^{(\ell,+)} = v^{(\ell,1)} \xleftarrow{\alpha} w \xleftarrow{\beta} v^{(\ell,2)}$ where $u^{(i,1)}$ and $v^{(\ell,2)}$ are nonempty strings and $v^{(\ell,1)}$ may be an empty string, or

c′) $w^{(i)} = w \to u^{(i,2)}$ and $w^{(\ell,+)} = v^{(\ell,1)} \xleftarrow{\beta} w \xleftarrow{\alpha} v^{(\ell,2)}$ where $u^{(i,2)}$ and $v^{(\ell,1)}$ are nonempty strings and $v^{(\ell,2)}$ are may be an empty string.

Here the orientation of the arrows labeled β is determined by the fact that $\mathrm{Ext}^1_{\Lambda_T}(M(w^{(\ell,+)}), M(w^{(i)})) = 0$, while the orientation of the arrows labeled α is determined by the structure of Q_T. Note that we cannot have $w^{(i)} = u^{(i,1)} \to w \leftarrow u^{(i,2)}$ for some nonempty strings $u^{(i,1)}$ and $u^{(i,2)}$, otherwise the structure of Q_T implies that

$$\mathrm{Hom}_{\mathcal{D}^b(\Lambda_T)}(M(w^{(\ell,+)}), M(w^{(i)})[1][-1]) = \mathrm{Hom}_{\Lambda_T}(M(w^{(\ell,+)}), M(w^{(i)})) \neq 0,$$

and this contradicts that $\mathcal{X}_2$ is a simple-minded collection. Using Lemma 3.1, it is straightforward to verify that in each of these cases the admissible curves γ_2^+ and γ_i are noncrossing.

Finally, assume $s_{w^{(2,+)}}$ and $s_{w^{(i)}}$ do not share an endpoint and γ_2^+ and γ_i are of the same color. As $\mathcal{X}_2$ is a simple-minded collection, we know that $\mathrm{Hom}_{\Lambda_T}(M(w^{(i)}), M(w^{(2,+)})) = 0$. Thus $\mathrm{Ext}^1_{\Lambda_T}(M(w^{(2,+)}), M(w^{(i)})) = 0$ by Theorem 4.7. We obtain the same family of cases as in the previous paragraph and, as above, it is routine to verify from these that γ_2^+ and γ_i are noncrossing. □

Proposition 8.9. Let $\mathcal{X} \in$ 2-smc(Λ_T). There exists $\mathbf{B}_\mathcal{X} = \{B_1, \ldots, B_k\} \in \mathrm{NCP}(T)$ with Kreweras complement $\mathrm{Kr}(\mathbf{B}_\mathcal{X}) = \{B'_1, \ldots, B'_\ell\}$ such that

i) $\mathrm{Seg}^{-1}(\mathcal{X}) = \bigsqcup_{i=1}^k \mathrm{Seg}(B_i)$

ii) $\mathrm{Seg}^0(\mathcal{X}) = \bigsqcup_{i=1}^\ell \mathrm{Seg}(B'_i)$.

Proof. i) Write $\mathcal{SEG}^{-1}(\mathcal{X}) = \bigsqcup_{i=1}^k \mathcal{SEG}_i^{-1}(\mathcal{X})$ where each $\mathcal{SEG}_i^{-1}(\mathcal{X})$ is a connected component of $\mathcal{SEG}^{-1}(\mathcal{X})$. Also, let $\mathrm{Seg}_i^{-1}(\mathcal{X})$ denote the set of segments defined by $\mathcal{SEG}_i^{-1}(\mathcal{X})$.

We claim that any two segments in $\mathrm{Seg}_i^{-1}(\mathcal{X})$ either have no common vertices or they agree only at an endpoint of each. Since $\mathrm{Hom}_{\mathcal{D}^b(\Lambda_T)}(X_s, X_t) = 0$ for any objects in $\mathcal{X}$ and since any $\mathcal{SEG}_i^{-1}(\mathcal{X})$ is connected, Lemma 5.4 implies that there are no segments in $\mathrm{Seg}_i^{-1}(\mathcal{X})$ that share an endpoint and agree along a segment.

