ORIENTED FLIP GRAPHS OF POLYGONAL SUBDIVISIONS AND NONCROSSING TREE PARTITIONS

ALEXANDER GARVER AND THOMAS MCCONVILLE

Abstract. Given a tree embedded in a disk, we introduce a simplicial complex of noncrossing geodesics supported by the tree, which we call the noncrossing complex. The facets of the noncrossing complex have the structure of an oriented flip graph. Special cases of these oriented flip graphs include the Tamari lattice, type A Cambrian lattices, Stokes posets of quadrangulations, and oriented exchange graphs of quivers mutation-equivalent to a type A Dynkin quiver. We prove that the oriented flip graph is a polygonal, congruence-uniform lattice. To do so, we express the oriented flip graph as a lattice quotient of a lattice of biclosed sets.

The facets of the noncrossing complex have an alternate ordering known as the shard intersection order. We prove that this shard intersection order is isomorphic to a lattice of noncrossing tree partitions, which generalizes the classical lattice of noncrossing set partitions. The oriented flip graph inherits a cyclic action from its congruence-uniform structure. On noncrossing tree partitions, this cyclic action generalizes the classical Kreweras complementation on noncrossing set partitions.

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1. Introduction

The purpose of this work is to understand the combinatorics associated with lattices of polygonal subdivisions (equivalently, partial triangulations) of a convex polygon. We refer to the lattices of polygonal subdivisions we study as **oriented flip graphs** (see Definition 3.10). Special cases of these posets include the Tamari lattice, type A Cambrian lattices [28], oriented exchange graphs of type A cluster algebras [5], and the Stokes poset of quadrangulations defined by Chapoton [8]. As a consequence of our work, we prove and generalize Chapoton's conjecture claiming that the Stokes poset is a lattice [8].

Rather than directly studying polygonal subdivisions, it turns out to be more convenient to formulate our theory in terms of trees that are dual to polygonal subdivisions of a polygon. That is, our work begins with the initial data of a tree T embedded in a disk so that its leaves lie on the boundary and its other vertices lie in the interior of the disk. This data gives rise to a simplicial complex of **noncrossing** sets of **arcs** on this tree that we call the **noncrossing complex**, $\Delta^{NC}(T)$. One of our main results is that the noncrossing complex is a pure and thin simplicial complex (see Corollaries 3.6 and 3.9). The combinatorics of $\Delta^{NC}(T)$ allow us to define our oriented flip graphs, which we denote by $\overrightarrow{FG}(T)$.

After defining oriented flip graphs, we turn our attention to understanding their lattice theoretic aspects. In Theorem 4.14, we show that for any tree T, the oriented flip graph $\overrightarrow{FG}(T)$ is a **congruence-uniform lattice**. In particular, any oriented flip graph is a lattice.

The Tamari lattice is a standard example of a congruence-uniform lattice [20]; see also [7], [28]. Nathan Reading gave a proof of congruence-uniformity of the Tamari lattice by proving that the weak order on permutations is congruence-uniform and applying the lattice quotient map from the weak order to the Tamari lattice defined by Björner and Wachs in [4]. To prove our congruence-uniformity result, we take a similar approach. We define a congruence-uniform lattice of **biclosed sets** of T, denoted $\text{Bic}(T)$. The essential lattice properties of $\text{Bic}(T)$ are given in Theorem 4.1. We then identify the oriented flip graph $\overrightarrow{FG}(T)$ with a sublattice and a lattice quotient of $\text{Bic}(T)$ (see Theorem 4.14). This method was applied to some other Tamari-like lattices in [18],[22]. The technique of studying a lattice by realizing it as a quotient lattice is not new, see for example [25], [26].

Congruence-uniform lattices admit an alternative poset structure called the **shard intersection order** [30]. For example, the shard intersection order of the Tamari lattice is the lattice of noncrossing set partitions [29]. We introduce a new family of objects called **noncrossing tree partitions** of T. We then prove in Theorem 5.13 that the poset of noncrossing tree partitions is a lattice under refinement. After that, we identify the shard intersection order of $\overrightarrow{FG}(T)$ with the lattice of noncrossing tree partitions of T, denoted $\text{NCP}(T)$ (Theorem 5.16).

Next, semidistributive lattices L, of which congruence-uniform lattices are an example, admit a lattice theoretically defined bijection $\text{Kr} : L \to L$, called the **Kreweras map**. We can interpret the Kreweras map on $\overrightarrow{FG}(T)$ as a bijection $\text{Kr} : \text{NCP}(T) \to \text{NCP}(T)$. We refer to $\text{Kr}(\mathbf{B})$ as the **Kreweras complement** of the noncrossing tree partition $\mathbf{B}$. In Corollary 5.12, we describe the intricate combinatorial conditions that two noncrossing tree partitions must satisfy in order for one to be the Kreweras complement of the other. Our notion of Kreweras complement generalizes the classical notion of the same name on noncrossing set partitions.

Lastly, the lattices that we study in this paper can be interpreted as certain lattices of subcategories of a finite dimensional algebra naturally associated to the given tree T. These so-called **tiling algebras**, denoted Λ_T, were introduced in [32]. In our concurrent paper [19], we interpret $\overrightarrow{FG}(T)$ as the lattice of torsion-free classes and $\text{NCP}(T)$ as the lattice of wide subcategories of the module category of Λ_T.

1.1. **Organization.** In Sections 2.1 and 2.2, we review the lattice theory that we will use throughout the paper to obtain our results. We consider Sections 2.1 and 2.2 to be expository; we do not believe any of the results therein to be new. We include theses sections to make the presentation more self-contained.

In Section 3, we introduce the noncrossing complex and the reduced noncrossing complex of arcs on a tree T. We then develop the combinatorics of these complexes, which is an important part of the definition of oriented flip graphs. In Section 4, we introduce the lattice of biclosed sets of T and we show how the oriented flip graph $\overrightarrow{FG}(T)$ is both a sublattice and quotient lattice of $\text{Bic}(T)$ (see Theorem 4.14).

In Section 5, we introduce noncrossing tree partitions of T, which generalize the classical noncrossing set partitions. We show that, as in the classical case, noncrossing tree partitions form a lattice $\text{NCP}(T)$ under refinement (Theorem 5.13). In fact, they form a meet-subsemilattice of an appropriately chosen partition lattice. Furthermore, we show that $\text{NCP}(T)$ is isomorphic to the shard intersection order of the oriented flip graph of T (Theorem 5.16).

In Section 6, we conclude the paper with applications of our work. We show that the top element of $\overrightarrow{FG}(T)$ is obtained by *rotating* arcs in the bottom element of $\overrightarrow{FG}(T)$ (see Theorem 6.7). This result recovers one of Brüstle and Qiu (see [6]) in the case when one considers triangulations of a disk, rather than partial triangulations of a disk. We then show that when T has interior vertices of degree exactly 3 (resp., 4), the oriented flip graph is isomorphic to an oriented exchange graph of a type A quiver (resp., a Stokes poset of quadrangulations) (see Proposition 6.9 (resp., Proposition 6.12)). By Proposition 6.12 and Theorem 4.14, we obtain that Stokes posets of quadrangulations are lattices, which was conjectured by Chapoton [8].

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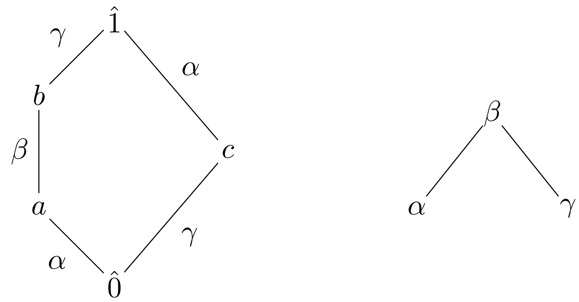


FIGURE 1. (left) A lattice with a CU-labeling (right) a poset of labels

2. PRELIMINARIES

2.1. **Lattices.** We will see that many properties of oriented flip graphs can be deduced from their lattice structure. In this section, we review some background on lattice theory, following [30]. Unless stated otherwise, we assume that all lattices considered are finite.

Given a poset $(P, \leqslant)$, the **dual** poset $(P^*, \leqslant^*)$ has the same underlying set with $x \leqslant^* y$ if and only if $y \leqslant x$. A **chain** in P is a totally ordered subposet of P. A chain $x_0 < \cdots < x_N$ is **saturated** if there does not exist $y \in P$ such that $x_{i-1} < y < x_i$ for some i. A saturated chain is **maximal** if x_0 is a minimal element of P and x_N is a maximal element of P.

A **lattice** is a poset for which any two elements x, y have a least upper bound $x \vee y$ called the **join** and a greatest lower bound $x \wedge y$ called the **meet**. Any finite lattice has a least and a greatest element, denoted $\hat{0}$ and $\hat{1}$, respectively. An element j is **join-irreducible** if $j \neq \hat{0}$ and whenever $j = x \vee y$ either $j = x$ or $j = y$ holds. **Meet-irreducible** elements are defined dually. Let $\mathrm{JI}(L)$ and $\mathrm{MI}(L)$ be the sets of join-irreducibles and meet-irreducibles of L, respectively.

For $A \subseteq L$, the expression $\bigvee A$ is **irredundant** if there does not exist a proper subset $A' \subsetneq A$ such that $\bigvee A' = \bigvee A$. Given $A, B \subseteq JI(L)$ such that $\bigvee A$ and $\bigvee B$ are irredundant and $\bigvee A = \bigvee B$, we set $A \preceq B$ if for all $a \in A$ there exists $b \in B$ with $a \leqslant b$. If $x \in L$ and $A \subseteq \mathrm{JI}(L)$ such that $x = \bigvee A$ is irredundant, we say $x = \bigvee A$ is a **canonical join-representation** for x if $A \preceq B$ for any other irredundant join-representation $x = \bigvee B,\ B \subseteq \mathrm{JI}(L)$. Dually, one may define **canonical meet-representations**.

In Figure 1, we show a lattice with 5 elements. The set of join-irreducibles is $\{a, b, c\}$. The top element $\hat{1}$ has two irredundant expressions as a join of join-irreducibles, namely $a \vee c = \hat{1}$ and $b \vee c = \hat{1}$. Since $\{a, c\} \preceq \{b, c\}$, the expression $a \vee c = \hat{1}$ is the canonical join-representation for $\hat{1}$.

A lattice L is **meet-semidistributive** if for any three elements $x, y, z \in L$, $x \wedge z = y \wedge z$ implies $(x \vee y) \wedge z = x \wedge z$. A lattice L is **semidistributive** if both L and L^* are meet-semidistributive. It is known that a lattice is semidistributive if and only if it has canonical join-representations and canonical meet-representations for each of its elements [17, Theorem 2.24].

A **lattice congruence** Θ is an equivalence relation such that if $x \equiv y \mod \Theta$ then $x \wedge z \equiv y \wedge z \mod \Theta$ and $x \vee z \equiv y \vee z \mod \Theta$ for all $x, y, z \in L$. If Θ is a lattice congruence of L, the set of equivalence classes L/Θ inherits a lattice structure from L. Namely, $[x] \vee [y] = [x \vee y]$ and $[x] \wedge [y] = [x \wedge y]$ for all $x, y \in L$. The lattice L/Θ is called a **lattice quotient** of L, and the natural map $L \to L/\Theta$ is a **lattice quotient map**. Although lattice quotients are easiest to describe in algebraic terms, it is often more useful to give the following order-theoretic definition.

Lemma 2.1. An equivalence relation Θ on a finite lattice L is a lattice congruence if

(1) every equivalence class of Θ is a closed interval of L, and
(2) the maps $x \mapsto \min[x]_\Theta$ and $x \mapsto \max[x]_\Theta$ are order-preserving.

Lemma 2.1 has been proven several times in the literature, e.g. [30, Proposition 9-5.2]. For our purposes, it is more convenient to use the following modification; see [18, Lemma 3.1] or [14, Lemma 4.2].

Lemma 2.2. Let L be a lattice with idempotent, order-preserving maps $\pi_\downarrow : L \to L,\ \pi^\uparrow : L \to L$. If for any $x \in L$

(1) $\pi_\downarrow(x) \leqslant x \leqslant \pi^\uparrow(x)$,
(2) $\pi_\downarrow(\pi^\uparrow(x)) = \pi_\downarrow(x)$,

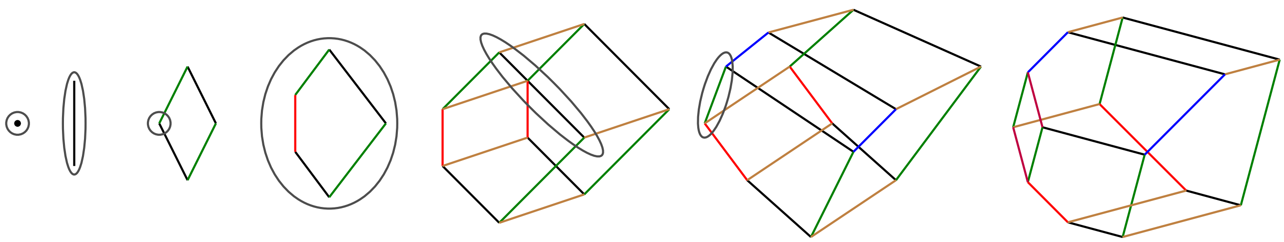


Figure 2. A sequence of interval doublings

(3) $\pi^{\uparrow}(\pi_{\downarrow}(x)) = \pi^{\uparrow}(x)$,

then the equivalence relation $x \equiv y \mod \Theta$ if $\pi_{\downarrow}(x) = \pi_{\downarrow}(y)$ is a lattice congruence.

If $\pi_{\downarrow} : L \to L$ and $\pi^{\uparrow} : L \to L$ are maps satisfying the conditions of the previous lemma, then $\pi_{\downarrow}(x) = \pi_{\downarrow}(y)$ if and only if $\pi^{\uparrow}(x) = \pi^{\uparrow}(y)$. Consequently, we could have defined the lattice congruence Θ using $\pi^{\uparrow}$ instead.

Given x, y in a poset P, we say y **covers** x, denoted $x \lessdot y$, if $x < y$ and there does not exist $z \in P$ such that $x < z < y$. We let $\mathrm{Cov}(P)$ denote the set of all covering relations of P. If P is finite, then the partial order on P is the transitive closure of its covering relations. In a finite lattice L, if $j \in \mathrm{JI}(L)$, then j covers a unique element j_*. Dually, if $m \in \mathrm{MI}(L)$, then m is covered by a unique element m^*. It should be clear from the context whether m^* is an element of the dual lattice L^* or is the unique element covering a meet-irreducible m. We describe the behavior of covering relations under lattice quotients in Lemma 2.3. A proof of this lemma may be found in Section 9-5 of [30].

Lemma 2.3. Let L be a lattice with a lattice congruence Θ.

(1) The interval $[[x]_\Theta, [y]_\Theta]$ in L/Θ is isomorphic to the quotient interval $[x, y]/\Theta$.
(2) If $(x, y) \in \mathrm{Cov}(L)$, then either $[x]_\Theta = [y]_\Theta$ or $([x]_\Theta, [y]_\Theta) \in \mathrm{Cov}(L/\Theta)$.
(3) If $x = \max[x]_\Theta$, then for each $[y]_\Theta$ with $([x]_\Theta, [y]_\Theta) \in \mathrm{Cov}(L/\Theta)$ there exists a unique $y' \in [y]_\Theta$ with $(x, y') \in \mathrm{Cov}(L)$.
(4) If $y = \min[y]_\Theta$, then for each $[x]_\Theta$ with $([x]_\Theta, [y]_\Theta) \in \mathrm{Cov}(L/\Theta)$ there exists a unique $x' \in [x]_\Theta$ with $(x', y) \in \mathrm{Cov}(L)$.

The set of lattice congruences $\mathrm{Con}(L)$ of a lattice L is partially ordered by refinement. The top element of $\mathrm{Con}(L)$ is the congruence that identifies all of the elements of L, whereas the bottom element does not identify any elements of L. It is known that $\mathrm{Con}(L)$ is a distributive lattice. By Birkhoff's representation theorem for distributive lattices, $\mathrm{Con}(L)$ is isomorphic to the poset of order-ideals of $\mathrm{JI}(\mathrm{Con}(L))$, where the set of join-irreducibles is viewed as a subposet of $\mathrm{Con}(L)$.

Given $x \lessdot y$ in L, let $\mathrm{con}(x, y)$ denote the most refined lattice congruence for which $x \equiv y$. These congruences are join-irreducible, and if L is finite, then every join-irreducible lattice congruence is of the form $\mathrm{con}(j_*, j)$ for some $j \in \mathrm{JI}(L)$ [17, Theorem 2.30]. Consequently, there is a natural surjective map of sets $\mathrm{JI}(L) \to \mathrm{JI}(\mathrm{Con}(L))$ given by $j \mapsto \mathrm{con}(j_*, j)$. Dually, there is a natural surjection $\mathrm{MI}(L) \to \mathrm{MI}(\mathrm{Con}(L))$ given by $m \mapsto \mathrm{con}(m, m^*)$. If both maps are bijections, then we say L is **congruence-uniform** (or **bounded**). Congruence-uniform lattices are the topic of the next section.

2.2. **Congruence-uniform lattices.** Given a subset I of a poset P, let $P_{\leqslant I} = \{x \in P : \ (\exists y \in I)\ x \leqslant y\}$. If I is a closed interval of a poset P, the **doubling** $P[I]$ of P at I is the induced subposet of $P \times 2$ consisting of the elements in $P_{\leqslant I} \times \{0\} \sqcup ((P \backslash P_{\leqslant I}) \cup I) \times \{1\}$. Some doublings are shown in Figure 2. Day proved that a lattice is congruence-uniform if and only if it may be constructed from a 1-element lattice by a sequence of interval doublings [11].

Let L be a lattice and P a poset. A function $\lambda : \mathrm{Cov}(L) \to P$ is a **CN-labeling** of L if L and its dual L^* satisfy the following condition (see [27]): For any elements $x, y, z \in L$ with $z \lessdot x,\ z \lessdot y$, and any maximal chains C_1, C_2 in $[z, x \vee y]$ with $x \in C_1$ and $y \in C_2$,

(CN1) the elements $x' \in C_1,\ y' \in C_2$ such that $x' \lessdot x \vee y$ and $y' \lessdot x \vee y$ satisfy

$$\lambda(x', x \vee y) = \lambda(z, y),\ \ \lambda(y', x \vee y) = \lambda(z, x);$$

(CN2) if $(u,v) \in \mathrm{Cov}(C_1)$ with $z < u,\ v < x \vee y$ then $\lambda(z,x) < \lambda(u,v)$ and $\lambda(z,y) < \lambda(u,v)$;
(CN3) the labels on $\mathrm{Cov}(C_1)$ are all distinct.