Suppose that $s_1, s_2 \in \mathrm{Seg}_i^{-1}(\mathcal{X})$ agree along a segment, but have no common endpoints. Let γ_1 and γ_2 be the edges of $\mathcal{SEG}_i^{-1}(\mathcal{X})$ whose segments are s_1 and s_2, respectively. Since $\mathcal{SEG}_i^{-1}(\mathcal{X})$ is a tree, let $(\gamma^{(1)}, \ldots, \gamma^{(r)})$ with $\gamma^{(j)} \in \mathcal{SEG}_i^{-1}(\mathcal{X})$ denote the unique sequence of edges connecting an endpoint of s_1 to an endpoint of s_2. Let $(s^{(1)}, \ldots, s^{(r)})$ with $s^{(j)} \in \mathrm{Seg}_i^{-1}(\mathcal{X})$ denote the sequence of segments defined by $(\gamma^{(1)}, \ldots, \gamma^{(r)})$. We assume $s^{(1)}$ (resp., $s^{(r)}$) agrees with s_1 (resp., s_2) at an endpoint, and, by the previous paragraph, we can assume that $s^{(j)}$ and $s^{(j+1)}$ agree only at endpoints for each j. Now from the structure of T, we have that $s^{(1)}$ agrees with s_1 along a segment or $s^{(r)}$ agrees with s_2 along a segment. In either situation we reach a contradiction.

We now have that each $\mathrm{Seg}_i^{-1}(\mathcal{X})$ is an inclusion-minimal set of segments. Since $\mathcal{SEG}_i^{-1}(\mathcal{X})$ is a connected component of $\mathcal{SEG}^{-1}(\mathcal{X})$, we observe that $\mathrm{Seg}_i^{-1}(\mathcal{X})$ is segment-connected. Thus for each $i \in [k]$, we define

$$B_i := \{v \in T : \ v \text{ is an endpoint of some segment in } \mathrm{Seg}_i^{-1}(\mathcal{X})\},$$

and we obtain that $\mathrm{Seg}_i^{-1}(\mathcal{X}) = \mathrm{Seg}(B_i)$. By Proposition 8.8, this implies that $\mathbf{B}_{\mathcal{X}} := \{B_1, \ldots, B_k\} \in \mathrm{NCP}(T)$.

The proof of ii) is similar so we omit it. We remark that the noncrossing tree partition corresponding to $\mathcal{SEG}^0(\mathcal{X}) = \bigsqcup_{i=1}^{\ell} \mathcal{SEG}_i^0(\mathcal{X})$ is defined as $\mathbf{B}' := \{B_1', \ldots, B_\ell'\}$ where

$$B_i' := \{v \in T : \ v \text{ is an endpoint of some segment in } \mathrm{Seg}_i^0(\mathcal{X})\}.$$

Lastly, we know that $\mathcal{SEG}(\mathcal{X})$ is a noncrossing tree by Proposition 8.8. Furthermore, we have that the green segments in $\mathrm{Seg}^{-1}(\mathcal{X}) = \bigsqcup_{i=1}^{k} \mathrm{Seg}(B_i)$ and the red segments in $\mathrm{Seg}^0(\mathcal{X}) = \bigsqcup_{i=1}^{\ell} \mathrm{Seg}(B_i')$ define a red-green tree. Thus [15, Corollary 5.12] implies that $\mathbf{B}' = \mathrm{Kr}(\mathbf{B}_{\mathcal{X}})$. ☐

8.3. **From noncrossing tree partitions to simple-minded collections.** In this section, we present the last two lemmas needed to show that the image of the map θ, as defined in Theorem 8.4, lies in 2-smc(Λ_T).

Lemma 8.10. Let $(\mathbf{B}, \mathrm{Kr}(\mathbf{B})) \in \mathrm{NCP}(T)^2$ and let $M(u)$ (resp., $M(v)$) be an indecomposable Λ_T-module whose corresponding segment appears in $\mathrm{Seg}(B)$ for some block B of $\mathbf{B}$ (resp., of $\mathrm{Kr}(\mathbf{B})$). Then

(1) $\mathrm{Hom}_{\mathcal{D}^b(\Lambda_T)}(M(u)[1], M(v)[k]) = 0$ for any $k \leqslant 0$,
(2) $\mathrm{Hom}_{\mathcal{D}^b(\Lambda_T)}(M(v), M(u)[1][k]) = 0$ for any $k \leqslant 0$.

Proof. For each part, we assume that $\mathbf{B}$ is not the top or bottom element of $\mathrm{NCP}(T)$, otherwise the statements hold vacuously. In each part, whenever we assume that $s_u = [y_1, y_2]$ and $s_v = [x_1, x_2]$ agree along a segment, we let $s_w = [a, b]$ denote the unique maximal segment along which they agree. Furthermore, we let γ_u and γ_v be admissible curves for s_u and s_v, respectively, that witness the fact that $s_u \in \mathrm{Seg}(B)$ for some block B of $\mathbf{B}$ and $s_v \in \mathrm{Seg}(B')$ for some block B' of $\mathrm{Kr}(\mathbf{B})$, and orient this curves from a to b.

(1) We have that $\mathrm{Hom}_{\mathcal{D}^b(\Lambda_T)}(M(u)[1], M(v)[k]) = \mathrm{Ext}^{k-1}_{\Lambda_T}(M(u), M(v)) = 0$, since $k - 1 \leqslant -1$.

(2) Since $\mathrm{Hom}_{\mathcal{D}^b(\Lambda_T)}(M(v), M(u)[1][k]) = \mathrm{Ext}^{k+1}_{\Lambda_T}(M(v), M(u)) = 0$ for $k \leqslant -2$, it is enough to show that $\mathrm{Hom}_{\Lambda_T}(M(v), M(u)) = 0$ and $\mathrm{Ext}^1_{\Lambda_T}(M(v), M(u)) = 0$.

We first show that $\mathrm{Hom}_{\Lambda_T}(M(v), M(u)) = 0$. Suppose that s_v and s_u have no common endpoints. We claim that $\nu := \{\{x_1, x_2\}, \{y_1, y_2\}, \{i\} : \ i \in V^o \backslash \{x_1, x_2, y_1, y_2\}\}$ is a noncrossing tree partition. Since γ_v and γ_u do not cross and since s_v and s_u have no common endpoints, we can replace γ_v with a red admissible curve γ_v' representing s_v that does not cross γ_u. Thus $\nu \in \mathrm{NCP}(T)$. Now by Lemma 7.2, we have that $\mathrm{Hom}_{\Lambda_T}(M(v), M(u)) = 0$.

Now suppose the segments s_u and s_v share an endpoint. Since $s_u \in \mathrm{Seg}(B)$ for some block B of $\mathbf{B}$ and $s_v \in \mathrm{Seg}(B')$ for some block B' of $\mathrm{Kr}(\mathbf{B})$, they are distinct and thus share exactly one endpoint. We can assume that s_u and s_v agree along some segment, otherwise we are done. Since s_u and s_v agree along s_w, we must have that $v = v' \leftrightarrow w$ and $u = u' \leftrightarrow w$ for some strings u' and v' in Λ_T, at least one of which is nonempty. Assume a is the common endpoint of s_u and s_v. By Lemma 3.1 (3), with $s_w = [a, b]$ playing the role of t and γ_v playing the role of γ, we have that γ_v either turns left at b or γ_u turns right at b. Thus either $v = v' \to w$ and $u = w$ or $v = w$ or $u \leftarrow w$. This implies that $\mathrm{Hom}_{\Lambda_T}(M(v), M(u)) = 0$.

Lastly, we show that $\mathrm{Ext}^1_{\Lambda_T}(M(v), M(u)) = 0$. By Proposition 4.4, we can restrict to the situation where s_u and s_v have at least one common vertex of T. By Proposition 4.5, we can assume that if s_u and s_v have only one vertex in common, then that vertex is an endpoint of each.

Assume s_u and s_v agree only at an endpoint. By Lemma 4.6, $\mathrm{Ext}^1_{\Lambda_T}(M(v), M(u)) \neq 0$ if and only if there exists an arrow $\alpha \in (Q_T)_1$ such that the string $(u \longleftrightarrow v) = (u \overset{\alpha}{\longleftarrow} v)$. Since $s_u \in B \in \mathbf{B}$ and $s_v \in B' \in \mathrm{Kr}(\mathbf{B})$, any admissible curve γ_u (resp., γ_v) leaves its endpoints from their right (resp., left). Thus the existence of such an arrow $\alpha \in (Q_T)_1$ implies that γ_u and γ_v leave their common endpoint from a common corner of T, and such a configuration is not allowed.