A lattice is **congruence-normal** if it has a CN-labeling. Alternatively, a lattice is congruence-normal if it may be constructed from a 1-element lattice by a doubling sequence of **order-convex sets**; see [27]. The following lemma shows that CN-labelings behave nicely under taking intervals or lattice quotients. The proof of this lemma is straightforward; see e.g. [27, Proposition 22].

Lemma 2.4. Let L be a congruence-normal lattice with CN-labeling $\lambda : \mathrm{Cov}(L) \to P$.

(1) Let Θ be a lattice congruence of L. Define an edge-labeling $\tilde{\lambda} : \mathrm{Cov}(L/\Theta) \to P$ by $\tilde{\lambda}([x]_\Theta, [y]_\Theta) = \lambda(x,y)$ whenever $(x,y) \in \mathrm{Cov}(L)$ and $x \not\equiv y \mod \Theta$. This labeling is well-defined and is a CN-labeling of L/Θ.
(2) The restriction of a CN-labeling to an interval $[x,y]$ is a CN-labeling of $[x,y]$.

We say $\lambda : \mathrm{Cov}(L) \to P$ is a **CU-labeling** if it is a CN-labeling, and
(CU1) $\lambda(j_*, j) \neq \lambda(j'_*, j')$ for $j, j' \in \mathrm{JI}(L),\ j \neq j'$, and
(CU2) $\lambda(m, m^*) \neq \lambda(m', m'^*)$ for $m, m' \in \mathrm{MI}(L),\ m \neq m'$.

For example, the colors on the edges of Figure 2 form a CU-labeling, where the color set is ordered $s \leqslant t$ if color s appears before t in the sequence of doublings.

In [27], Reading characterized congruence-normal lattices as those lattices that admit a CN-labeling. From his proof, it is straight-forward to show that a lattice is congruence-uniform if and only if it admits a CU-labeling.

Proposition 2.5. A lattice is congruence-uniform if and only if it admits a CU-labeling.

If $x \lessdot y$ and $w \lessdot z$, then covers (x,y) and (w,z) are **perspective** if either $y \wedge w = x$ and $y \vee w = z$ or $x \wedge z = w$ and $x \vee z = y$. Such a notion is useful for lattice congruences. Namely, if (x,y) and (w,z) are perspective and Θ is a lattice congruence, then $x \equiv y \mod \Theta$ if and only if $w \equiv z \mod \Theta$.

If (x,y) and (w,z) are persective covering relations, then we set $(x,y) \leqslant (w,z)$ if $x \leqslant w$ and $y \leqslant z$. It is not difficult to see that if (x,y) is minimal in this ordering, then y is a join-irreducible element, and x is its unique lower cover. Indeed, if y is not join-irreducible, there exists an element $y' < y$ such that $y' \nleqslant x$. Taking y' to be minimal with this property, we may conclude that (x,y) is perspective with (x',y') for any choice of cover $x' \lessdot y'$ and $(x',y') < (x,y)$. Dually, if (x,y) is maximal in this ordering, then x is meet-irreducible and y is its unique upper cover.

For an element x, let $\lambda_\downarrow(x) = \{\lambda(y,x) :\ y \in L, y \lessdot x\}$. Dually, let $\lambda^\uparrow(x) = \{\lambda(x,y) :\ y \in L, x \lessdot y\}$.

Lemma 2.6. Let L be a congruence-uniform lattice with CU-labeling $\lambda : \mathrm{Cov}(L) \to P$. For any $s \in P$, there is a unique join-irreducible $j \in \mathrm{JI}(L)$ and meet-irreducible $m \in \mathrm{MI}(L)$ such that $\lambda(j_*, j) = s = \lambda(m, m^*)$. Moreover, for any covering relation $x \lessdot y$, $\lambda(x,y) = s$ exactly when (x,y) is perspective with (j_*, j) and with (m, m^*).

Proof. Let $s \in P$ be given, and let j be minimal such that $s \in \lambda_\downarrow(j)$, and let $w \in L$ with $\lambda(w,j) = s$. If j is not join-irreducible, then there exists some z covered by j distinct from w. By (CN1), there exists an element $w' < j$ such that $\lambda(w \wedge z, w') = s$, which is a contradiction to the minimality of j. Hence, j is join-irreducible.

Let $x, y \in L$ such that $x \lessdot y$ and $\lambda(x,y) = s$. If y is join-irreducible, then $y = j$ by (CU1). Otherwise, by the previous argument, (x,y) is perspective with some cover (x_1, y_1) such that $y > y_1$. Applying this several times, we get a sequence $y > y_1 > \cdots > y_N$ and covers (x_i, y_i) such that (x,y) is perspective with (x_i, y_i) for all i. This terminates if y_N is minimal. But that forces $y_N = j$, so (x,y) is perspective with (j_*, j).

Now let $j \in \mathrm{JI}(L)$ and $(x,y) \in \mathrm{Cov}(L)$ such that (j_*, j) and (x,y) are perspective. If y is a join-irreducible, then $y = j$ holds. Otherwise, we may construct a sequence $(x_i, y_i) \in \mathrm{Cov}(L)$ such that any two covers are perspective, $\lambda(x_i, y_i) = \lambda(x,y)$ and $y_1 > y_2 > \cdots > y_N$ with $y_N \in \mathrm{JI}(L)$. Since perspective pairs induce the same lattice congruence, we have $\mathrm{con}(j_*, j) = \mathrm{con}(x_N, y_N)$, so $j = y_N$.

Dually, if m is maximal such that $s \in \lambda^\uparrow(m)$, then it follows from a similar argument that m is meet-irreducible. Furthermore for any $(x,y) \in \mathrm{Cov}(L)$, then (x,y) and (m, m^*) are perspective if and only if $\lambda(x,y) = s = \lambda(m, m^*)$. □

Lemma 2.6 shows that a CU-labeling is essentially unique if it exists. Using the proof, one can construct the following labeling.

Corollary 2.7. If L is a congruence-uniform lattice, then the edge-labeling $\lambda : \mathrm{Cov}(L) \to \mathrm{JI}(\mathrm{Con}(L))$ where $\lambda(x,y) = \mathrm{con}(x,y)$ is a CU-labeling.

It is known that congruence-uniformity is preserved under lattice quotients; see e.g. [17, Theorem 2.43]. Preservation of CU-labelings under lattice quotients gives another proof.

Corollary 2.8. If L is a (finite) congruence-uniform lattice and Θ is a lattice congruence, then L/Θ is congruence-uniform.

The CU-labeling of a congruence-uniform lattice encodes the sequence of doublings that create it. This labeling has the side effect of encoding canonical join-representations and canonical meet-representations as follows.

Proposition 2.9. Let L be a congruence-uniform lattice with CU-labeling λ. For $x \in L$, the canonical join-representation of x is $\bigvee D$, where D is the set of join-irreducibles $j \in \mathrm{JI}(L)$ such that $\lambda(j_*, j) \in \lambda_\downarrow(x)$. Dually, for $x \in L$, the canonical meet-representation of x is $\bigwedge U$, where U is the set of meet-irreducibles $m \in \mathrm{MI}(L)$ such that $\lambda(m, m^*) \in \lambda^\uparrow(x)$.

Proof. We prove that $x = \bigvee D$ is a canonical join-representation of x. The dual statement may be proved similarly.

We first show that the equality $x = \bigvee D$ holds. For $j \in D$, the pair (j_*, j) is perspective with some cover (c, x), so $j < x$. Hence, $\bigvee D \leqslant x$. If they are unequal, then there exists an element c covered by x for which $\bigvee D \leqslant c$. But (c, x) is perspective with (j_*, j) for some $j \in D$, which implies $j \nleqslant c$. Hence, $x = \bigvee D$.

Now suppose $\bigvee D$ is redundant, and let $j_0 \in D$ such that $x = \bigvee D \backslash \{j_0\}$. Let c_0 be the element covered by x with $\lambda(c_0, x) = \lambda((j_0)_*, j_0)$. Since $c_0 < \bigvee D\backslash\{j_0\}$, there exists $j_1 \in D\backslash\{j_0\}$ where $c_0 \vee j_1 = x$. Let c_1 be covered by x with $\lambda(c_1, x) = \lambda((j_1)_*, j_1)$. By (CN1), there exists c_1' with $c_0 \wedge c_1 \lessdot c_1' \leqslant c_0$ such that $\lambda(c_0 \wedge c_1, c_1') = \lambda((j_1)_*, j_1)$. But this means $j_1 \leqslant c_1' \leqslant c_0$ holds, which is a contradiction.

Now let $E \subseteq \mathrm{JI}(L)$ such that $x = \bigvee E$ is irredundant, and suppose $D \neq E$. Let $j_0 \in D\backslash E$, and let c_0 be the element covered by x such that $\lambda((j_0)_*, j_0) = \lambda(c_0, x)$. Since $c_0 < \bigvee E$, there exists $j' \in E$ such that $j' \nleqslant c_0$. Since $j' \neq j_0$, the cover (j'_*, j') is not perspective with (c_0, x). In particular, $c_0 \wedge j' < j'_*$ holds. Let a_0 be an element covering $c_0 \wedge j'$ with $a_0 < j'$. Then $a_0 \vee c_0 = x$, so $(c_0 \wedge j', a_0)$ and (c_0, x) are perspective. This means $j_0 \leqslant a_0 < j'$. Hence, $D \preceq E$, as desired. □

Lemma 2.10. Let L be a congruence-uniform lattice with CU-labeling λ. For $x \in L$, there exists a unique element y such that $\lambda^\uparrow(x) = \lambda_\downarrow(y)$.

Proof. We prove the lemma by induction on $|L|$. If $|L| = 1$, then the statement is immediate. If not, let L' be a congruence-uniform lattice with interval I such that $L'[I] \cong L$. Let Θ be the lattice congruence whose equivalence classes are the fibers of $L \to L'$. Let s be the label in each Θ-equivalence class.

For $x \in L$, if $x = \max[x]_\Theta$, then the upper covers of x in L are in correspondence with the upper covers of $[x]_\Theta$ in L/Θ. This correspondence preserves labels. Hence, there is a unique element $[y]_\Theta$ in L/Θ with $\lambda_\downarrow([y]_\Theta) = \lambda^\uparrow([x]_\Theta)$. Taking y to be the minimum element in $[y]_\Theta$, we have

$$\lambda_\downarrow(y) = \lambda_\downarrow([y]_\Theta) = \lambda^\uparrow([x]_\Theta) = \lambda^\uparrow(x).$$

By the uniqueness of $[y]_\Theta$, if y is not unique in L, then there exists an element y' such that $y' \neq \min[y']_\Theta$. But $s \in \lambda_\downarrow(y')$ and $s \notin \lambda^\uparrow(x)$. Hence, the element y is unique in L.

Now let x be an element of L such that $x \neq \max[x]_\Theta$. Then the upper covers of x are in correspondence with upper covers of $[x]_\Theta$ restricted to the interval I and one additional element, $\max[x]_\Theta$. Since $s \in \lambda^\uparrow(x)$, any element y with $\lambda_\downarrow(y) = \lambda^\uparrow(x)$ satisfies $[y]_\Theta \in I$ and $y = \max[y]_\Theta$. Since I inherits a CU-labeling from L/Θ, there exists a unique element $[y]_\Theta$ in I whose lower covers in I have the same labels as the upper covers of $[x]_\Theta$ (restricted to I). Taking $y = \max[y]_\Theta$, we deduce that $\lambda_\downarrow(y) = \lambda^\uparrow(x)$. The uniqueness of y follows from the uniqueness of $[y]_\Theta$. □

We define the **Kreweras map** $\mathrm{Kr} : L \to L$ by $\mathrm{Kr}(x) = y$ where x and y are defined as in Lemma 2.10. A dual statement to Lemma 2.10 shows that Kr is a bijection. A special case of this bijection was originally defined by Kreweras on the lattice of noncrossing set partitions [21]. Using a standard bijection between noncrossing partitions and bracketings of a word, the bijection defined by Kreweras is equivalent to the Kreweras map on the Tamari lattice.

Lemma 2.10 may be restated using Proposition 2.9 to define a bijection $L \to L$ that switches canonical join-representations with canonical meet-representations. In these terms, this bijection can be shown to exist more generally for semidistributive lattices [2].

Lemma 2.11. Let L be a congruence-uniform lattice with CU-labeling $\lambda : \mathrm{Cov}(L) \to P$. Let $[x, y]$ be an interval of L for which $y = \bigvee_{i=1}^{l} a_i$ where $\{a_1, \ldots, a_l\}$ are a subset of the elements covering x. Then there exist elements $c_1, \ldots, c_l$ covered by y such that $x = \bigwedge_{i=1}^{l} c_i$ and $\lambda(x, a_i) = \lambda(c_i, y)$ for all i.

Proof. Since the restriction of a CU-labeling to an interval $[x,y]$ is a CU-labeling of $[x,y]$, we may assume $x = \hat{0},\ y = \hat{1}$. Let U be the set of meet-irreducibles $m \in \mathrm{MI}(L)$ such that $\lambda(m, m^*) \in \lambda^{\uparrow}(\hat{0})$. Then $\hat{0} = \bigwedge U$ is a canonical meet-representation. Then $\mathrm{Kr}(\hat{0}) = \bigvee\{\kappa(m) :\ m \in U\}$ is a canonical join-representation. But $\{\kappa(m) :\ m \in U\}$ is the set of atoms of L, so

$$\hat{1} = \mathrm{Kr}(\hat{0}) = \bigvee\{\kappa(m) :\ m \in U\} = \bigvee A$$

where A is the set of atoms of L. As this is the canonical join-representation of $\hat{1}$, we must have $A = \{a_1, \ldots, a_l\}$, and there exist $c_1, \ldots, c_l$ covered by y with $\lambda(\hat{0}, a_i) = \lambda(c_i, \hat{1})$ for all i. As each c_i is meet-irreducible, we have $\kappa(c_i) = a_i$ for all i. Hence, $x = \bigwedge_{i=1}^{l} c_i$. □

Given a congruence-uniform lattice L, the shard intersection order can be defined from a CU-labeling $\lambda : \mathrm{Cov}(L) \to S$ as follows. For $x \in L$, let $y_1, \ldots, y_k$ be the set of elements in L such that $(y_i, x) \in \mathrm{Cov}(L)$. Define

$$\psi(x) = \{\lambda(w,z) :\ \bigwedge_{i=1}^{k} y_i \leqslant w \lessdot z \leqslant x\}.$$

The **shard intersection order** $\Psi(L)$ is the collection of sets $\psi(x)$ for $x \in L$, ordered by inclusion. The shard intersection order was defined at this level of generality by Nathan Reading following Theorem 1-7.24 in [30].

The poset $\Psi(L)$ derives its name from a related construction on hyperplane arrangements. If $\mathcal{A}$ is a real, central, simplicial hyperplane arrangement, then the poset of regions with respect to any choice of fundamental chamber is a semidistributive lattice. Each hyperplane is divided into several cones, called **shards**. The **shard intersection order** is the poset of intersections of shards, ordered by reverse inclusion. When the poset of regions is a congruence-uniform lattice, the resulting poset is isomorphic to $\Psi(L)$. However, while any shard intersection order coming from a congruence-uniform poset of regions is a lattice, this does not hold for arbitrary congruence-uniform lattices [23]. For example, if L is a 3-element chain, its shard intersection order $\Psi(L)$ does not have a unique maximum element, so it is not a lattice.

3. The noncrossing complex

In this section, we introduce the noncrossing complex of arcs on a tree. This simplicial complex gives rise to a pure, thin simplicial complex that we refer to as the reduced noncrossing complex. We use the facets of the reduced noncrossing complex to define our main object of study, the oriented flip graph of a tree.

A **tree** is a finite, connected acyclic graph. Any tree may be embedded in a disk D^2 in such a way that a vertex is on the boundary if and only if it is a leaf. Unless specified otherwise, we will assume that any tree comes equipped with such an embedding. We will refer to non-leaf vertices of a tree as **interior vertices**. We assume that any interior vertex of a tree has degree at least 3. We say two trees T and T' are **equivalent** if there is an ambient isotopy $F : [0,1] \times D^2 \to D^2$ such that $F(0,x) = x$ and $F(1,x) \in D^2 \backslash T'$ for all $x \in D^2 \backslash T$. Roughly speaking, we only consider the combinatorial structure of the embedded tree—its vertices, edges, faces, and orientation—rather than the precise embedding of the tree.

A tree T embedded in D^2 determines a collection of 2-dimensional **faces**. By definition, a **face** of T is the closure of a connected component of $D^2 \backslash T$. A **corner** of a tree is a pair (v, F) consisting of an interior vertex v and a face F containing v. We let $\mathrm{Cor}(T)$ denote the set of corners of T. The embedding that accompanies T also endows each interior vertex with a cyclic ordering. Given two corners $(u, F), (u, G) \in \mathrm{Cor}(T)$, we say that (u, G) is **immediately clockwise** (resp., **immediately counterclockwise**) from (u, F) if F and G are incident to a common edge and G is clockwise (resp., counterclockwise) from F according to the cyclic ordering at u.

An **acyclic path** (or **chordless path**) supported by a tree T is a sequence $(v_0, \ldots, v_t)$ of pairwise distinct vertices of T such that v_i and v_j are adjacent if and only if $|i - j| = 1$. We typically identify acyclic paths with their underlying vertex sets; that is, we do not distinguish between acyclic paths of the form $(v_0, \ldots, v_t)$ and $(v_t, \ldots, v_0)$. We will refer to v_0 and v_t as the **endpoints** of the acyclic path $(v_0, \ldots, v_t)$. Note that an acyclic path is determined by its endpoints, and thus we can write $[v_0, v_t] = (v_0, \ldots, v_t)$. As an acyclic path $(v_0, \ldots, v_t)$ defines a subgraph of T (namely, the induced subgraph on the vertices $v_0, \ldots, v_t$), it makes sense to refer to an **edge** of $(v_0, \ldots, v_t)$. Additionally, if $(v_0, \ldots, v_t)$ and $(v_t, \ldots, v_{t+s})$ are acyclic paths that agree only at v_t and where $[v_0, v_{t+s}]$ is an acyclic path, we define their **composition** as $[v_0, v_t] \circ [v_t, v_{t+s}] := [v_0, v_{t+s}]$.

An **arc** $p = (v_0, \ldots, v_t)$ is an acyclic path whose endpoints are distinct leaves and any two edges (v_{i-1}, v_i) and (v_i, v_{i+1}) are incident to a common face. We say p **traverses a corner** or **contains a corner** (v, F) if $v = v_i$ for for some $i = 1, \ldots, t-1$ and F is the face that is incident to both (v_{i-1}, v_i) and (v_i, v_{i+1}). Since an arc p divides

D^2 into two components, it determines two disjoint subsets of the set of faces of T that we will call **regions**. We let $\operatorname{Reg}(p, F)$ denote the region defined by p that contains the face F.