Now assume s_u and s_v agree along a segment, but they have no common endpoints. Now we can write $u = u^{(1)} \leftrightarrow w \leftrightarrow u^{(2)}$ and $v = v^{(1)} \leftrightarrow w \leftrightarrow v^{(2)}$ for some strings $u^{(1)}, u^{(2)}, v^{(1)}$, and $v^{(2)}$ in Λ_T where

i) $u^{(1)}$ and $u^{(2)}$ are nonempty or
ii) $v^{(1)}$ and $v^{(2)}$ are nonempty or
iii) $u^{(1)}$ and $v^{(2)}$ are nonempty and $u^{(2)}$ and $v^{(1)}$ are empty or
iv) $v^{(1)}$ and $u^{(2)}$ are nonempty and $u^{(1)}$ and $v^{(2)}$ are empty.

Suppose we are in case i). Since s_u and s_v are noncrossing and since $u^{(1)}$ and $u^{(2)}$ are nonempty, we have from Lemma 3.1 (1) (with s_w playing the role of t) that γ_u either turns left at a and b or turns right at a and b. Thus $u = u^{(1)} \leftarrow w \leftarrow u^{(2)}$ or $u = u^{(1)} \rightarrow w \rightarrow u^{(2)}$. By Theorem 4.7, we have that $\mathrm{Ext}^1_{\Lambda_T}(M(v), M(u)) = 0$. In case ii), the analogous arguments shows that $v = v^{(1)} \leftarrow w \leftarrow v^{(2)}$ or $v = v^{(1)} \rightarrow w \rightarrow v^{(2)}$. Thus Theorem 4.7 implies that $\mathrm{Ext}^1_{\Lambda_T}(M(v), M(u)) = 0$

Suppose we are in case iii). We have from Lemma 3.1 (2) (with s_w playing the role of t, γ_v playing the role of γ, and γ_u playing the role of γ') that either γ_v turns left at b and γ_u turns right at a or γ_v turns right at b and γ_u turns left at a. This implies that either $u = u^{(1)} \leftarrow w$ and $v = w \rightarrow v^{(2)}$ or $u = u^{(1)} \rightarrow w$ and $v = w \leftarrow v^{(2)}$. By Theorem 4.7, we have that $\mathrm{Ext}^1_{\Lambda_T}(M(v), M(u)) = 0$. The analogous argument can be used in case iv). □

Lemma 8.11. Let $(\mathbf{B}, \mathrm{Kr}(\mathbf{B})) \in \mathrm{NCP}(T)^2$. Then the objects

$$\{M(u)[1] : s_u \in \mathrm{Seg}(B) \text{ where } B \in \mathbf{B}\} \sqcup \{M(v) : s_v \in \mathrm{Seg}(B') \text{ where } B' \in \mathrm{Kr}(\mathbf{B})\} \subseteq \mathcal{D}^b(\Lambda_T)$$

form a thick subcategory of $\mathcal{D}^b(\Lambda_T)$.

Proof. Let $\mathscr{T}$ denote the smallest triangulated category that contains the objects in the statement of the lemma and that is closed under taking summands of its objects. Note that $M(u) \in \mathscr{T}$ for each $s_u \in \mathrm{Seg}(B)$ where $B \in \mathbf{B}$ because $\mathscr{T}$ is closed under taking shifts of objects. Since $\{M(i) : i \in (Q_T)_0\}$ is a simple-minded collection, it is enough to show that every indecomposable Λ_T-module belongs to $\mathscr{T}$. To do so, we use what we call admissible sequences of segments.

We say $(s_{u^{(1)}}, \ldots, s_{u^{(k)}})$ is an **admissible sequence** of segments for $s = [a, b]$ if the following hold:

i) $M(u^{(i)}) \in \mathscr{T}$ for each $i \in [k]$,
ii) s_{i-1} and s_i are segments that share an endpoint,
iii) vertex a (resp., b) is an endpoint of s_1 (resp., s_k).

Observe that every segment $s = [a, b]$ has an admissible sequence of segments $(s_{u^{(1)}}, \ldots, s_{u^{(k)}})$ of length at most n given by the sequence of segments connecting a and b in the red-green tree defined by $(\mathbf{B}, \mathrm{Kr}(\mathbf{B}))$. We remark that the vertices a' and b' of T that are the endpoints shared by $s_{u^{(i)}}$ and $s_{u^{(i+1)}}$ and by $s_{u^{(j)}}$ and $s_{u^{(j+1)}}$, respectively, define a segment $[a', b'] \in \mathrm{Seg}(T)$ for any i and j satisfying $1 \leqslant i < j < k$. This follows from the fact that $s = [a, b] \in \mathrm{Seg}(T)$.

We prove that if every $s_u \in \mathrm{Seg}(T)$ with an admissible sequence $(s_{u^{(1)}}, \ldots, s_{u^{(k)}})$ has the property that $M(u) \in \mathscr{T}$, then every $s_v \in \mathrm{Seg}(T)$ with an admissible sequence $(s_{v^{(1)}}, \ldots, s_{v^{(k+1)}})$ has $M(v) \in \mathscr{T}$. If $s_u \in \mathrm{Seg}(T)$ has an admissible sequence $(s_{u^{(1)}})$, then $s_u = s_{u^{(1)}}$ and so $M(u) \in \mathscr{T}$.

Now assume that every $s_u \in \mathrm{Seg}(T)$ with an admissible sequence $(s_{u^{(1)}}, \ldots, s_{u^{(k)}})$ has the property that $M(u) \in \mathscr{T}$. Let $s_v = [a, b] \in \mathrm{Seg}(T)$ be any segment and let $(s_{v^{(1)}}, \ldots, s_{v^{(k+1)}})$ be an admissible sequence for s_v. Observe that in $(s_{v^{(1)}}, \ldots, s_{v^{(k+1)}})$ there exists $i \in [k]$ such that, without loss of generality, $s_{v^{(i)}}$ and $s_{v^{(i+1)}}$ are distinct segments that satisfy one of the following:

- $\mathrm{supp}(M(v^{(i)})) \cap \mathrm{supp}(M(v^{(i+1)})) = \varnothing$ or
- $\mathrm{supp}(M(v^{(i)})) \cap \mathrm{supp}(M(v^{(i+1)})) \neq \varnothing$.

Suppose that $\mathrm{supp}(M(v^{(i)})) \cap \mathrm{supp}(M(v^{(i+1)})) = \varnothing$. Note that $s_{v^{(i)}}$ and $s_{v^{(i+1)}}$ agree only at an endpoint. By the properties of admissible sequences, this implies that $s_{v^{(i)}} \circ s_{v^{(i+1)}} \in \mathrm{Seg}(T)$. Now we have that up to reversing the roles of $v^{(i)}$ and $v^{(i+1)}$, there is a nonsplit extension $0 \to M(v^{(i)}) \to M(v^{(i)} \leftarrow v^{(i+1)}) \to M(v^{(i+1)}) \to 0$. This means there is a triangle in $\mathcal{D}^b(\Lambda_T)$ given by $M(v^{(i)}) \to M(v^{(i)} \leftarrow v^{(i+1)}) \to M(v^{(i+1)}) \to M(v^{(i)})[1]$ so $M(v^{(i)} \leftarrow v^{(i+1)}) \in \mathscr{T}$. We obtain an admissible sequence $(s_{v^{(1)}}, \ldots, s_{v^{(i-1)}}, s_{(v^{(i)} \leftarrow v^{(i+1)})}, s_{v^{(i+2)}}, \ldots, s_{v^{(k+1)}})$ for s_v of length k. By induction, we obtain that $M(v) \in \mathscr{T}$.

Now suppose that $\mathrm{supp}(M(v^{(i)})) \cap \mathrm{supp}(M(v^{(i+1)})) \neq \varnothing$. Since $s_{v^{(i)}}$ and $s_{v^{(i+1)}}$ share an endpoint, it is easy to see that there is nonzero morphism $f : M(v^{(i)}) \to M(v^{(i+1)})$ or a nonzero morphism $f : M(v^{(i+1)}) \to M(v^{(i)})$. Without loss of generality, we assume the former. We obtain a triangle in $\mathcal{D}^b(\Lambda_T)$ given by $M(v^{(i)}) \xrightarrow{f} M(v^{(i+1)}) \to \mathrm{Cone}(f) \to M(v^{(i)})[1]$ whose long exact sequence reduces to the following exact sequence

$$0 \longrightarrow \underbrace{H^{-1}(\mathrm{Cone}(f))}_{\cong \ker(f)} \longrightarrow H^0(M(v^{(i)})) \xrightarrow{f} H^0(M(v^{(i+1)})) \longrightarrow \underbrace{H^0(\mathrm{Cone}(f))}_{\cong \mathrm{coker}(f)} \longrightarrow 0.$$