A **segment** is an acyclic path consisting of at least two vertices and with the same incidence condition that is required of arcs, but whose endpoints are *not* leaves. Observe that interior vertices of T are not considered to be segments.

Example 3.1. Let T denote the tree shown in Figure 3 and let $p = (7, 10, 11, 12, 5)$ be the arc of T shown in blue. The arc p contains the corners $(10, F_2)$, $(11, F_5)$, and $(12, F_5)$. The two regions defined by p are $\operatorname{Reg}(p, F_1) = \{F_1, F_2, F_3, F_6, F_7, F_8\}$ and $\operatorname{Reg}(p, F_4) = \{F_4, F_5\}$.

Definition 3.2. We say that two arcs p and q are **crossing** along a segment s if

i) each vertex of s appears in p and in q and

ii) if R_p and R_q are regions defined by p and q, respectively, then $\mathrm{R}_p \nsubseteq \mathrm{R}_q$ and $\mathrm{R}_q \nsubseteq \mathrm{R}_p$.

We say they are **noncrossing** otherwise. The **noncrossing complex** $\Delta^{NC}(T)$ is defined to be the abstract simplicial complex whose simplices are pairwise noncrossing collections of arcs supported by the tree T.

Example 3.3. Let T denote the tree shown in Figure 3. Let $p = (7, 10, 11, 12, 5)$ and $q = (6, 10, 11, 9, 1)$ denote the arcs of T shown in blue and red, respectively. The arcs p and q cross along the segment $s = (10, 11)$ shown in purple.

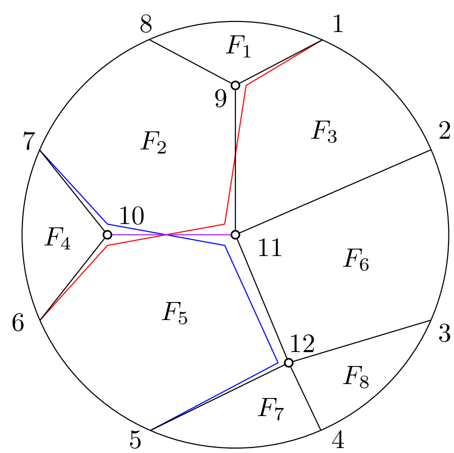


FIGURE 3

If p is an arc whose vertices all lie in a common face, then p is noncrossing with every arc supported by T. We call such an arc a **boundary arc**. Observe that boundary arcs are exactly those arcs that define a region consisting of a single face. This implies that the faces of T are in bijection with boundary arcs of T. The **reduced noncrossing complex** $\widetilde{\Delta}^{NC}(T)$ is the abstract simplicial complex consisting of the faces of $\Delta^{NC}(T)$ containing no boundary arcs.

We now introduce a partial ordering on arcs that contain a particular corner of T, which will enable us to understand the combinatorial structure of the noncrossing complex and the reduced noncrossing complex of T. Let $\mathcal{F}$ be a face of $\Delta^{NC}(T)$ and let (v, F) be a corner that is contained in at least one arc of $\mathcal{F}$. The arcs of $\mathcal{F}$ that contain (v, F) are partially ordered in the following way: $p \leqslant_{(v,F)} q$ if and only if $\operatorname{Reg}(p, F) \subseteq \operatorname{Reg}(q, F)$.

Lemma 3.4. If $\mathcal{F}$ is a face of $\Delta^{NC}(T)$ and (v, F) is a corner contained in at least one arc of $\mathcal{F}$, then the partially ordered set $(\{p \in \mathcal{F} : \ p \text{ contains } (v, F)\}, \leqslant_{(v,F)})$ is a linearly ordered set.

Proof. Since any pair $p_1, p_2 \in \{p \in \mathcal{F} : \ p \text{ contains } (v, F)\}$ is noncrossing, one has that $p_1 \leqslant_{(v,F)} p_2$ or $p_2 \leqslant_{(v,F)} p_1$. Thus $(\{p \in \mathcal{F} : \ p \text{ contains } (v, F)\}, \leqslant_{(v,F)})$ is a linearly ordered set. □

It follows from Lemma 3.4 that the partially ordered set $(\{p \in \mathcal{F} : \ p \text{ contains } (v, F)\}, \leqslant_{(v,F)})$ has a unique maximal element, which we will denote by $p(v, F)$. We say that an arc p of $\mathcal{F}$ is **marked** at (v, F) if $p = p(v, F)$.

The following proposition enables us to show that the simplicial complex $\widetilde{\Delta}^{NC}(T)$ is **pure** (i.e., any two facets have the same cardinality) in Corollary 3.6 and **thin** (i.e., every codimension 1 simplex is a face of exactly two facets) in Corollary 3.9.

Proposition 3.5. Let $\mathcal{F}$ be a face of $\Delta^{NC}(T)$, let $p \in \mathcal{F}$, and let $\text{Reg}_1, \text{Reg}_2$ denote the regions defined by p.

1. The arc p is marked at some corner of T.
2. In p is not a boundary arc, then p is marked at a corner in Reg_1 and at a corner in Reg_2.
3. Assume that p is marked at two distinct corners $(v, F), (w, G) \in \text{Cor}(T)$ and that F and G belong to the same region defined by p. Then there exists an arc $p' \notin \mathcal{F}$ that contains (v, F) and (w, G') where $G' \neq G$ and where $\mathcal{F} \cup \{p'\} \in \Delta^{NC}(T)$.

Proof. (1) Let $(v, F) \in \text{Cor}(T)$ be a corner contained in p. If $p = p(v, F)$, then we are done. Otherwise, let $q \in \mathcal{F}$ be the arc containing (v, F) such that $p \lessdot_{(v,F)} q$. Let w be an interior vertex at which p and q separate, let (w, G) be the corner traversed by p at w, and let $p' = p(w, G) \in \mathcal{F}$. Since p' and q are noncrossing and $p \leqslant_{(w,G)} p'$, p' must contain the corner (v, F) and $G \in \text{Reg}(p, F)$. Now this implies $p \leqslant_{(v,F)} p'$ so $p \leqslant_{(v,F)} p' <_{(v,F)} q$. Thus $p = p'$.

(2) In the proof of (1), we showed that if p contains a corner (w_i, G_i) with $G_i \in \text{Reg}_i$, then there exists a corner (v_i, F_i) with $F_i \in \text{Reg}_i$ such that $p = p(v_i, F_i)$. If p is not a boundary arc, then it contains such a corner (w_i, G_i) with $G_i \in \text{Reg}_i$ for $i = 1, 2$.

(3) Assume that p contains two distinct corners $(v, F), (w, G) \in \text{Cor}(T)$ where $p = p(v, F)$ and $p = p(w, G)$ and where F and G belong to the same region defined by p. Let G' be the face containing w such that $G \cap G'$ is an edge of the segment $[v, w]$. We can assume that at least one arc of $\mathcal{F}$ contains $(w, G') \in \text{Cor}(T)$, otherwise define p' to be the boundary arc corresponding to G' and we obtain that $\mathcal{F} \cup \{p'\} \in \Delta^{NC}(T)$.

Let $q := p(w, G') \in \mathcal{F}$. The arc p is expressible as the composition $p = [v_0, v] \circ [v, w] \circ [w, w_0]$. Similarly, q is the composition $q = [v_1, w] \circ [w, w_1]$ where $[w, w_1]$ and p do not agree along any edges. Let p' be the arc $p' := [v_0, w] \circ [w, w_1]$. Clearly, p' and p do not cross.

Next, we show that $\mathcal{F} \cup \{p'\} \in \Delta^{NC}(T)$. Let $q' \in \mathcal{F}$ and suppose that q' and p' cross along a segment s. It is enough to assume that s is contained in either $[v_0, w]$ or $[w, w_1]$. If s is contained in $[v_0, w]$, then since p and p' agree along $[v_0, w]$ we have that q' and p cross along s, a contradiction. Similarly, q' and p' cannot cross along a segment s contained in $[w, w_1]$. We conclude that $\mathcal{F} \cup \{p'\} \in \Delta^{NC}(T)$.

Lastly, if p' was an element of $\mathcal{F}$, this would contradict $p = p(v, F)$ since $p <_{(v,F)} p'$. We obtain that $p' \notin \mathcal{F}$. □

Corollary 3.6. The simplicial complex $\widetilde{\Delta}^{NC}(T)$ is pure.

Proof. Assume $\mathcal{F} \in \Delta^{NC}(T)$ is a facet. By Proposition 3.5 (1), each boundary arc of $\mathcal{F}$ is marked at exactly one corner of T. Similarly, by Proposition 3.5 (2), each nonboundary arc of $\mathcal{F}$ is marked at exactly two corners of T. We also know that $\#\{\text{boundary arcs of } \mathcal{F}\} = \#\{\text{faces of } T\}$. We obtain that

$$\begin{aligned} \#\text{Cor}(T) &= \#\{\text{boundary arcs in } \mathcal{F}\} + 2\#\{\text{nonboundary arcs in } \mathcal{F}\} \\ &= \#\{\text{faces of } T\} + 2\#\{\text{nonboundary arcs in } \mathcal{F}\}. \end{aligned}$$

Thus $\#\{\text{nonboundary arcs in } \mathcal{F}\} = \frac{1}{2}\left(\#\text{Cor}(T) - \#\{\text{faces of } T\}\right)$. As the latter number is independent of $\mathcal{F}$, we have that $\Delta^{NC}(T)$ is pure and thus so is $\widetilde{\Delta}^{NC}(T)$. □

Proposition 3.7. If $\mathcal{F}$ is a facet of $\Delta^{NC}(T)$ and $p \in \mathcal{F}$ is not a boundary arc, then there exists a unique arc $q \notin \mathcal{F}$ such that $(\mathcal{F}\backslash\{p\}) \cup \{q\}$ is a facet of $\Delta^{NC}(T)$. Moreover, if p is marked at two distinct corners $(v, F), (u, G) \in \text{Cor}(T)$, then $[v, u]$ is the unique longest segment along which p and q cross.

We refer to the operation sending a facet $\mathcal{F}$ of $\Delta^{NC}(T)$ (or $\widetilde{\Delta}^{NC}(T)$) to the new facet $(\mathcal{F}\backslash\{p\}) \cup \{q\}$ of $\Delta^{NC}(T)$ (or $\widetilde{\Delta}^{NC}(T)$), in the sense of Proposition 3.7, as a **flip** of $\mathcal{F}$ at p and denote it by μ_p. We define the **flip graph** of T, denoted $FG(T)$, to be the graph whose vertices are facets of $\widetilde{\Delta}^{NC}(T)$ and such that two vertices are connected by an edge if and only if the corresponding facets can be obtained from each other by a single flip.

Example 3.8. Figure 4 shows an example of performing a flip on a facet of a tree T. A black dot appears on an arc if it is the largest arc in that facet containing the corresponding corner. The boundary arcs of T are $(1, 5, 2)$, $(1, 5, 6, 4)$, $(2, 5, 6, 3)$, and $(3, 4, 6)$. These appear in gold. Flipping the green arc produces the red arc.

Proof of Proposition 3.7. By Proposition 3.5 (2), there exist distinct corners $(v_1, F_1), (v_2, F_2) \in \text{Cor}(T)$ contained in p where $F_i \in \text{Reg}_i$ and such that $p = p(v_i, F_i)$ for $i = 1, 2$. Orienting p from v_1 to v_2, we assume without loss of generality that p turns left at v_1 and right at v_2. Let p_1 and p_2 be arcs of $\mathcal{F}\backslash\{p\} \in \Delta^{NC}(T)$ where p_1 and p_2 are marked at $p(v_1, F_1)$ and $p(v_2, F_2)$, respectively, with respect to the other arcs of $\mathcal{F}\backslash\{p\}$. Since $\mathcal{F}$ is a facet, it contains each boundary arc. As p is not a boundary arc, there do exist the desired arcs p_1 and p_2 in $\mathcal{F}\backslash\{p\}$.

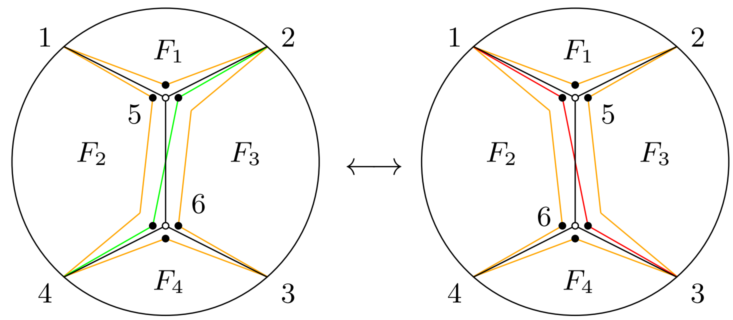


FIGURE 4. The two facets of $\Delta^{NC}(T)$.

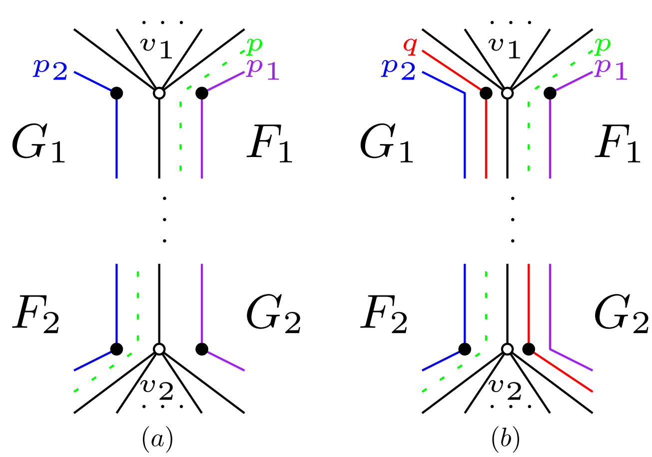


FIGURE 5. In (a), we show part of the face $\mathcal{F}\backslash\{p\}$ and we indicate corners at which p_1 and p_2 are marked by black dots. In (b), we show the part of the face $\mathcal{F}\backslash\{p\} \cup \{q\}$ and we indicate corners at which p_1, p_2, and q are marked by black dots. In (a) and (b), we indicate where the arc p appeared before it was removed.

Claim 1: In the face $\mathcal{F}\backslash\{p\} \in \Delta^{NC}(T)$, one has that $p_1 = p(v_2, G_2)$ and $p_2 = p(v_1, G_1)$ where G_i is the unique face of T such that (v_i, G_i) is immediately clockwise from (v_i, F_i) (see Figure 5).

To see that *Claim 1* holds, we show that $p_1 = p(v_2, G_2)$ and the proof that $p_2 = p(v_1, G_1)$ is similar. Write $p_1 = s_1 \circ [v_1, w_1]$ and $p(v_2, G_2) = s_2 \circ [v_2, w_2]$ where $s_1, [v_1, w_1], s_2$, and $[v_2, w_2]$ are acyclic paths of T, w_1 and w_2 are leaf vertices of T, and where we require that $[v_1, w_1]$ and $[v_2, w_2]$ each contain part of the segment $[v_1, v_2]$.

Now consider the arc $p' := s_1 \circ [v_1, v_2] \circ s_2$. Since p_1 (resp., $p(v_2, G_2)$) does not cross any arcs of $\mathcal{F}$ along s_1 (resp., s_2), the same is true for p'. Similarly, p does not cross any arcs of $\mathcal{F}$ along $[v_1, v_2]$ so the same is true for p'. As $\mathcal{F}$ is a facet of $\Delta^{NC}(T)$, we have that $p' \in \mathcal{F}$. Now it is clear that $p' = p(v_1, F_1)$ and $p' = p(v_2, G_2)$. This completes the proof of *Claim 1*.

Next, let $p_1 = s_1 \circ [v_2, w_1]$ and let $p_2 = [w_2, v_1] \circ s_2$ for some acyclic paths s_1 and s_2 and some leaf vertices w_1 and w_2 of T. Define $q := [w_2, v_1] \circ [v_1, v_2] \circ [v_2, w_1] \neq p$. By *Claim 1* and the proof of Proposition 3.5 (3), we have that $(\mathcal{F}\backslash\{p\}) \cup \{q\} \in \Delta^{NC}(T)$. Furthermore, it is clear that $q = p(v_1, G_1) = p(v_2, G_2)$ in $(\mathcal{F}\backslash\{p\}) \cup \{q\}$ and that $[v_1, v_2]$ is the unique longest segment along which p and q cross.

Next, we show that $\mathcal{F}$ and $(\mathcal{F}\backslash\{p\}) \cup \{q\}$ are the only faces of $\Delta^{NC}(T)$ that contain $\mathcal{F}\backslash\{p\}$. Note that from this it also follows that $(\mathcal{F}\backslash\{p\}) \cup \{q\}$ is a facet of $\Delta^{NC}(T)$. Suppose there exists an arc $p' \notin \mathcal{F}\backslash\{p\}$ such that $(\mathcal{F}\backslash\{p\}) \cup \{p'\}$ is a facet. We see immediately that p' contains the segment $[v_1, v_2]$. Also, $p' = p(v_2, F_2) = p(v_1, F_1)$ or $p' = p(v_2, G_2) = p(v_1, G_1)$, otherwise by combining Proposition 3.5 (3) and *Claim 1* we have that $(\mathcal{F}\backslash\{p\}) \cup \{p'\}$ is not a facet. The following claim shows that if $p' = p(v_2, F_2) = p(v_1, F_1)$ (resp., $p' = p(v_2, G_2) = p(v_1, G_1)$), then $p' = p$ (resp., $p' = q$).

Claim 2: Let $p = [u_2, v_1] \circ [v_1, v_2] \circ [v_2, u_1]$.

i) If p' contains the corner (v_2, F_2), then p' and p agree along $[u_1, v_2] \circ [v_2, v_1]$.
ii) If p' contains the corner (v_1, F_1), then p' and p agree along $[u_2, v_1] \circ [v_1, v_2]$.

iii) If p' contains the corner (v_2, G_2), then p' and q agree along $[w_1, v_2] \circ [v_2, v_1]$.
iv) If p' contains the corner (v_1, G_1), then p' and q agree along $[w_2, v_1] \circ [v_1, v_2]$.

We prove part *i)* of *Claim 2*, and the proofs of the other parts are analogous. Suppose there exists an interior vertex $x \in [u_1, v_2]$ where p and p' separate. Let $(x, H) \in \text{Cor}(T)$ be the corner contained in p. Since $p = p(v_1, F_1) = p(v_2, F_2)$ in $\mathcal{F}$ and since $\mathcal{F}$ is a facet, there exists an arc $a \in \mathcal{F} \backslash \{p\}$ where $a = p(x, H)$ in $\mathcal{F}$. There are two cases: $H \in \text{Reg}(p, F_2)$ or $H \in \text{Reg}(p, F_1)$ (see Figure 6).