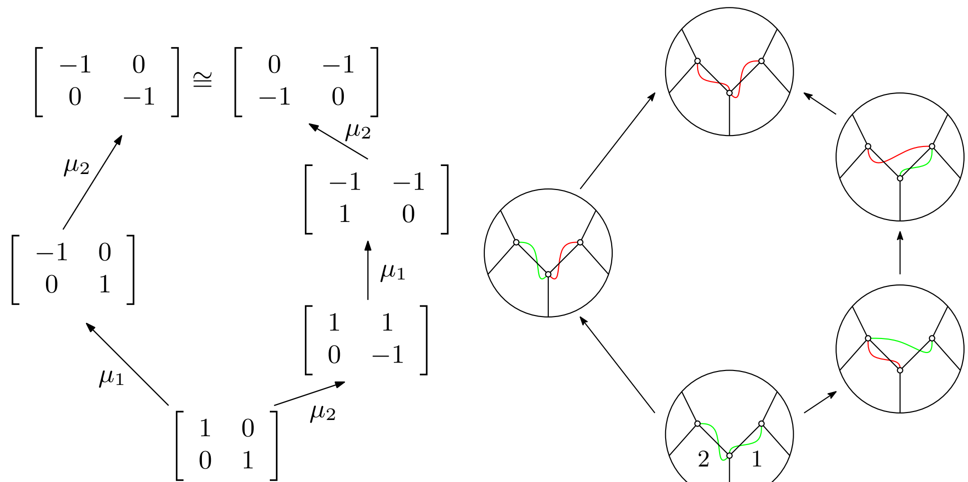


FIGURE 7. The **c**-matrices of $Q = 2 \leftarrow 1$ and the corresponding noncrossing tree partitions with their Kreweras complements.

We now have that $\text{Cone}(f) = M(w^{(1)})[1] \oplus M(w^{(2)})$ where $\text{supp}(M(w^{(1)})) = \text{supp}(M(v^{(i)}))\backslash\text{supp}(M(v^{(i+1)}))$ and $\text{supp}(M(w^{(2)})) = \text{supp}(M(v^{(i+1)}))\backslash\text{supp}(M(v^{(i)}))$. If $\text{supp}(M(w^{(1)})) = \varnothing$ (resp., $\text{supp}(M(w^{(2)})) = \varnothing$), one checks that $(s_{v^{(1)}}, \ldots, s_{v^{(i-1)}}, s_{w^{(2)}}, s_{v^{(i+2)}}, \ldots, s_{v^{(k+1)}})$ (resp., $(s_{v^{(1)}}, \ldots, s_{v^{(i-1)}}, s_{w^{(1)}}, s_{v^{(i+2)}}, \ldots, s_{v^{(k+1)}})$) is an admissible sequence for s_v of length k. By induction, we obtain that $M(v) \in \mathscr{T}$.

Finally, suppose that both $\text{supp}(M(w^{(1)})) \neq \varnothing$ and $\text{supp}(M(w^{(2)})) \neq \varnothing$. Since $\mathscr{T}$ is closed under taking summands of its objects, we have that $M(w^{(1)}), M(w^{(2)}) \in \mathscr{T}$. From the properties of admissible sequences, we have that the vertices a' and b' of T that are the endpoints shared by $s_{v^{(i-1)}}$ and $s_{v^{(i)}}$ and by $s_{v^{(i+1)}}$ and $s_{u^{(i+2)}}$, respectively, define a segment $[a', b'] \in \text{Seg}(T)$. This implies that $s_{w^{(1)}} \circ s_{w^{(2)}} \in \text{Seg}(T)$. Thus, up to reversing the roles of $w^{(1)}$ and $w^{(2)}$, there is a nonsplit extension $0 \to M(w^{(1)}) \to M(w^{(1)} \leftarrow w^{(2)}) \to M(w^{(2)}) \to 0$. This extension defines a triangle in $\mathcal{D}^b(\Lambda_T)$ given by $M(w^{(1)}) \to M(w^{(1)} \leftarrow w^{(2)}) \to M(w^{(2)}) \to M(w^{(1)})[1]$. Thus $M(w^{(1)} \leftarrow w^{(2)}) \in \mathscr{T}$. We obtain an admissible sequence $(s_{v^{(1)}}, \ldots, s_{v^{(i-1)}}, s_{w^{(1)} \leftarrow w^{(2)}}, s_{v^{(i+2)}}, \ldots, s_{v^{(k+1)}})$ for s_v of length k. By induction, we obtain that $M(v) \in \mathscr{T}$. □

We believe that there exists a generalization of our descriptions of torsion pairs, wide subcategories, and 2-term simple-minded collections in greater generality, namely, in the generality of gentle algebras. We are working to find a suitable analogue of the oriented flip graph and the noncrossing tree partitions that will model the combinatorics of these representation theoretic objects in such generality.

9. Classification of **c**-matrices

We now apply our work to obtain a combinatorial classification of the c-matrices of quivers Q_T where the internal vertices of T are all of degree 3. The **c**-matrices [12] of a quiver Q are related to noncrossing partitions of finite Coxeter groups [23] and many important objects in representation theory [3]. In [3], the **c**-matrices of quivers were interpreted representation theoretically as certain simple-minded collections in the bounded derived category of a finite dimensional algebra Λ. Our result is that **c**-matrices of Q_T are classified by noncrossing tree partitions of T paired with their Kreweras complement.

Theorem 9.1. Assume that T is a tree whose internal vertices are of degree 3.

(1) The map $\varphi : \text{Seg}(T) \to \mathbf{c}\text{-vec}(Q)^+ := \{\text{positive } \mathbf{c}\text{-vectors of } Q_T\}$, defined by $s \mapsto (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geqslant 0}$, where $a_i := 1$ if the edge corresponding to vertex $i \in (Q_T)_0$ appears in s and $a_i := 0$ otherwise, is a bijection.

(2) The map $\{(\mathbf{B}, \text{Kr}(\mathbf{B}))\}_{\mathbf{B} \in \text{NCP}(T)} \to \mathbf{c}\text{-mat}(Q)$, defined by sending $(\mathbf{B}, \text{Kr}(\mathbf{B}))$ to the **c**-matrix C whose negative **c**-vectors are $\{-\varphi(s) : s \in \text{Seg}(B) \text{ where } B \in \mathbf{B}\}$ and whose positive **c**-vectors are $\{\varphi(s) : s \in \text{Seg}(B') \text{ where } B' \in \text{Kr}(\mathbf{B})\}$, is a bijection (see Figure 7).

Proof. (1) By Corollary 4.3, there is a bijection between segments of T and the indecomposable modules of Λ_T. This bijection sends a segment s to a string module $M(w)$ of Λ_T where $w = w_1 \leftrightarrow \cdots \leftrightarrow w_k$ has the property that each vertex $w_i \in (Q_T)_0$ corresponds to an edge of T whose vertices both appear in s. Now consider the map

$\underline{\dim}: \Lambda_T\text{-mod} \to \mathbb{Z}^n_{\geqslant 0}$. By [8, Theorem 6], the restriction $\underline{\dim}: \text{ind}(\Lambda_T\text{-mod}) \to \mathbf{c}\text{-vec}(Q)^+$ is a bijection. As the composition $s \mapsto \underline{\dim}(M(w))$ agrees with the map in the assertion, this completes the proof.

(2) By Theorem 8.4, there is a bijective map

$$(\mathbf{B}, \text{Kr}(\mathbf{B})) \xmapsto{\theta} \{M(u)[1] : s_u \in \text{Seg}(B) \text{ where } B \in \mathbf{B}\} \sqcup \{M(v) : s_v \in \text{Seg}(B') \text{ where } B' \in \text{Kr}(\mathbf{B})\}$$

where the latter belongs to 2-smc(Λ_T). Define a map $\Phi : \text{2-smc}(\Lambda_T) \to \mathbf{c}\text{-mat}(Q)$ by

$$\{X_1, \ldots, X_n\} \mapsto \{\underline{\dim}(X_1), \ldots, \underline{\dim}(X_n)\}$$

where $\underline{\dim} : \mathcal{D}^b(\Lambda_T) \to \mathbb{Z}^n$ is defined as $\underline{\dim}(X_i) := \sum_{j\in\mathbb{Z}}(-1)^j \underline{\dim}(X_i^j)$. The latter map was shown to be a bijection in [3]. Using the proof of (1), we see that

$$(\mathbf{B}, \text{Kr}(\mathbf{B})) \xmapsto{\Phi\circ\theta} \{-\varphi(s_u) : s_u \in \text{Seg}(B) \text{ where } B \in \mathbf{B}\} \sqcup \{\varphi(s_v) : s_v \in \text{Seg}(B') \text{ where } B' \in \text{Kr}(\mathbf{B})\}$$

and the result follows. □

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