Without loss of generality, we assume $H \in \text{Reg}(p, F_2)$. If a contains (v_2, F_2), then $\text{Reg}(p, F_2) = \text{Reg}(p, H) \subsetneq \text{Reg}(a, H) = \text{Reg}(a, F_2)$, contradicting that $p = p(v_2, F_2)$ in $\mathcal{F}$. Thus a does not contain (v_2, F_2). This implies that there exists $y \in [x, v_2]$ such that p and a separate at y. Since a and p are noncrossing and since $p <_{(x,H)} a$, any edge of a that is not an edge of p is only incident to faces in $\text{Reg}(p, F_1)$. We conclude that p' and a cross along $[x, y]$, a contradiction. ☐

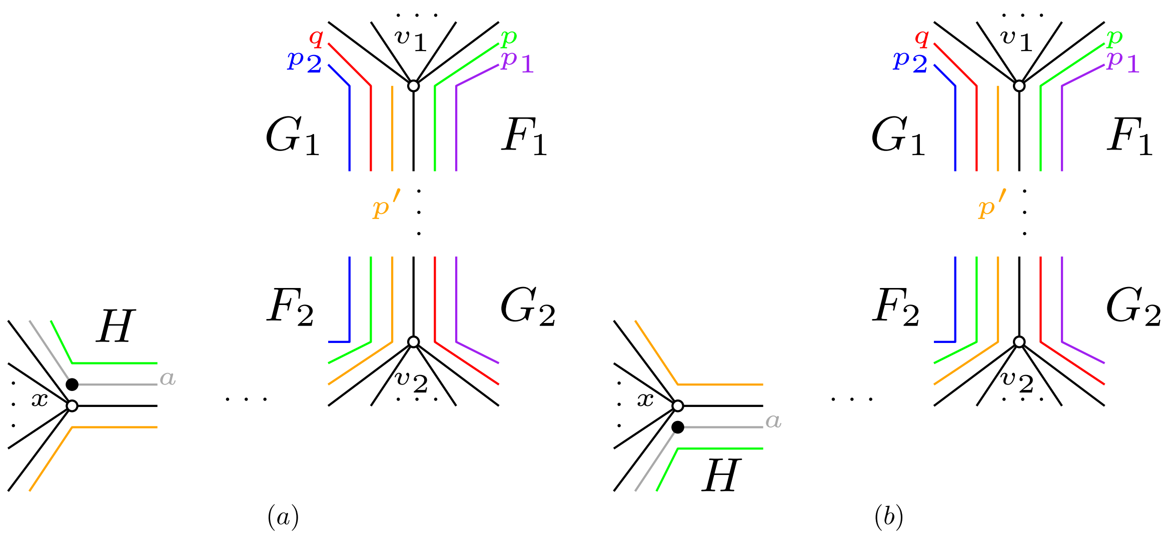


FIGURE 6. The configuration of arcs in the setting of Lemma 3.8 *i)*. Note that in this situation we do not know if p' contains (v_1, F_2) or (v_1, F_1), which is why it appears to terminate at v_1 in (a) and (b). The arc $a = p(x, H)$ has the property that $H \in \text{Reg}(p, F_2)$ or $H \in \text{Reg}(p, F_1)$. We indicate that a is marked at corner (x, H) by marking it with a black dot in (a) and (b).

Corollary 3.9. The simplicial complex $\widetilde{\Delta}^{NC}(T)$ is thin.

Proof. The simplicial complex $\widetilde{\Delta}^{NC}(T)$ is thin since Proposition 3.7 shows that any nonboundary arc may be flipped into a unique other arc. ☐

We now define the following object, which is fundamental to our work in this paper.

Definition 3.10. Let $\mathcal{F}_1, \mathcal{F}_2 \in FG(T)$ and assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are connected by an edge in $FG(T)$. Let $\mathcal{F}_2 = \mu_p \mathcal{F}_1$ where $\mathcal{F}_2 = \mathcal{F}_1 \backslash \{p\} \cup \{q\}$. If $p = p(u, F) = p(v, G)$ and $q = p(u, F') = p(v, G')$, we orient the edge connecting $\mathcal{F}_1$ and $\mathcal{F}_2$ so that $\mathcal{F}_1 \longrightarrow \mathcal{F}_2$ if the corner (u, F') (resp., (v, G')) is immediately clockwise from the corner (u, F) (resp., (v, G)) about vertex u (resp., v). Otherwise, we orient the edge so that $\mathcal{F}_2 \longrightarrow \mathcal{F}_1$. We refer to the resulting directed graph as the **oriented flip graph** of T and denote it by $\overrightarrow{FG}(T)$.

Additionally, any edge of $\overrightarrow{FG}(T)$ connecting $\mathcal{F}$ and $\mu_p \mathcal{F}$ is naturally labeled by the segment determined by the marked corners of p in $\mathcal{F}$ (or in $\mu_p \mathcal{F}$).

Remark 3.11. There is a canonical bijection between the facets of the noncrossing complex and the reduced noncrossing complex given by removing the boundary arcs of a facet of the noncrossing complex. Using this bijection, we will at times equivalently think of vertices of $\overrightarrow{FG}(T)$ as facets of the noncrossing complex rather than as facets of the reduced noncrossing complex.

Example 3.12. In Figure 7, we show the oriented flip graph (without edge labels) of the tree T from Figure 3. Here the vertices are facets of the reduced noncrossing complex $\widetilde{\Delta}^{NC}(T)$.

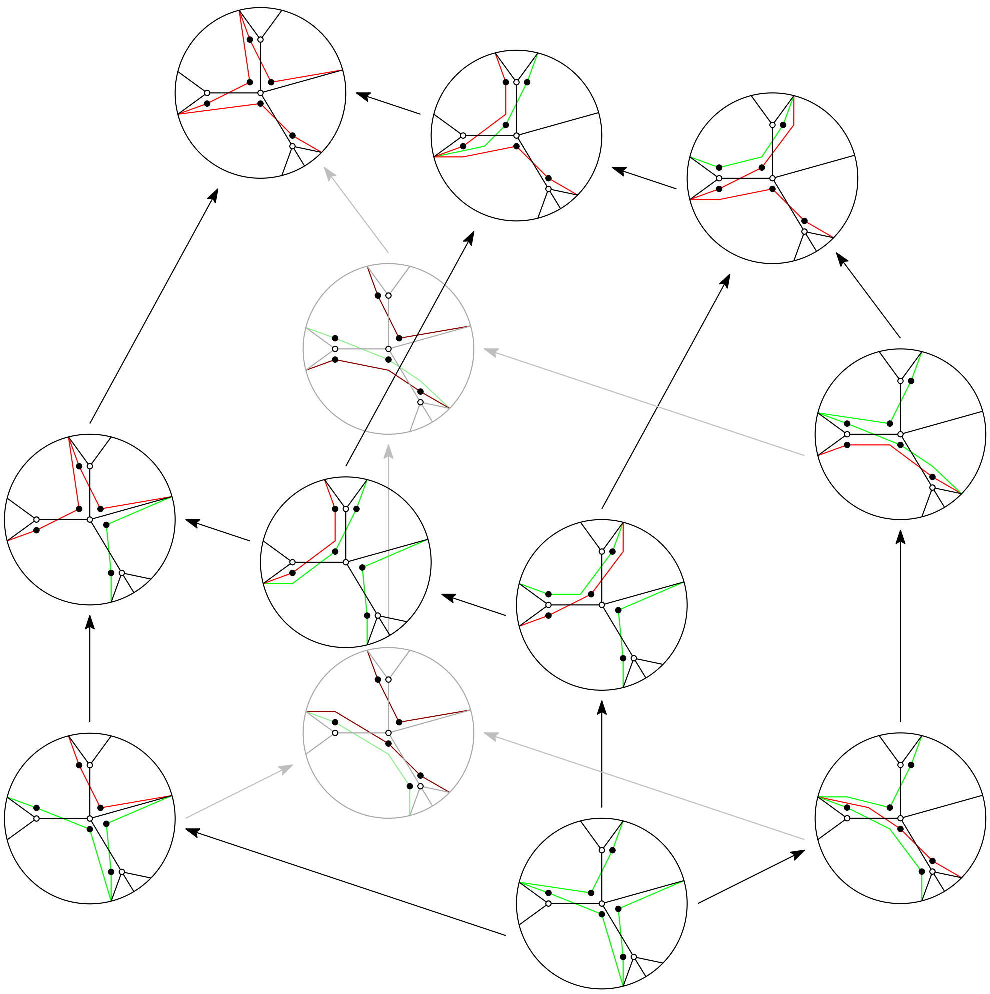


FIGURE 7. An example of an oriented flip graph. Since the vertices are facets of $\widetilde{\Delta}^{NC}(T)$, the boundary arcs are not shown here. Consequently, each face of T has exactly one corner that is not marked.

4. Sublattice and quotient lattice description of the oriented flip graph

In this section, we identify the oriented flip graph $\overrightarrow{FG}(T)$ as both a sublattice and quotient lattice of another lattice. In Section 4.1 we define a closure operator on segments, and introduce a poset of biclosed sets of segments, denoted $\mathrm{Bic}(T)$. It was shown in [18] that $\mathrm{Bic}(T)$ is a congruence-uniform lattice. We define a distinguished lattice congruence Θ on $\mathrm{Bic}(T)$.

In Section 4.3, we define maps $\eta : \mathrm{Bic}(T) \to \overrightarrow{FG}(T)$ and $\phi : \overrightarrow{FG}(T) \to \mathrm{Bic}(T)$. The map η is a surjective lattice map such that $\eta(X) = \eta(Y)$ exactly when $X \equiv Y \mod \Theta$. The map ϕ is a lattice map such that $\eta \circ \phi$ is the identity on $\overrightarrow{FG}(T)$. Since congruence-uniformity and polygonality are preserved by lattice quotient maps, we deduce that $\overrightarrow{FG}(T)$ is a congruence-uniform and polygonal lattice.

4.1. **Biclosed collections of segments.** Let $\mathrm{Seg}(T)$ be the set of segments supported by a tree T. For $X \subseteq \mathrm{Seg}(T)$, we say X is **closed** if given $s, t \in X$ with $s \circ t \in \mathrm{Seg}(T)$, one has $s \circ t \in X$. If X is any subset of $\mathrm{Seg}(T)$, its **closure** $\overline{X}$ is the smallest closed set containing X. Say X is **biclosed** if X and $\mathrm{Seg}(T)\backslash X$ are both closed. For example, the collection of red segments in the left part of Figure 9 is biclosed. We let $\mathrm{Bic}(T)$ denote the poset of biclosed subsets of $\mathrm{Seg}(T)$, ordered by inclusion.

The following theorem may be proved by a minor modification of [18, Theorem 5.4], which is stated in terms of the acyclic paths of a graph. Here, the set $\mathrm{Seg}(T)$ of segments is partially ordered by inclusion of segments.

Theorem 4.1. The poset $\mathrm{Bic}(T)$ is a semidistributive, congruence-uniform, and polygonal lattice. Furthermore:

(1) For $X, Y \in \mathrm{Bic}(T)$, if $X \subsetneq Y$ then there exists $y \in Y \backslash X$ with $X \cup \{y\} \in \mathrm{Bic}(T)$.

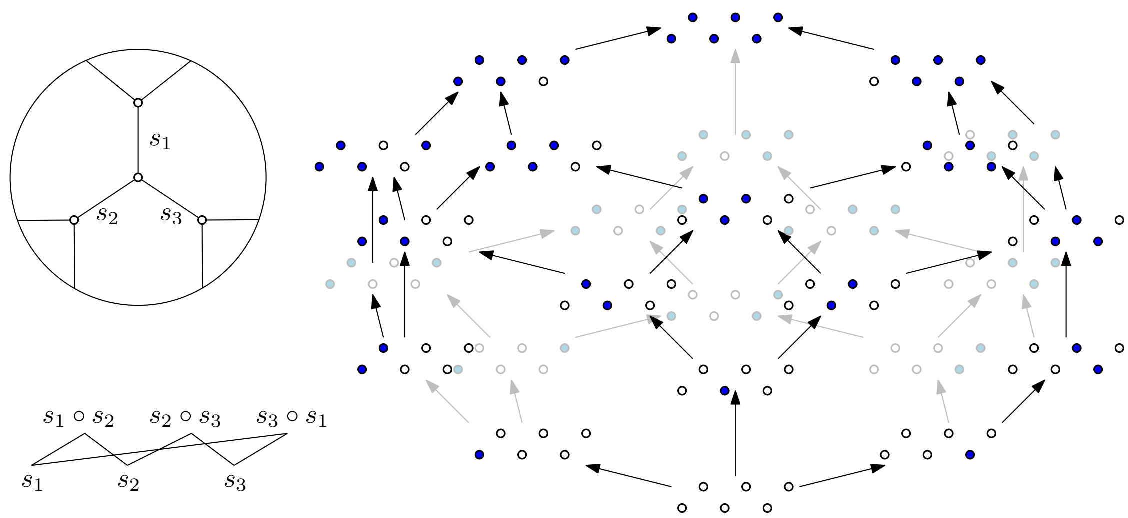


FIGURE 8. A tree T, its poset of segments $\mathrm{Seg}(T)$, and the lattice of biclosed sets of segments $\mathrm{Bic}(T)$.

(2) For $W, X, Y \in \mathrm{Bic}(T)$ with $W \subseteq X \cap Y$, the set $W \cup \overline{(X \cup Y)\backslash W}$ is biclosed.
(3) The edge-labeling $\lambda : \mathrm{Cov}(\mathrm{Bic}(T)) \to \mathrm{Seg}(T)$ where $\lambda(X, Y) = s$ if $Y\backslash X = \{s\}$ is a CN-labeling.

Taking $W = \varnothing$ in (2), we deduce that $X \vee Y = \overline{X \cup Y}$. As complementation is a duality for the lattice of biclosed sets, a dual formula may be given for the meet as follows.

Lemma 4.2. For $X, Y \in \mathrm{Bic}(T)$, we have $X \wedge Y = \mathrm{Seg}(T)\backslash\overline{(\mathrm{Seg}(T)\backslash X) \cup (\mathrm{Seg}(T)\backslash Y)}$.

A lattice of biclosed sets of segments is given in Figure 8 (see also the upper lattice in Figure 7 in [18]). The elements of this lattice of biclosed sets are written as subsets of the poset $\mathrm{Seg}(T)$. The Hasse diagram of this lattice is the skeleton of a zonotope with 26 vertices. Although one can find examples where $\mathrm{Bic}(T)$ is isomorphic to the weak order on permutations, Figure 8 shows this is not true for all trees T.

A poset is **graded** if every maximal chain contains the same number of elements. Theorem 4.1(1) implies that $\mathrm{Bic}(T)$ is graded by cardinality.

Any subset S' of a closure space S inherits a closure operator $X \mapsto (\overline{X} \cap S')$. In general, biclosed subsets of S' may not be biclosed as subsets of S. For spaces of segments, some intervals of $\mathrm{Bic}(T)$ are isomorphic to $\mathrm{Bic}(S')$ for some subset S' of segments. We state this precisely as the following proposition.

Proposition 4.3. Let $W \subseteq \mathrm{Seg}(T)$ be a biclosed set of segments, and let $s_1, \ldots, s_k \in \mathrm{Seg}(T)\backslash W$ such that $W \cup \{s_i\}$ is biclosed for all i. Let $\{B_1, \ldots, B_l\}$ be the finest partition on $\{s_1, \ldots, s_k\}$ such that if $s_i \circ s_j$ is a segment then s_i and s_j lie in the same block. Then the interval $[W, W \cup \overline{\{s_1, \ldots, s_k\}}]$ is isomorphic to $\mathrm{Bic}(\overline{B_1}) \times \cdots \times \mathrm{Bic}(\overline{B_l})$.

Proof. We first prove that the sets $W, \overline{B_1}, \ldots, \overline{B_l}$ are all disjoint. Suppose $W \cap \overline{B_i}$ is nonempty for some i, and let $t \in W \cap \overline{B_i}$ be of minimum length. Since $s_j \notin W$ for all j, t must be a concatenation $t_1 \circ t_2$ of elements of $\overline{B_i}$. By minimality, t_1 and t_2 are not in W. But W is co-closed, a contradiction.

Now suppose there are two blocks, say B_1, B_2, such that $\overline{B_1} \cap \overline{B_2}$ contains an element t. Then t is the concatenation of some elements of B_1 and of some elements of B_2. Relabeling if necessary, let $s_i \in B_i$, $t_i \in \overline{B_i}$ for $i = 1, 2$ such that $s_1 \circ t_1 = t = s_2 \circ t_2$. Then either s_1 is a subsegment of s_2 or vice versa. Without loss of generality, we assume $s_1 \subsetneq s_2$. Let s' be the segment such that $s_1 \circ s' = s_2$. Since $W \cup \{s_1\}$ is closed, s' must not be in W. But s' is in W since $W \cup \{s_2\}$ is co-closed. Hence, we have shown that the closures of the blocks are disjoint.

Since the biclosed property is preserved under restriction, the map $X \mapsto (X \cap \overline{B_1}, \ldots, X \cap \overline{B_l})$ from $[W, W \cup \overline{\{s_1, \ldots, s_k\}}$ to $\mathrm{Bic}(\overline{B_1}) \times \cdots \times \mathrm{Bic}(\overline{B_l})$ is well-defined. It remains to show that the inverse is also well-defined. Namely, given $(X_1, \ldots, X_l) \in \mathrm{Bic}(\overline{B_1}) \times \cdots \times \mathrm{Bic}(\overline{B_l})$, we prove that $W \cup \bigcup_{i=1}^{l} X_i$ is biclosed in $\mathrm{Seg}(T)$. Suppose this does not always hold, and choose $(X_1, \ldots, X_l)$ minimal such that $W \cup \bigcup_{i=1}^{l} X_i$ is not biclosed. Let $X = \bigcup_{i=1}^{l} X_i$. Since $X \neq \varnothing$, there is some nonempty X_j. As $\mathrm{Bic}(\overline{B_j})$ is graded, there is some $s \in X_j$ such that $X_j\backslash\{s\}$ is biclosed. By the minimality assumption, $(W \cup X)\backslash\{s\}$ is biclosed.

Assume $W \cup X$ is not co-closed. Then there exist segments t, t' not in $W \cup X$ such that $s = t \circ t'$. As X_j is co-closed in $\overline{B_j}$, the segment t is not in $\overline{\{s_1, \ldots, s_k\}}$. Since $W \cup \{s_i\}$ is co-closed for any i, the segment s can be factored as $s_i \circ s'$ for some $s_i \in X_j$ and $s' \in \overline{\{s_1, \ldots, s_k\}}$. There are two cases to consider: either t is contained in s_i or s_i is contained in t.

If $t \subsetneq s_i$, then there exists a segment t'' with $t \circ t'' = s_i$. Since $W \cup \{s_i\}$ is co-closed, t'' is in W. However, $t'' \circ s' = t'$, $s' \in W \cup \overline{B_j}$ and $t' \notin W \cup \overline{B_j}$. This contradicts the fact that $W \cup \overline{B_j}$ is closed.

If $s_i \subsetneq t$, then there exists a segment t'' with $s_i \circ t'' = t$. Since $W \cup \{s_i\}$ is closed, t'' is not in W. However, $t'' \circ t' = s'$, $s' \in W \cup \overline{B_j}$ and $t'', t' \notin W \cup \overline{B_j}$. This contradicts the fact that $W \cup \overline{B_j}$ is co-closed.

Now assume $W \cup X$ is not closed. Then there exist segments $s' \in W \cup X$, $t \notin W \cup X$ such that $s \circ s' = t$. Since X_j is closed and segments in blocks B_i with $i \neq j$ cannot be concatenated with s, the segment s' is in W. After relabeling, we may assume $s = s_1 \circ \cdots \circ s_m$ for some $m \leqslant k$. Since $W \cup \{s_m\}$ is closed, the segment $s_m \circ s'$ is in W. Similarly, $s_i \circ \cdots \circ s_m \circ s'$ is in W for any i. This contradicts the assumption that $t \notin W$. □

We may refer to intervals of $\mathrm{Bic}(T)$ as in Proposition 4.3 as **facial intervals**.

4.2. **A lattice congruence on biclosed sets.** In this section, we define a lattice congruence Θ on $\mathrm{Bic}(T)$. The quotient lattice $\mathrm{Bic}(T)/\Theta$ will be shown to be isomorphic to $\overrightarrow{FG}(T)$ in Section 4.3.

Let $s = (v_0, \ldots, v_l)$ be a segment, and orient the segment from v_0 to v_l. Let C_s be the set of segments $(v_i, \ldots, v_j)$ such that

- if $i > 0$ then s turns right at v_i, and
- if $j < l$ then s turns left at v_j.

We note that s is always in C_s since the above conditions are vacuously true. Furthermore if $t \in C_s$, then $C_t \subseteq C_s$. Let K_s be the set of segments $(v_i, \ldots, v_j)$ such that

- if $i > 0$ then s turns left at v_i, and
- if $j < l$ then s turns right at v_j.

From these definitions, it is clear that whenever there is a composition $s = s_1 \circ s_2$, either $s_1 \in C_s$ and $s_2 \in K_s$ or vice versa. Furthermore, if $t \in C_s$, then $C_t \subseteq C_s$. Similarly, if $t \in K_s$, then $K_t \subseteq K_s$. Combining these statements, we obtain the following lemma, which is used frequently in our proofs.

Lemma 4.4. Let $s, t \in \mathrm{Seg}(T)$ such that $t \in C_s$. If $t = t_1 \circ t_2$, then either $t_1 \in C_s$ or $t_2 \in C_s$. The same statement holds replacing C_s with K_s.

Applying Lemma 4.4 recursively, we get a similar result for any decomposition of a segment.

Lemma 4.5. If $s = s_1 \circ \cdots \circ s_l$ is any decomposition of s, then $s_i \in C_s$ and $s_j \in K_s$ for some i, j.

We frequently apply Lemma 4.5 in the special case where $l = 2$, but the statement for general l is sometimes used as well. Another simple yet important fact is that C_s and K_s are biclosed sets.

Lemma 4.6. For any segment s, the sets C_s and K_s are biclosed.

Proof. For a segment s. Let s', t and t' be segments such that $s' = t \circ t'$. If $s' \in C_s$ then either $t \in C_s$ or $t' \in C_s$ by Lemma 4.4. Hence, the set C_s is co-closed.

On the other hand, if t and t' are composable subsegments of s, then they cannot both lie in C_s. To see this, fix an orientation on $s = (v_0, \ldots, v_l)$, and let $t = (v_i, \ldots, v_j)$, $t' = (v_j, \ldots, v_k)$ for some i, j, k where $0 \leqslant i < j < k \leqslant l$. If $t \in C_s$ then s turns right at v_j, whereas if $t' \in C_s$, then s turns left at v_j.

The fact that K_s is biclosed follows from a similar argument. □

Given a tree T embedded in a disk, we let $T^\vee$ be a **reflection** of T (i.e., $T^\vee$ is the image of T under a Euclidean reflection performed on D^2). The choice of reflection is immaterial since the noncrossing complex and oriented flip graph are invariant under rotations of T. The tree $T^\vee$ has the same set of segments and defines the same noncrossing complex as T. Since reflection switches left and right, $\overrightarrow{FG}(T^\vee)$ has the opposite orientation of $\overrightarrow{FG}(T)$, and for any segment s, $C_{s^\vee} = K_s^\vee$. Let $\pi_\downarrow, \pi^\uparrow$ be functions on $\mathrm{Bic}(T)$ such that for $X \in \mathrm{Bic}(T)$,

$$\pi_\downarrow(X) = \{s \in X : \ C_s \subseteq X\}$$

$$\pi^\uparrow(X) = \{s \in \mathrm{Seg}(T) : \ K_s \cap X \neq \varnothing\}$$

These maps are closely related to the maps labeled $\pi_\downarrow$ and $\pi^\uparrow$ in [18]. For completeness, we prove their main properties here.

Lemma 4.7. For $X \in \mathrm{Bic}(T)$, both $\pi_\downarrow(X)$ and $\pi^\uparrow(X)$ are biclosed.

Proof. We first show that $\pi_\downarrow(X)$ is co-closed. Let $s \in \pi_\downarrow(X)$. By definition, this means that $C_s \subseteq X$. If $t \in C_s$, then $C_t \subseteq C_s \subseteq X$, so $t \in \pi_\downarrow(X)$. Hence, $C_s \subseteq \pi_\downarrow(X)$. Now, if $s = t \circ u$, then either $t \in C_s$ or $u \in C_s$, so either $t \in \pi_\downarrow(X)$ or $u \in \pi_\downarrow(X)$. Hence, $\pi_\downarrow(X)$ is co-closed.

Next, we show that $\pi_\downarrow(X)$ is closed. Let $s, t \in \pi_\downarrow(X)$ such that $s \circ t$ is a segment. For $u \in C_{s\circ t}$ if u is a subsegment of s or t, then $u \in C_s$ or $u \in C_t$, respectively. Otherwise, $u = u' \circ u''$ where u' is a subsegment of s and u'' is a subsegment of t. In this case $u' \in C_s$ and $u'' \in C_t$. In either case, $u \in X$ holds. Consequently $s \circ t \in \pi_\downarrow(X)$.

We have now established that $\pi_\downarrow(X)$ is biclosed. The fact that $\pi^\uparrow(X)$ is biclosed may be proved by a similar argument. Alternatively, it follows from the fact that $\pi_\downarrow(X)$ is biclosed and Lemma 4.8(1). ☐

Lemma 4.8. For $X, Y \in \mathrm{Bic}(T)$:

(1) $\pi_\downarrow(\mathrm{Seg}(T^\vee)\backslash X^\vee) = \mathrm{Seg}(T^\vee)\backslash \pi^\uparrow(X)^\vee$,
(2) $\pi_\downarrow(\pi^\uparrow(X)) = \pi_\downarrow(X)$,
(3) $\pi^\uparrow(\pi_\downarrow(X)) = \pi^\uparrow(X)$,
(4) $\pi_\downarrow(X) \subseteq X \subseteq \pi^\uparrow(X)$,
(5) $\pi_\downarrow(\pi_\downarrow(X)) = \pi_\downarrow(X)$,
(6) $\pi^\uparrow(\pi^\uparrow(X)) = \pi^\uparrow(X)$,
(7) if $X \subseteq Y$, then $\pi_\downarrow(X) \subseteq \pi_\downarrow(Y)$ and $\pi^\uparrow(X) \subseteq \pi^\uparrow(Y)$.

Proof. Both (4) and (7) are clear from the definitions. (3) and (6) follow from (2) and (5) by taking the complement of the reflection of X and applying (1). It remains to prove (1), (2), and (5).

For (1), we have the following set of equalities:

$$\begin{aligned}
\pi_\downarrow(\mathrm{Seg}(T^\vee)\backslash X^\vee) &= \{s^\vee \in \mathrm{Seg}(T^\vee)\backslash X^\vee : \ C_{s^\vee} \subseteq \mathrm{Seg}(T^\vee)\backslash X^\vee\} \\
&= \{s^\vee \in \mathrm{Seg}(T^\vee) : \ C_{s^\vee} \subseteq \mathrm{Seg}(T^\vee)\backslash X^\vee\} \\
&= \{s^\vee \in \mathrm{Seg}(T^\vee) : \ K_s^\vee \subseteq \mathrm{Seg}(T^\vee)\backslash X^\vee\} \\
&= \{s^\vee \in \mathrm{Seg}(T^\vee) : \ K_s \subseteq \mathrm{Seg}(T)\backslash X\} \\
&= \mathrm{Seg}(T^\vee)\backslash\{s^\vee \in \mathrm{Seg}(T^\vee) : \ K_s \cap X \neq \varnothing\} \\
&= \mathrm{Seg}(T^\vee)\backslash \pi^\uparrow(X)^\vee.
\end{aligned}$$

For (2), the reverse inclusion is clear. Suppose $\pi_\downarrow(\pi^\uparrow(X)) \neq \pi_\downarrow(X)$ and let $s \in \pi_\downarrow(\pi^\uparrow(X))\backslash \pi_\downarrow(X)$ be of minimum length. Since $C_t \subseteq C_s$ for $t \in C_s$, this implies $s \in \pi^\uparrow(X)\backslash X$. Let $u \in K_s \cap X$. Then either $s = t \circ u, s = u \circ t'$, or $s = t \circ u \circ t'$ holds for some segments $t, t' \in C_s$. But this implies $s \in X$, a contradiction.

For (5), the inclusion $\pi_\downarrow(\pi_\downarrow(X)) \subseteq \pi_\downarrow(X)$ is clear. Let $s \in \pi_\downarrow(X)$. Then $C_s \subseteq X$ holds. If $t \in C_s$, then $C_t \subseteq C_s$ and $t \in \pi_\downarrow(X)$. Consequently, $C_s \subseteq \pi_\downarrow(X)$, so $s \in \pi_\downarrow(\pi_\downarrow(X))$. ☐

For the remainder of the paper, we let Θ be the equivalence relation on $\mathrm{Bic}(T)$ such that $X \equiv Y \mod \Theta$ if $\pi_\downarrow(X) = \pi_\downarrow(Y)$. Using Lemmas 2.2 and 4.8, we deduce the following proposition.

Proposition 4.9. The equivalence relation Θ is a lattice congruence on $\mathrm{Bic}(T)$.

4.3. **Map from biclosed sets to the oriented flip graph.** In this section, we define a surjective map $\eta : \mathrm{Bic}(T) \to \overrightarrow{FG}(T)$ and prove that it is a lattice quotient map.

Let $X \in \mathrm{Bic}(T)$. Given a corner (v, F), let $p_{(v,F)} = p_{(v,F)}(X)$ be the (unique) arc supported by T such that for any interior vertex u of $p_{(v,F)}$ distinct from v, the following condition holds:

- Orienting $p_{(v,F)}$ from v to u, the arc $p_{(v,F)}$ turns left at u if and only if $[v, u]$ is in X.

Observe that $p_{(v,F)}$ contains the corner (v, F). Here, we write $p_{(v,F)}(X)$ to emphasize that $p_{(v,F)}$ does depend on X. However, we will typically just write $p_{(v,F)}$ as it will be clear from the context which biclosed set is playing the role of X.

In Lemmas 4.10 and 4.11, we prove that the this collection of arcs is a facet of the noncrossing complex. Before proving this, we set up some notation.

For an arc $p = (v_0, \ldots, v_l)$ oriented from v_0 to v_l, let C_p be the set of segments $(v_i, \ldots, v_j)$, $0 < i < j < l$ such that

- p turns right at v_i, and
- p turns left at v_j.

Define K_p in the same way, switching the roles of left and right.

Lemma 4.10. Let X and $\{p_{(v,F)}\}_{(v,F)}$ be defined as above. For any $p \in \{p_{(v,F)}\}_{(v,F)}$, $C_p \subseteq X$ and $K_p \cap X = \varnothing$.

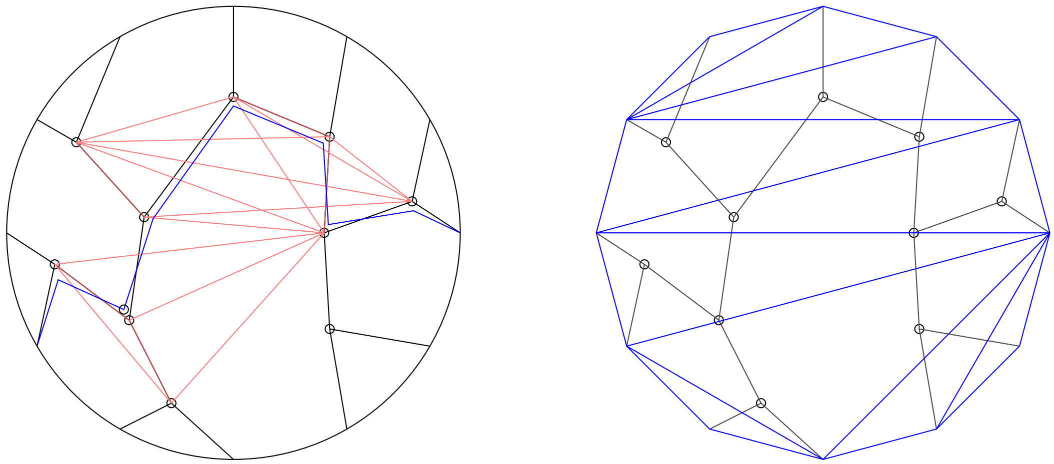


FIGURE 9. (left) A blue arc defined by η at the circled corner with respect to the red biclosed set of segments; (right) The triangulation defined by η

Proof. Let $p = p_{(v,F)}$ for some corner (v, F) of T. Let $s \in C_p$, and set $s = [u, w]$. We show that $s \in X$ by considering several cases based on the location of v relative to s.

Case 1: If v is an endpoint of s, then $s \in X$ by the defining rule of $p_{(v,F)}$.

Case 2: If v is in the interior of s, then $s = [u, v] \circ [v, w]$. Since p extends left through both endpoints of s, both $[u, v]$ and $[v, w]$ are in X. Since X is closed, this implies $s \in X$.

Case 3: If v is not in s, then there exists a segment $[v, u]$ such that $[v, u] \circ [u, w]$ is a segment of p. Since p extends left through both endpoints of s, $[v, w] \in X$ but $[v, u] \notin X$. Since X is co-closed, this implies $s \in X$.

As these three cases exhaust the possibilities for v, we may conclude that $s \in X$ holds, as desired. The fact that $K_p \cap X = \varnothing$ follows from a dual argument. □

Lemma 4.11. Given $X \in \mathrm{Bic}(T)$, the set $\{p_{(v,F)}\}_{(v,F)}$ is a facet of $\Delta^{NC}(T)$. Moreover, $p_{(v,F)}$ is the arc marked at the corner (v, F).

Proof. Let (v, F), (v', F') be two corners of T and let $p_1 = p_{(v,F)}$ and $p_2 = p_{(v',F')}$. Suppose p_1 and p_2 cross along a segment s. By taking s to be the longest segment along which p_1 and p_2 agree, we may assume that p_1 leaves each of the endpoints of s to the right while p_2 leaves s to the left. Then $s \in K_{p_1}$ and $s \in C_{p_2}$. By Lemma 4.10, $K_{p_1} \cap X = \varnothing$ and $C_{p_2} \subseteq X$, a contradiction.

Let $\mathcal{F} = \{p_{(v,F)}\}_{(v,F)}$, and let (v, F) be a corner of T. Let $q \in \mathcal{F}$ be the arc marked at (v, F). That q exists follows from the fact that every corner of T is contained in some arc of $\mathcal{F}$. If $q \neq p_{(v,F)}$, then they agree on some segment $[v, w]$ and diverge at w. Orient both arcs from v to w. Since $p_{(v,F)}$ and q are noncrossing, we know that $p_{(v,F)}$ turns in the same direction at both v and w whereas q turns in different directions. We consider two possible cases for q:

Case 1: If q turns left at v and right at w, then $[v, w] \notin X$ since $K_q \cap X = \varnothing$. As $p_{(v,F)}$ turns left at w, this contradicts the rule defining $p_{(v,F)}$.

Case 2: If q turns right at v and left at w, then $[v, w] \in X$ since $C_q \subseteq X$. As $p_{(v,F)}$ turns right at w, this again contradicts the rule defining $p_{(v,F)}$.

In either case, we obtain a contradiction. Hence, $p_{(v,F)}$ is the arc marked at (v, F).

It remains to show that $\mathcal{F}$ is maximal. If not, then there exist two corners $(v, F), (v', F')$ such that $p_{(v,F)} = p_{(v',F')}$ and $\mathrm{Reg}(p_{(v,F)}, F) = \mathrm{Reg}(p_{(v',F')}, F')$. Let $p = p_{(v,F)}$, and orient the arc p from v toward v'. Since $\mathrm{Reg}(p, F) = \mathrm{Reg}(p, F')$, the arc p turns in the same direction at v and v'. We claim that a contradiction is met if p turns right at both v and v' or left at both vertices.

If p turns right at both v and v', then $[v, v'] \in X$ by the definition of $p_{(v',F')}$ but $[v, v'] \notin X$ by definition of $p_{(v,F)}$. If p turns left at both v and v', then $[v, v'] \in X$ by definition of $p_{(v,F)}$ but $[v, v'] \notin X$ by definition of $p_{(v',F')}$. In either case, we obtain a contradiction. Hence, $\mathcal{F}$ is a facet of $\Delta^{NC}(T)$. □

We let $\eta : \mathrm{Bic}(T) \to \overrightarrow{FG}(T)$ be the map $\eta(X) = \{p_{(v,F)}\}_{(v,F)}$ where $X \in \mathrm{Bic}(T)$ and arcs $p_{(v,F)}$ are defined as above. Recall from Remark 3.11 that even though $\eta(X)$ is a facet of $\Delta^{NC}(T)$, it is uniquely identified with a facet of $\widetilde{\Delta}^{NC}(T)$. Thus it makes sense to regard $\eta(X)$ as an element of $\overrightarrow{FG}(T)$. An example of this map is given in Figure 9.

For any $\mathcal{F} \in \overrightarrow{FG}(T)$, let $\phi(\mathcal{F}) = \overline{\bigcup_{p\in\mathcal{F}} C_p}$.

Lemma 4.12. For $X \in \mathrm{Bic}(T)$ and $\mathcal{F} \in \overrightarrow{FG}(T)$,

(1) $\phi(\eta(X)) = \pi_\downarrow(X)$, and
(2) $\eta(\phi(\mathcal{F})) = \mathcal{F}$.

Proof. (1): Let $X \in \mathrm{Bic}(T)$ be given and set $\mathcal{F} = \eta(X)$. If $s \in \phi(\mathcal{F})$, then there exist $s_1, \ldots, s_l$ such that $s = s_1 \circ \cdots \circ s_l$ and $s_i \in C_p$ for some $p \in \mathcal{F}$. For each i, $C_{s_i} \subseteq C_p \subseteq X$, so $s_i \in \pi_\downarrow(X)$. As $\pi_\downarrow(X)$ is closed, this implies $s \in \pi_\downarrow(X)$. Hence $\phi(\eta(X)) \subseteq \pi_\downarrow(X)$.

We prove the reverse inclusion $\pi_\downarrow(X) \subseteq \phi(\eta(X))$ by induction on the length of the segments in $\pi_\downarrow(X)$. Let $s \in \pi_\downarrow(X)$ and assume that $t \in \phi(\eta(X))$ whenever $t \in C_s$ and $t \neq s$. Let v be an endpoint of s. Orienting s away from v, let F be the face to the right of s that is incident to v. Let $p = p_{(v,F)}$ and orient p in the same direction as s. Let v' be the last vertex along s at which s and p meet. Let $t = [v, v']$. If v' is an endpoint of s, then p must turn left at v' by definition, and $s \in C_p$. If v' is not an endpoint of s, we consider two cases:

(i) If s turns left at v', then $t \in C_s$. By the inductive hypothesis, $t \in \phi(\eta(X))$ holds, which contradicts the definition of $p_{(v,F)}$.

(ii) If s turns right at v', then $s = t \circ t'$ and $t \in C_p$. Since $t' \in C_s$, $t' \in \phi(\eta(X))$. Hence $s \in \phi(\eta(X))$ holds.

(2): Let $\mathcal{F} \in \overrightarrow{FG}(T)$ and set $X = \phi(\mathcal{F})$. Let (v, F) be a corner of T. Let p be the arc in $\eta(\phi(\mathcal{F}))$ marked at (v, F) and let q be the arc in $\mathcal{F}$ marked at (v, F). We prove that $p = q$ and conclude that $\eta(\phi(\mathcal{F})) = \mathcal{F}$.

Suppose p and q diverge at some vertex v'. Orient both paths from v to v'. Let $s = [v, v']$.

Assume p turns left at v' and q turns right at v'. By the definition η, the fact that p turns left at v' implies that $s \in \phi(\mathcal{F})$ holds. By the definition of ϕ, there must exist segments $s_1, \ldots, s_l$ such that $s = s_1 \circ \cdots \circ s_l$ and $s_i \in C_{q_i}$ for some arcs $q_i \in \mathcal{F}$. Orient each q_i in the same direction as q. We may assume $v \in s_1$ and $v' \in s_l$. Let v_i be the first vertex of s_i for each i. Since q_1 and q do not cross and q is marked at (v, F), we conclude that both q_1 and q turn left at v_2. By similar reasoning, q_2 and q both turn left at v_3. By induction, q turns left at v', a contradiction.

Now assume p turns right at v' and q turns left. Then $s \notin \phi(\mathcal{F}) = \overline{\bigcup_{q'\in\mathcal{F}} C_{q'}}$. Since $s \notin C_q$, q must turn left at v. Let F' be the face to the right of q containing v and the first edge of s. Let q' be the arc of $\mathcal{F}$ marked at (v, F'). Then q' and q agree after v. Hence $s \in C_{q'}$, a contradiction. □

By Lemma 4.12, the equivalence relation on $\mathrm{Bic}(T)$ induced by η is equal to Θ. That is, $X \equiv Y \mod \Theta$ holds if and only if $\eta(X) = \eta(Y)$. By Proposition 4.9, we may identify the facets of the noncrossing complex with the elements of the quotient lattice $\mathrm{Bic}(T)/\Theta$. It remains to show that this ordering is isomorphic to $\overrightarrow{FG}(T)$. To this end, it is enough to check that the Hasse diagram of $\mathrm{Bic}(T)/\Theta$ is $\overrightarrow{FG}(T)$, as in the following lemma. Recall the edge-labeling of the oriented flip graph from Definition 3.10.

Lemma 4.13. The Hasse diagram of $\overrightarrow{FG}(T)$ is isomorphic to that of $\mathrm{Bic}(T)/\Theta$. More precisely, we have the following.

(1) Let $X \in \mathrm{Bic}(T)$ such that $X = \pi_\downarrow(X)$. If s is a segment in X such that $X \backslash \{s\}$ is biclosed, then $\eta(X\backslash\{s\}) \xrightarrow{s} \eta(X)$.
(2) Let $\mathcal{F} \in \overrightarrow{FG}(T)$. If $\mathcal{F}\backslash\{p\} \cup \{\tilde{p}\} \xrightarrow{s} \mathcal{F}$ for some arcs $p, \tilde{p}$ and segment s, then $\phi(\mathcal{F})\backslash\{s\}$ is biclosed and $\eta(\phi(\mathcal{F})\backslash\{s\}) = \mathcal{F}\backslash\{p\} \cup \{\tilde{p}\}$.

Proof. (1): Let $X \in \mathrm{Bic}(T)$ such that $X = \pi_\downarrow(X)$. Let s be a segment in X such that $X\backslash\{s\}$ is biclosed. Then $C_s \subseteq X$ since $X = \pi_\downarrow(X)$. If $t \in K_s\backslash\{s\}$, then there exist (possibly empty) segments $s_1, s_2 \in C_s$ such that $s = s_1 \circ t \circ s_2$. Since $s_1, s_2 \in X$ and $X\backslash\{s\}$ is biclosed, this means $t \notin X$. Hence, $K_s \cap X = \{s\}$.

We identify two arcs p and $\tilde{p}$ that are involved in the flip between $\eta(X)$ and $\eta(X\backslash\{s\})$ as follows. Let v, v' be the endpoints of s. Orient s from v to v'. Let F be the face to the right of s incident to v and the first edge of s. Let p be the arc of $\eta(X)$ marked at (v, F). Since $C_s \subseteq X$ and $K_s \cap X = \{s\}$, the arc p contains s and turns left at v'. Let F' be the face left of s incident to v' and the last edge of s. Let p' be the arc of $\eta(X)$ marked at (v', F'). Reversing the orientation on s, the previous argument implies that p' contains s.

We claim that $p = p'$. If not, then p and p' must diverge at a vertex v''. Let $t = [v', v'']$ and $u = [v, v'']$. Without loss of generality, we may assume that $s \circ t = u$. Since p and p' diverge at v'', the definition of η implies that either $t \in X$ or $u \in X$, but not both. Since X is closed, we must have $u \in X$ and $t \notin X$. But then $X\backslash\{s\}$ is not co-closed, a contradiction.

Let $\tilde{p}$ be the arc obtained by flipping p in $\eta(X)$. Then p and $\tilde{p}$ meet along s. We show that $\eta(X\backslash\{s\}) = \eta(X)\backslash\{p\} \cup \{\tilde{p}\}$.

Let G be the face left of s containing v and the first edge of s. Similarly, let G' be the face right of s containing v' and the last edge of s. Let q be the arc marked at (v, G) in $\eta(X)$, and let q' be the arc marked at (v', G') in $\eta(X)$.

By the definition of η, the only arcs that can be different between $\eta(X)$ and $\eta(X\backslash\{s\})$ are those arcs marked at $(v, F), (v, G), (v', F')$, or (v', G'). Just as we proved that p is the arc in $\eta(X)$ marked at (v, F) and (v', F'), a similar argument shows that $\tilde{p}$ is the arc in $\eta(X\backslash\{s\})$ marked at (v, G) and (v', G').

We show that q is in $\eta(X\backslash\{s\})$ and is marked at (v', F'). Similarly we claim that q' is in $\eta(X\backslash\{s\})$ and is marked at (v, F). As these two proofs are nearly identical, we only write the first.

Let $\tilde{q}$ be the arc in $\eta(X\backslash\{s\})$ marked at (v, F). We claim that $\tilde{q} = q'$. Starting from v and heading towards v', the arc $\tilde{q}$ is the same as p until reaching v' by the definition of η. Then since s was removed from X, the arc $\tilde{q}$ turns right at v'.

Suppose $\tilde{q}$ diverges from q' at a vertex v'' where v'' is reached after v'. Let t denote the segment $[v', v'']$. If $\tilde{q}$ turns left at v'', then $s \circ t$ is in X. As $X\backslash\{s\}$ is biclosed, this implies $t \in X$. But then q' turns left at v'', contradicting the assumption that q' and $\tilde{q}$ diverge at v''. If $\tilde{q}$ instead turns right at v'', then $s \circ t$ is not in X. Since X is closed, this implies t is not in X. But then q' turns right at v'', again contradicting the assumption about v''.

Finally, we may consider the possibility that $\tilde{q}$ diverges from q' at a vertex v'' before v. This possibility is disproven in the same manner as above, switching the roles of q' and $\tilde{q}$.

We have now established that $\eta(X)$ and $\eta(X\backslash\{s\})$ differ by at most one arc. Indeed, they must differ by exactly one arc since $X = \pi_{\downarrow}(X)$ is the minimum element in its equivalence class. Hence, $\eta(X\backslash\{s\}) = \eta(X)\backslash\{p\} \cup \{\tilde{p}\}$ holds, as desired.

(2): Let $\mathcal{F} \in \overrightarrow{FG}(T)$. Assume $\mathcal{F}\backslash\{p\} \cup \{\tilde{p}\} \xrightarrow{s} \mathcal{F}$ for some arcs $p, \tilde{p}$ and segment s. Then $s \in C_p$, so $s \in \phi(F)$.

Suppose $\phi(\mathcal{F})\backslash\{s\}$ is not closed. Then there exist segments $t, u \in \phi(\mathcal{F})$ such that $t \circ u = s$. We may assume $t \in K_s$ and $u \in C_s$. Let v, v' be the endpoints of t. Assume t and u meet at v', and orient the arcs containing t from v to v'. Let $t_1, \ldots, t_l$ be segments such that $t = t_1 \circ \cdots \circ t_l$ and $t_i \in C_{p_i}$ for some arcs $p_i \in \mathcal{F}$. We assume v is in t_1 and v' is in t_l. For each i, let v_i be the first vertex in t_i with the orientation induced by t. Since p_1 and $\tilde{p}$ do not cross along t_1, $\tilde{p}$ must turn left at v_2. Similarly, $\tilde{p}$ turns left at $v_3, \ldots, v_l$. But since $t \in K_s$, $\tilde{p}$ turns right at v', so it crosses p_l, a contradiction. We deduce that $\phi(\mathcal{F})\backslash\{s\}$ is closed.

Suppose $\phi(\mathcal{F})\backslash\{s\}$ is not co-closed. Then there exist segments t, u such that $t \notin \phi(\mathcal{F})$, $u \in \phi(\mathcal{F})$, and $s \circ t = u$. Since $\pi_{\downarrow}(\phi(\mathcal{F})) = \phi(\mathcal{F})$, we deduce that $s \in C_u$ and $t \in K_u$. Let $u_1, \ldots, u_l$ be segments with arcs $p_1, \ldots, p_l$ in $\mathcal{F}$ such that $u_i \in C_{p_i}$ and $u = u_1 \circ \cdots \circ u_l$. Orient u from u_1 to u_l. By similar reasoning as before, since $\tilde{p}$ and p_i do not cross along u_i for each i, if u_i is a subsegment of s, then $\tilde{p}$ turns left at the end of u_i. As $t \notin \phi(\mathcal{F})$, there exists some segment u_j that is neither a subsegment of s or t. Let v' be the common endpoint of s and t, and let v be the endpoint of u_j contained in s. Since $s \in C_u$, u_j turns left at v. Hence, p_j turns right at v and left at v', whereas $\tilde{p}$ turns left at v and right at v'. But this means $\tilde{p}$ and p_j cross along $[v, v']$, an impossibility.

Therefore, $\phi(\mathcal{F})\backslash\{s\}$ is biclosed. From (1), the equality $\eta(\phi(\mathcal{F})\backslash\{s\}) = \mathcal{F}\backslash\{p\} \cup \{\tilde{p}\}$ holds. □

Theorem 4.14. The maps η and ϕ identify $\overrightarrow{FG}(T)$ as a quotient lattice and a sublattice of $\mathrm{Bic}(T)$ as follows.

(1) The map η is a surjective lattice map such that $\eta(X) = \eta(Y)$ if and only $X \equiv Y \mod \Theta$.
(2) The map ϕ is an injective lattice map whose image is $\pi_{\downarrow}(\mathrm{Bic}(T))$.

Proof. We have already established that η is lattice quotient map. It remains to show that ϕ preserves the lattice operations.

Let $\mathcal{F}, \mathcal{F}' \in \overrightarrow{FG}(T)$. Since η is a lattice map,

$$\begin{aligned}\mathcal{F} \vee \mathcal{F}' &= \eta(\phi(\mathcal{F})) \vee \eta(\phi(\mathcal{F}')) \\ &= \eta(\phi(\mathcal{F}) \vee \phi(\mathcal{F}')) \\ &\leqslant \eta(\phi(\mathcal{F} \vee \mathcal{F}')) \\ &= \mathcal{F} \vee \mathcal{F}'.\end{aligned}$$

Hence, $\eta(\phi(\mathcal{F}) \vee \phi(\mathcal{F}')) = \eta(\phi(\mathcal{F} \vee \mathcal{F}'))$. Since $\phi(\mathcal{F} \vee \mathcal{F}')$ is minimal in its Θ-equivalence class, $\phi(\mathcal{F} \vee \mathcal{F}') \leqslant \phi(\mathcal{F}) \vee \phi(\mathcal{F}')$. Since ϕ is order-preserving, the reverse inequality also holds. Thus, ϕ preserves joins.

Since ϕ is order-preserving, $\phi(\mathcal{F} \wedge \mathcal{F}') \leqslant \phi(\mathcal{F}) \wedge \phi(\mathcal{F}')$ holds. Let $X = \phi(\mathcal{F}) \wedge \phi(\mathcal{F}')$. Since

$$\eta(\phi(\mathcal{F} \wedge \mathcal{F}')) = \mathcal{F} \wedge \mathcal{F}' = \eta\phi(\mathcal{F}) \wedge \eta\phi(\mathcal{F}'),$$

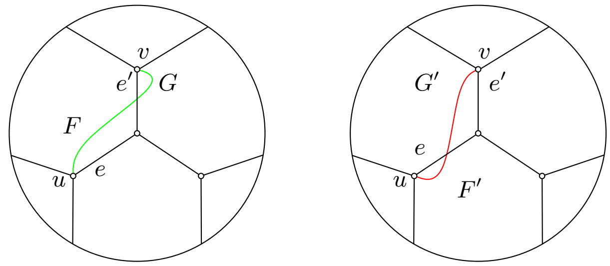


FIGURE 10. A green admissible curve and a red admissible curve for the segment $[u, v]$

it suffices to show that $\pi_\downarrow(X) = X$.

Let $s \in X$ and $t \in C_s$. Since $\phi(\mathcal{F}) = \pi_\downarrow(\phi(\mathcal{F}))$, $C_s \subseteq \phi(\mathcal{F}) \cap \phi(\mathcal{F}')$. By Lemma 4.2, if $t \notin X$ then there exist $u_1, \ldots, u_l \notin \phi(\mathcal{F}) \cap \phi(\mathcal{F}')$ such that $t = u_1 \circ \cdots \circ u_l$. But $u_i \in C_t$ for some i by Lemma 4.5. Since $C_t \subseteq C_s$, we deduce $u_i \in \phi(\mathcal{F}) \cap \phi(\mathcal{F}')$, a contradiction. □

By Lemma 2.4 and Theorem 4.1(3), it follows that the labeling $\mathcal{F}' \xrightarrow{s} \mathcal{F}$ of the covering relations of $\overrightarrow{FG}(T)$ by segments is a CN-labeling. To see that this is a CU-labeling, we observe that if there is a flip $\mathcal{F}' \xrightarrow{s} \mathcal{F}$, then $\eta(C_s) \leqslant \mathcal{F}$. The following corollary is a consequence of Proposition 2.9.

Corollary 4.15. The canonical join-representation of a element $\mathcal{F} \in \overrightarrow{FG}(T)$ is

$$\mathcal{F} = \bigvee_{\substack{s \in S \\ \exists \mathcal{F}' \xrightarrow{s} \mathcal{F}}} \eta(C_s).$$

5. Noncrossing tree partitions

In this section, we introduce noncrossing tree partitions, which are partitions of the interior vertices of a tree embedded in a disk whose blocks are noncrossing as defined in Section 5.1. In Section 5.2, we define a bijection on the set of noncrossing tree partitions, which we call Kreweras complementation. The equivalence of this definition of Kreweras complementation with the lattice-theoretic definition in Section 2.1 is given in Section 5.3. Our main result in this section is that the lattice of noncrossing tree partitions is isomorphic to the shard intersection order of $\overrightarrow{FG}(T)$, which we prove in Section 5.5.

5.1. **Admissible curves.** Fix a tree $T = (V, E)$ embedded in a disk D^2 with the Euclidean metric. Let V^o denote the set of interior vertices of T. We fix a small $\epsilon > 0$ such that the ϵ-ball centered at any interior vertex of T is contained in D^2, and no two such ϵ-balls intersect. For each corner (v, F), we fix a point $z(v, F)$ in the interior of F of distance ϵ from v. Let

$$T_\epsilon = T \cup \bigcup_{v \in V^o} \{x \in D^2 : \ |x - v| < \epsilon\}.$$

In words, T_ϵ is the embedded tree T plus the open ϵ-ball around each interior vertex. If s is a segment of T, let s_ϵ denote the set of points on an edge of s of distance at least ϵ from any interior vertex of T.

It will be convenient to represent segments as certain curves in the disk as follows. A **flag** is a triple (v, e, F) of a vertex v incident to an edge e, which is incident to a face F. Orienting e away from v, we say a flag is **green** if F is left of e. Otherwise, the flag is **red**. Let (u, e, F), (v, e', G) be two green flags such that $[u, v]$ is a segment containing the edges e, e' as in Figure 10. A **green admissible curve** $\gamma : [0, 1] \to D^2$ for $[u, v]$ is a simple curve for which $\gamma(0) = z(u, F)$, $\gamma(1) = z(v, G)$ and $\gamma([0, 1]) \subseteq D^2 \backslash (T_\epsilon \backslash [u, v]_\epsilon)$. Similarly, if (u, e, F') and (v, e', G') are red flags, then a **red admissible curve** is defined the same way, with $\gamma(0) = z(u, F')$, $\gamma(1) = z(v, G')$. We say a segment is **green** if it is represented by a green admissible curve. Similarly, a segment is **red** if it is represented by a red admissible curve. We may also refer to an **admissible curve** for a segment without specifying a color. Such a curve may be either green or red.

If a colored segment s is represented by a curve with endpoints $z(u, F)$ and $z(v, G)$, we say that (u, F) and (v, G) are the **endpoints** of s, and we write $\mathrm{Endpt}(s) = \{(u, F), (v, G)\}$. If S is a collection of colored segments, we let $\mathrm{Endpt}(S) = \bigcup_{s \in S} \mathrm{Endpt}(s)$. We refer to corners or vertices as the endpoints of a segment at different parts of this paper. The distinction should be clear from the context.

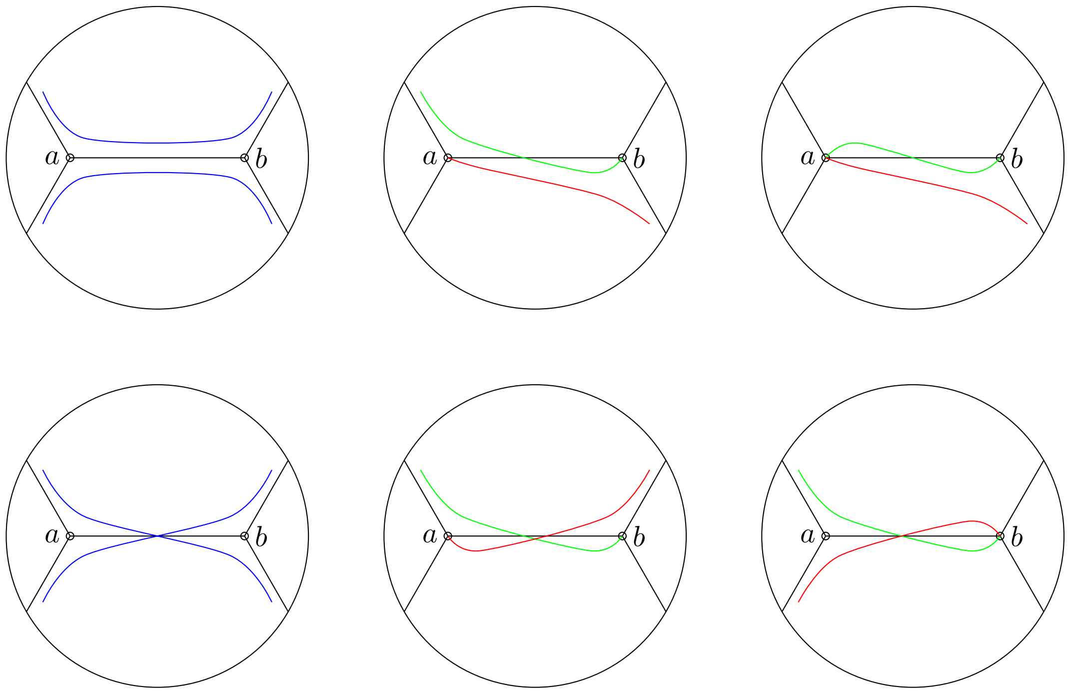


Figure 11. Several examples of crossing and noncrossing admissible curves representing segments supported by the tree.

Given an interior vertex $v \in V^o$ incident to faces F and F', let $\alpha_v^{F,F'} : [0,1] \to D^2$ be a simple path contained in $F \cup F'$ with $\alpha_v^{F,F'}(0) = z(v,F)$, $\alpha_v^{F,F'}(1) = z(v,F')$ and $|\alpha_v^{F,F'}(t) - v| = \epsilon$ for $t \in [0,1]$. We use the paths $\alpha_v^{F,F'}$ to concatenate admissible curves.

Two colored segments are **noncrossing** if they admit admissible curves that do not intersect. Otherwise, they are **crossing**. We remark that if two curves share an endpoint $z(u,F)$ then they are considered to be crossing. To determine whether two colored segments s,t cross, one must check whether the endpoints of t lie in different connected components of $(D^2 \backslash (T_\epsilon \backslash t_\epsilon)) \backslash \gamma$ for some admissible curve γ for s. We will find it convenient to distinguish several cases of crossings as in the following lemma. The first three cases correspond to the three columns of Figure 11.

Lemma 5.1. Let γ and γ' be two (red or green) admissible curves corresponding to segments s and s' that meet along a common segment t. Let $t = [a,b]$ and orient γ and γ' from a to b. Assume that γ and γ' do not share a corner. Then γ and γ' are noncrossing if and only if one of the following holds:

(1) γ (or γ') does not share an endpoint with t, and γ turns left (or right) at both endpoints of t;
(2) γ starts at a and turns left (resp., right) at b, and γ' ends at b and turns right (resp., left) at a;
(3) γ and γ' both start at a (resp., both end at b) where γ leaves a to (resp., arrives at b from) the left, and γ turns left at b (resp., a) or γ' turns right at b (resp., a).
(4) γ and γ' both start at a and end at b but are different colors.

If γ and γ' are both red admissible or both green admissible, then the third case does not occur.

Lemma 5.2. If red segments s and s' are noncrossing, then $K_s \cap C_{s'}$ is empty.

Proof. Suppose $K_s \cap C_{s'}$ contains an element $[u,v]$. Orient s and s' from u to v. Then s either starts at u or turns left at u, and it either ends at v or turns right at v. On the other hand, s' either starts at u or turns right at u, and it either ends at v or turns left at v. In each case, the segments s and s' are crossing. □

By Proposition 5.8, the same result holds for two green segments. For segments of different colors, we have the following lemma. Its proof is omitted as it is similar to that of Lemma 5.2.

Lemma 5.3. If a green segment s and a red segment s' are noncrossing, then $K_s \cap C_{s'}$ is empty.

We remark that Lemma 5.3 is asymmetric in red and green.

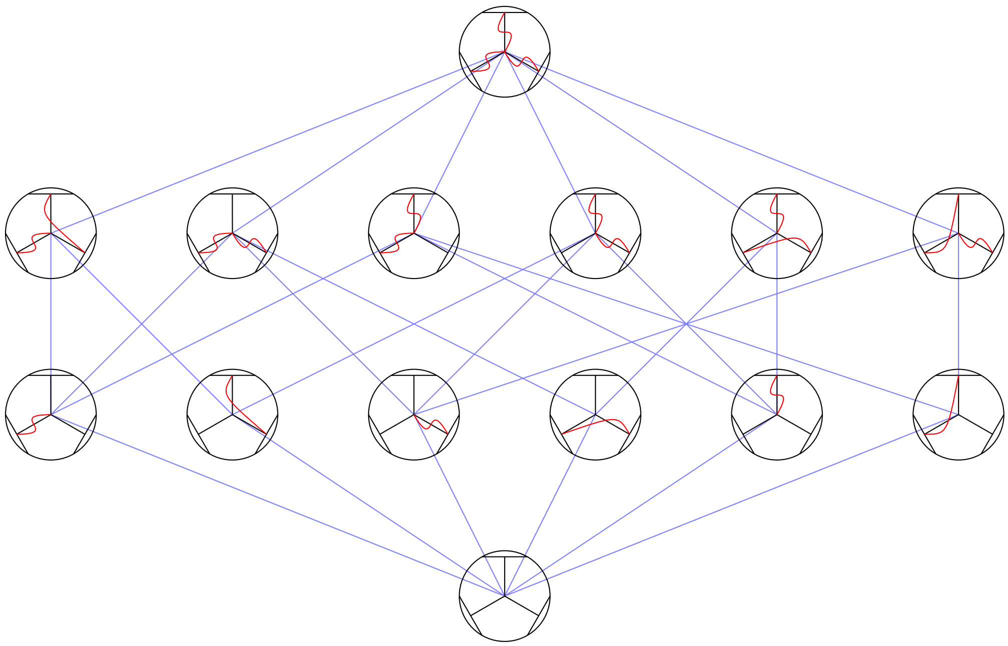


FIGURE 12. A lattice of noncrossing tree partitions

For $B \subseteq V^o$, let $\mathrm{Seg}(B)$ be the set of **inclusion-minimal** segments whose endpoints lie in B. That is, there do not exist distinct segments $s, t \in \mathrm{Seg}(B)$ where t is a subsegment of s. We say B is **segment-connected** if for any two elements u, v of B, there exists a sequence $u = u_0, \ldots, u_N = v$ of elements of B such that $[u_{i-1}, u_i] \in \mathrm{Seg}(B)$ for all i. If $\mathbf{B} = \{B_1, \ldots, B_l\}$ is a partition of V^o, we let $\mathrm{Seg}(\mathbf{B}) = \bigcup_{i=1}^{l} \mathrm{Seg}(B_i)$. We let $\mathrm{Seg}_g(\mathbf{B})$ (resp., $\mathrm{Seg}_r(\mathbf{B})$) denote the same set of segments, all colored green (resp., red).

A **noncrossing tree partition B** is a set partition of V^o such that any two segments of $\mathrm{Seg}_r(\mathbf{B})$ are noncrossing and each block of $\mathbf{B}$ is segment-connected. Note that we intentionally define noncrossing tree partitions using only red segments. Let $\mathrm{NCP}(T)$ be the poset of noncrossing tree partitions of T, ordered by refinement. We give an example of $\mathrm{NCP}(T)$ in Figure 12 where T is the tree whose biclosed sets appear in Figure 8. We remark that the lattice of noncrossing tree partitions is not isomorphic to the lattice of noncrossing set partitions in this example.

We record some basic properties of noncrossing tree partitions in the rest of this section.

Lemma 5.4. Let $\mathbf{B}$ be a noncrossing tree partition containing a block B. If $u, v \in B$ are distinct vertices such that $[u, v]$ is not a segment, then there exists a vertex $w \in B$ distinct from u and v such that $w \in [u, v]$.

Proof. Let $w \in [u, v]$ such that $[u, w]$ is a segment of maximum length. Since B is segment-connected, there exists a sequence $u = u_0, u_1, \ldots, u_l = v$ of elements of B such that $[u_{i-1}, u_i]$ is a segment for all i. We further assume that each segment $[u_{i-1}, u_i]$ is in $\mathrm{Seg}(B)$ and that l is minimal with this property.

Since w is in $[u, v]$, there exists some segment $[u_{i-1}, u_i]$ containing w such that $u_{i-1} \in [u, w]$. Then $w \in [u_i, u_{i+1}]$, so the noncrossing property forces $w \in B$. □

Lemma 5.5. Let $\mathbf{B}$ be a noncrossing tree partition. If B is a block of $\mathbf{B}$, then for any distinct vertices $u, v \in B$, there exists a sequence $u = u_0, \ldots, u_l = v$ such that $[u_{i-1}, u_i]$ is in $\mathrm{Seg}(B)$ for all i and $[u, v] = [u_0, u_1] \circ \cdots \circ [u_{l-1}, u_l]$.[1]

Proof. Let B be a block of $\mathbf{B}$ with at least two elements, and fix distinct vertices $u, v \in B$. We proceed by induction on the length of $[u, v]$. Among the vertices of $[u, v]$, let u_1 be the element of $B \backslash \{u\}$ minimizing the length of $[u, u_1]$. By Lemma 5.4, $[u, u_1]$ is a segment. By assumption, it is inclusion-minimal, so it is in $\mathrm{Seg}(B)$. If $u_1 \neq v$, then by the inductive hypothesis, there exists a sequence $u_1, u_2, \ldots, u_l$ of elements of B such that $u_l = v$ and $[u_{i-1}, u_i] \in \mathrm{Seg}(B)$ for all i. □

Lemma 5.6. Let b be a vertex in a green segment $[a, c]$, and let $[d, e]$ be some green segment that crosses $[a, c]$. Then either $[a, b]$ or $[b, c]$ crosses $[d, e]$, where $[a, b]$ and $[b, c]$ are both green.

[1] Note that $[u, v]$ is an acyclic path, but not necessarily a segment.

Proof. Suppose neither $[a,b]$ nor $[b,c]$ crosses $[d,e]$. Let γ_1, γ_2 and γ' be green admissible curves for $[a,b], [b,c]$, and $[d,e]$, respectively. Let e, e' be the edges of $[a,b]$ and $[b,c]$ incident to b. Orienting e and e' away from b, let F be the face left of e and F' the face left of e'. Then γ_1 (resp., γ_2) has an endpoint at $z(b,F)$ (resp., $z(b,F')$). Since $[a,c]$ is a segment, the faces F and F' are adjacent. Let $\gamma = \gamma_1 \circ \alpha_b^{F,F'} \circ \gamma_2$. Then γ is a green admissible curve for $[a,c]$ and γ does not intersect γ', a contradiction. ☐

5.2. **Kreweras complementation.** In this section, we define a bijection on $\mathrm{NCP}(T)$, which we call Kreweras complementation.

We define a function $\rho : \overrightarrow{FG}(T) \to \mathrm{NCP}(T)$ as follows. Let $\mathcal{F} \in \overrightarrow{FG}(T)$, and let S be the set of segments for which there exists $\mathcal{F}'$ with $\mathcal{F}' \xrightarrow{s} \mathcal{F}$. Since the arcs in $\mathcal{F}$ are pairwise noncrossing, there exists a realization by simple curves $\{\gamma_p : p \in \mathcal{F}\}$ such that the following conditions hold.

- If s is the largest segment contained in an arc $p \in \mathcal{F}$, then the image of γ_p is contained in $D^2 \backslash (T_\epsilon \backslash s_\epsilon)$.
- For distinct $p, q \in \mathcal{F}$, γ_p and γ_q are disjoint except possibly at the endpoints.
- For $p \in \mathcal{F}$, if γ_p is marked at (v,F), then γ_p contains the point $z(v,F)$.

For $s \in S$, let p be the arc marked at the endpoints of s. If p is marked at the corners (v,F), (v',F'), we let γ_s be the subpath of γ_p with endpoints $z(v,F)$ and $z(v',F')$. Since s is a lower label of $\mathcal{F}$, the curve γ_s is a red admissible curve for s. Since γ_p and γ_q are disjoint for distinct arcs p, q, the collection $\{\gamma_s : s \in S\}$ is a noncrossing set of red admissible curves. Hence, S defines a noncrossing tree partition $\mathbf{B}$ where $S = \mathrm{Seg}(\mathbf{B})$.

Given $\mathcal{F}$ and $\mathbf{B}$ as above, we set $\rho(\mathcal{F}) = \mathbf{B}$. We prove that ρ is a bijection.

Proposition 5.7. The map ρ is a bijection.

Proof. This proof makes frequent use of Lemmas 4.4 and 4.5.

Given $\mathbf{B} \in \mathrm{NCP}(T)$, let

$$\tilde{\phi}(\mathbf{B}) = \overline{\bigcup_{s \in \mathrm{Seg}(\mathbf{B})} C_s}.$$

Since C_s is a biclosed set, $\tilde{\phi}(\mathbf{B})$ is biclosed by Theorem 4.1. By Corollary 4.15, we have $\phi = \tilde{\phi} \circ \rho$. Since $\eta \circ \phi$ is the identity on $\overrightarrow{FG}(T)$, we have $(\eta \circ \tilde{\phi}) \circ \rho = \eta \circ \phi = \mathrm{id}_{\overrightarrow{FG}(T)}$. Hence, ρ is an injective function.

To show that ρ is surjective, it suffices to prove that $\tilde{\phi}$ is injective and its image is $\pi_\downarrow(\mathrm{Bic}(T))$. The latter statement is clear since $C_s \in \pi_\downarrow(\mathrm{Bic}(T))$ for any segment s, and $\pi_\downarrow(\mathrm{Bic}(T))$ is a sublattice of $\mathrm{Bic}(T)$ by Theorem 4.14.

Let $\mathbf{B} \in \mathrm{NCP}(T)$ and set $X = \tilde{\phi}(\mathbf{B})$. Let $S = \{s \in \mathrm{Seg}(T) : X \backslash \{s\} \in \mathrm{Bic}(T)\}$. To prove that $\tilde{\phi}$ is injective, we show that $\mathrm{Seg}(\mathbf{B}) = S$.

Suppose $S \backslash \mathrm{Seg}(\mathbf{B})$ is nonempty, and let $t \in S \backslash \mathrm{Seg}(\mathbf{B})$. Since $X \backslash \{t\}$ is closed, the segment t is not the concatenation of any two segments in X. Consequently, $t \in C_s$ for some $s \in \mathrm{Seg}(\mathbf{B})$ and $s \neq t$. Then $s = t^{(1)} \circ t \circ t^{(2)}$ where $t^{(1)}$ or $t^{(2)}$ is a nonempty segment (or both). Moreover, since $X \backslash \{t\}$ is co-closed and $s \in X$, at least one of $t^{(1)}$ or $t^{(2)}$ is in X. We may assume without loss of generality that $t^{(1)}$ is a segment in X. Since $t \in C_s$, we have $t^{(1)} \in K_s$. By definition, $t^{(1)} = t_1 \circ \cdots \circ t_l$ where each t_i is in $C_{s'_i}$ for some segments $s'_i \in \mathrm{Seg}(\mathbf{B})$. By Lemma 4.5, at least one of the t_i is in K_s. But this means $K_s \cap C_{s'_i}$ contains a proper subsegment of s. Hence, s and s'_i are distinct segments such that $K_s \cap C_{s'_i}$ is nonempty, so they are crossing by Lemma 5.2. This contradicts the fact that s and s'_i are in $\mathrm{Seg}(\mathbf{B})$.

Now assume $\mathrm{Seg}(\mathbf{B}) \backslash S$ is nonempty, and let $s \in \mathrm{Seg}(\mathbf{B}) \backslash S$. Then $X \backslash \{s\}$ is not biclosed.

Suppose $X \backslash \{s\}$ is not closed. Then there exists $t, t' \in X$ such that $s = t \circ t'$. Without loss of generality, we may assume $t \in C_s$ and $t' \in K_s$. Since $t' \in X$, there exist a decomposition $t' = t_1 \circ \cdots \circ t_l$ such that for all i, $t_i \in C_{s'_i}$ for some $s'_i \in \mathrm{Seg}(\mathbf{B})$. Since t' is in K_s, so is some t_i. Hence, $K_s \cap C_{s'_i}$ is nonempty for some $s'_i \in \mathrm{Seg}(\mathbf{B})$, and we again deduce that s and s'_i are crossing.

Suppose $X \backslash \{s\}$ is not co-closed. Then there exists $t \in \mathrm{Seg}(T) \backslash X$ such that $s \circ t \in X$. We choose the segment t to be minimal with those properties.

Suppose $s \circ t \in C_{s'}$ for some $s' \in \mathrm{Seg}(\mathbf{B})$. Since $t \notin X$, t is not in $C_{s'}$. Hence, $t \in K_{s \circ t}$ and $s \in C_{s \circ t}$. Now this implies that $s \in C_{s \circ t} \subseteq C_{s'}$. We obtain that $s \in K_s \cap C_{s'}$ so s and s' are crossing.

Now assume $s \circ t = s_1 \circ \cdots \circ s_l$, $l > 1$ where for all i, $s_i \in C_{s'_i}$ for some $s'_i \in \mathrm{Seg}(\mathbf{B})$. Since $t \notin X$ and $X = \pi_\downarrow(X)$, the segment t is not in $C_{s_1 \circ \cdots \circ s_l}$. Consequently, $s \in C_{s_1 \circ \cdots \circ s_l}$ and $t \in K_{s_1 \circ \cdots \circ s_l}$. We consider two cases: either s_l is a proper subsegment of t or t is a proper subsegment of s_l. We note that t is not equal to s_l since $t \notin X$ and $s_l \in X$.

If s_l is a subsegment of t, then there exists a segment t' such that $s \circ t' = s_1 \circ \cdots \circ s_{l-1}$. Then $t' \in X$ by minimality of t. But $t = t' \circ s_l$, contrary to X being closed.

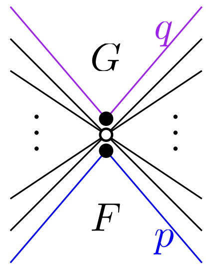


FIGURE 13. Two noncrossing arcs p and q with marked corners at a common vertex satisfy $\text{Reg}(p,F) \subseteq \text{Reg}(q,F)$ and $\text{Reg}(q,G) \subseteq \text{Reg}(p,G)$.

If t is a subsegment of s_l, then there exists t' such that $t' \circ t = s_l$. Then $t \in K_{s_l}$ since $t \in K_{s_1 \circ \cdots \circ s_l}$, so $t' \in C_{s_l}$. Since $s_l \in C_{s'}$ for some $s' \in \text{Seg}(\mathbf{B})$, we have $t' \in C_{s'}$. Since t' is a proper subsegment of s that shares an endpoint with s, either $t' \in K_s$ or $t' \in C_s$. If $t' \in K_s$, then s and s' cross by Lemma 5.2. If $t' \in C_s$, then $s_1 \circ \cdots \circ s_{l-1} \in K_s$. In particular, some s_i is in K_s. But $s_i \in C_{s'}$ for some $s' \in \text{Seg}(\mathbf{B})$, which again contradicts that s and s' are noncrossing.

We have established that S and $\text{Seg}(\mathbf{B})$ are identical. Hence, the map $\tilde{\phi}$ is injective, and the result follows. ☐

Proposition 5.8. For $s, t \in \text{Seg}(T)$, if s and t are noncrossing as green segments, then they are noncrossing as red segments.

Proof. Let γ_s and γ_t be green admissible curves for s and t that do not intersect. Suppose s has corners (u,F), (v,G) as a green segment and (u,F'), (v,G') as a red segment. Define γ'_s to be the curve $\alpha_v^{G,G'}\gamma_s\alpha_u^{F',F}$. We apply a homotopy to γ'_s so that γ'_s is a simple curve and $z(u,F')$ and $z(v,G')$ are the unique points of distance at most ϵ from some interior vertex of T. Then γ'_s is a red admissible curve for s. If γ'_t is defined in a similar manner, then it is a red admissible curve for t that does not intersect γ'_s. Hence, s and t are noncrossing as red segments. ☐

By Proposition 5.7, any noncrossing tree partition corresponds to some facet $\mathcal{F}$ of the noncrossing complex. Taking the set of green of $\mathcal{F}$ produces another noncrossing tree partition due to Proposition 5.8. We summarize this result as follows.

Theorem 5.9. Let $\mathbf{B}$ be a noncrossing tree partition, and let $\mathcal{F} = \rho^{-1}(\mathbf{B})$. Setting

$$S = \{s \in \text{Seg}(T) : \exists \mathcal{F} \xrightarrow{s} \mathcal{F}'\},$$

we have $S = \text{Seg}(\mathbf{B}')$ for some noncrossing tree partition $\mathbf{B}'$.

The noncrossing tree partition $\mathbf{B}'$ of Theorem 5.9 is called the **Kreweras complement** of $\mathbf{B}$. Kreweras complementation is a bijection $\text{Kr} : \text{NCP}(T) \to \text{NCP}(T)$.

5.3. **Red-green trees.** A **red-green tree** $\mathcal{T}$ is a collection of pairwise noncrossing colored segments such that every pair of vertices in V^o is connected by a sequence of curves in $\mathcal{T}$. The segments in $\mathcal{T}$ are allowed to be red or green. Let $\mathcal{T}_r$ (resp., $\mathcal{T}_g$) be the subset of red (resp., green) segments of $\mathcal{T}$.

That red-green trees are actual trees (i.e., acyclic) will be a consequence of Theorem 5.11.

Given $\mathcal{F} \in \overrightarrow{FG}(T)$, let $S_r = \{s : \exists \mathcal{F}' \xrightarrow{s} \mathcal{F}\}$ and $S_g = \{s : \exists \mathcal{F} \xrightarrow{s} \mathcal{F}'\}$.

Lemma 5.10. The graph on the interior vertices of T whose edge set is $S_r \cup S_g$ is acyclic.

Proof. Let $s_1, \ldots, s_\ell \in S_r \cup S_g$ be a sequence of distinct segments that form a **path** on a subset of the interior vertices of T (i.e., for each $i = 1, \ldots, \ell - 1$ the segments s_i and s_{i+1} have one common endpoint), and let $p_1, \ldots, p_\ell \in \mathcal{F}$ be the corresponding arcs. Now let (v_1, F_1) be the marked corner of p_1 where v_1 is the common endpoint of s_1 and s_2. It is enough to show that $\text{R}_1 \subseteq \cdots \subseteq \text{R}_\ell$ where $\text{R}_i := \text{Reg}(p_i, F_1)$. Since the segments $s_1, \ldots, s_\ell$ are pairwise distinct, these inclusions will be proper inclusions.

By Figure 13, we see that $\text{R}_1 \subseteq \text{R}_2$. Letting (v_1, F_1') and (v_2, F_2) denote the marked corners of p_2 where v_2 is the common endpoint of s_2 and s_3, we see $F_1 \notin \text{Reg}(p_2, F_1')$. We now have that $F_1 \in \text{Reg}(p_2, F_2)$ so $\text{R}_2 = \text{Reg}(p_2, F_2)$. Using Figure 13, we obtain $\text{R}_2 \subseteq \text{R}_3$. Continuing with this argument, we obtain the desired result. ☐

Theorem 5.11. The sets S_r and S_g form the red and green segments of a red-green tree. Conversely, every red-green tree is of this form.

Proof. In the same way that a nonintersecting collection of red admissible curves for segments of S_r was constructed in the definition of ρ in Section 5.2, one may construct a family of nonintersecting red and green admissible curves for $S_r \cup S_g$. It remains to show that the graph on the interior vertices of T with edge set $S_r \cup S_g$ is connected. Indeed, we have $\#S_r \cup S_g = \#V^o - 1$ since the set of segments $S_r \cup S_g$ comes from a facet of the reduced noncrossing complex, which is pure. Connectedness then follows from Lemma 5.10.

Now let $\mathcal{T}$ be a red-green tree. Then $\mathcal{T}_r$ is the set of minimal segments of a noncrossing tree partition $\mathbf{B}$. Let $X = \tilde{\phi}(\mathbf{B})$, where $\tilde{\phi}$ is the map to biclosed sets from the proof of Proposition 5.7. By definition, $X = \overline{\bigcup_{s\in\mathcal{T}_r} C_s}$. We prove that

$$\mathrm{Seg}(T)\backslash\pi^{\uparrow}(X) = \overline{\bigcup_{s\in\mathcal{T}_g} K_s}.$$

Since $\bigvee_{s\in\mathcal{T}_g} \eta(K_s)$ is the canonical join-representation of an element in $\overrightarrow{FG}(T^\vee)$, this equality uniquely identifies $\mathcal{T}_g$.

By definition, the set $\mathrm{Seg}(T)\backslash\pi^{\uparrow}(X)$ consists of segments t for which $K_t \cap X$ is empty. We first show that $\overline{\bigcup_{s\in\mathcal{T}_g} K_s}$ is a subset of $\mathrm{Seg}(T)\backslash\pi^{\uparrow}(X)$. To this end, it suffices to show that $K_s \cap X = \emptyset$ holds whenever $s \in \mathcal{T}_g$. If not, then let $s \in \mathcal{T}_g$ such that $K_s \cap X$ is nonempty, and let $t \in K_s \cap X$. Since $t \in X$, there exist segments $t_1, \ldots, t_l,\ s_1, \ldots, s_l$ such that $t = t_1 \circ \cdots \circ t_l$ and $t_i \in C_{s_i}$ for all i. Then $t_i \in K_t$ for some i. Since $K_t \subseteq K_s$, t_i is in K_s. But since s_i and s do not cross, $K_s \cap C_{s_i}$ is empty by Lemma 5.3, a contradiction.

Now we prove that $\mathrm{Seg}(T)\backslash\pi^{\uparrow}(X)$ is a subset of $\overline{\bigcup_{s\in\mathcal{T}_g} K_s}$. Let $t = [u, v]$ be a segment for which $K_t \cap X = \emptyset$. Since $\mathcal{T}$ is a red-green tree, there is a path in $\mathcal{T}$ with edges $s_1, \ldots, s_l$ such that s_1 starts at u and s_l ends at v. We consider two cases: either t is the concatenation of $s_1, \ldots, s_l$ (i.e. $t = s_1 \circ \cdots \circ s_l$), or it is not.

Assume that t is not equal to the concatenation of $s_1, \ldots, s_l$. Then there exists a vertex w incident to an edge e such that two adjacent segments s_i, s_{i+1} both contain e and share an endpoint at w. Then s_i and s_{i+1} must have different colors. Up to reversing the order of the segments, we may assume s_i is red and s_{i+1} is green. Let $[w', w]$ be the largest common subsegment of s_i and s_{i+1}. Since s_i and s_{i+1} are noncrossing, the segment $[w', w]$ is in $K_{s_i} \cap C_{s_{i+1}}$. Let s'_i, s'_{i+1} be segments such that $s'_i \circ [w', w] = s_i$ and $s'_{i+1} \circ [w', w] = s_{i+1}$. It is possible that $[w', w]$ is equal to s_i or s_{i+1} (but not both), in which case s'_i or s'_{i+1} is a lazy path and thus not a segment.

If s'_i is a segment and $i > 1$, we claim that it does not cross s_{i-1}. Indeed, if s'_i and s_{i-1} cross, then s_{i-1} must be green and $C_{s'_i} \cap K_{s_{i-1}}$ is nonempty. But this implies $C_{s_i} \cap K_{s_{i-1}}$ is nonempty, a contradiction.

If s'_i is not a segment and $i > 1$, we claim that s'_{i+1} does not cross s_{i-1}. If s'_{i+1} and s_{i-1} do cross, then s_{i-1} must be red and $K_{s'_{i+1}} \cap C_{s_{i-1}}$ is nonempty. But this implies $K_{s_{i+1}} \cap C_{s_{i-1}}$ is nonempty, a contradiction.

Hence, $s_1, \ldots, s'_i, s'_{i+1}, \ldots, s_l$ is a sequence of red and green segments connecting the endpoints of t such that the red segments are in $\bigcup_{s\in\mathcal{T}_r} C_s$ and the green segments are in $\bigcup_{s\in\mathcal{T}_g} K_s$. Moreover, adjacent segments are noncrossing. Proceeding inductively, we may assume that t is the concatenation of noncrossing colored segments $t_1, \ldots, t_l$ where each t_i is either a red segment in $\bigcup_{s\in\mathcal{T}_r} C_s$ or a green segment in $\bigcup_{s\in\mathcal{T}_g} K_s$. If $t_1, \ldots, t_l$ are all green segments, then $t \in \overline{\bigcup_{s\in\mathcal{T}_g} K_s}$, as desired.

Assume at least one segment is red, and let $t_i, \ldots, t_j$ be a maximal subsequence of red segments. We prove that $t_i \circ \cdots \circ t_j$ is in K_t. If $i > 1$, then t_{i-1} is a green segment not crossing t_i such that the concatenation $t_{i-1} \circ t_i$ is a segment. This implies $t_i \in K_{t_{i-1}\circ t_i}$. Similarly, if $j < l$, then t_{j+1} is a green segment not crossing t_j, and t_j is in $K_{t_j \circ t_{j+1}}$. Hence, $t_i \circ \cdots \circ t_j$ is in K_t. But this implies t_m is in K_t for some $i \leqslant m \leqslant j$. As $t_m \in X$, this contradicts the assumption that $K_t \cap X$ is empty. □

Since ρ is a bijection that only depends on the red segments of a facet, Theorem 5.11 gives a bijection between noncrossing tree partitions and red-green trees. This correspondence encodes Kreweras complementation in a nice way.

Corollary 5.12. Let $\mathbf{B}$ be a noncrossing tree partition. There exists a unique red-green tree $\mathcal{T}$ whose set of red segments is $\mathrm{Seg}(\mathbf{B})$. Moreover, the set of green segments of $\mathcal{T}$ is $\mathrm{Seg}(\mathrm{Kr}(\mathbf{B}))$.

5.4. **Lattice property.** Let $\Pi(V^o)$ be the lattice of all set partitions of V^o, ordered by refinement. Recall that the meet of any two set partitions is their common refinement. We prove that $\mathrm{NCP}(T)$ is a meet-subsemilattice of $\Pi(V^o)$ in Theorem 5.13. Since $\mathrm{NCP}(T)$ is a finite poset with a top and bottom element, this implies that it is a lattice.

Theorem 5.13. The poset $\mathrm{NCP}(T)$ is a lattice.

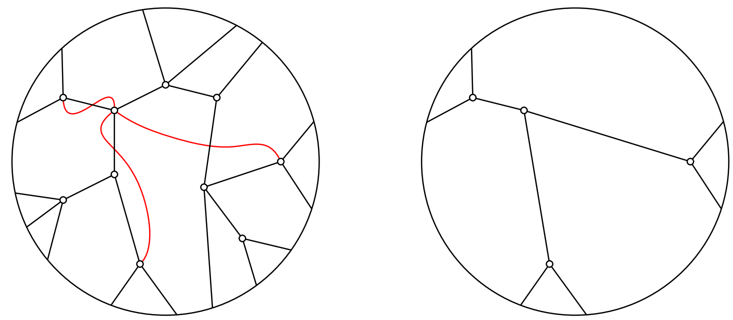


FIGURE 14. An example of a segment-connected subset and the contracted tree it defines.

Proof. Let $\mathbf{B}, \mathbf{B}'$ be two noncrossing tree partitions, and let $\mathbf{B}''$ be the common refinement of $\mathbf{B}$ and $\mathbf{B}'$. We claim that $\mathbf{B}''$ is a noncrossing tree partition and deduce that $\mathrm{NCP}(T)$ is a meet-subsemilattice of $\Pi(V^o)$. We first prove that every block of $\mathbf{B}''$ is segment-connected.

Let B'' be a block of $\mathbf{B}''$, and let $u, v \in B''$. There exist blocks $B \in \mathbf{B}$, $B' \in \mathbf{B}'$, each containing u and v. We prove by induction that there exists a sequence $u = u_0, \ldots, u_l = v$ of elements of B'' such that $[u_{i-1}, u_i]$ is a segment for all i. Among vertices of $[u, v]$, choose u_1 such that $[u, u_1]$ is a segment of maximum length. If $u_1 = v$, we are done. Since B is segment-connected, there exists a sequence $u = w_0, \ldots, w_m = v$ of elements of B such that $[w_{i-1}, w_i]$ is a segment for all i. Moreover, these segments may be chosen so that $[u, v]$ is the concatenation of the segments $[w_{i-1}, w_i]$. Then u_1 is a vertex in $[w_{i-1}, w_i]$ for some i. As $[w_{i-1}, w_i]$ is a segment, this forces $u_1 = w_{i-1}$ or $u_1 = w_i$. Hence, $u_1 \in B$. By a similar argument $u_1 \in B'$ so u_1 is an element of B''. By induction, we conclude that B'' is segment-connected.

Let $S = \bigcup_{B'' \in \mathbf{B}''} \mathrm{Seg}(B'')$ and suppose $[a, b], [c, d] \in S$ such that $[a, b]$ and $[c, d]$ are crossing.

Assume that these segments share a common endpoint, say $b = c$, then they intersect in a common segment $[b, e]$. As $\mathbf{B}$ and $\mathbf{B}'$ are noncrossing tree partitions, there exist blocks $B \in \mathbf{B}$, $B' \in \mathbf{B}'$ such that $a, b, d, e \in B$ and $a, b, d, e \in B'$. Hence, $e \in B''$. But $[b, e]$ is a subsegment of $[a, b]$ and $[b, d]$, contradicting the minimality of segments in $\mathrm{Seg}(B'')$.

Now assume that the endpoints are all distinct. Let B_1'', B_2'' be blocks in $\mathbf{B}''$ such that $a, b \in B_1''$ and $c, d \in B_2''$. Since $\mathbf{B}''$ is the common refinement of $\mathbf{B}$ and $\mathbf{B}'$, we may assume without loss of generality that $\mathbf{B}$ contains distinct blocks B_1 and B_2 such that $a, b \in B_1$ and $c, d \in B_2$. Since $\mathbf{B}$ is noncrossing, either $[a, b] \notin \mathrm{Seg}(B_1)$ or $[c, d] \notin \mathrm{Seg}(B_2)$. Suppose $[a, b] \notin \mathrm{Seg}(B_1)$. Then there exists $a_1 \in [a, b]$ such that $[a, a_1] \in \mathrm{Seg}(B_1)$. Then either $[a, a_1]$ or $[a_1, b]$ cross $[c, d]$ by Lemma 5.6. By induction, there exists segments $[a', b'] \in \mathrm{Seg}(B_1)$, $[c', d'] \in \mathrm{Seg}(B_2)$ such that $[a', b']$ and $[c', d']$ cross, a contradiction. □

5.5. **Shard intersection order.** In this section, we prove that the shard intersection order of $\overrightarrow{FG}(T)$ is naturally isomorphic to $\mathrm{NCP}(T)$.

Let B be a segment-connected subset of T^o, and let $S = \mathrm{Seg}_r(B)$. We define the **contracted tree** T_B such that

- B is the set of interior vertices of T_B,
- S is the set of interior edges of T_B, and
- for edges e with one endpoint u in B and the other endpoint not between two vertices of B, there is an edge from u to the boundary in the direction of e.

Example 5.14. Let T denote the tree shown in Figure 14 on the left, let B denote the noncrossing tree partition of T also shown there. Then the contracted tree T_B is shown on the right in Figure 14.

As in Proposition 4.3, we may compute the facial intervals of $\overrightarrow{FG}(T)$ as follows.

Proposition 5.15. Let $\mathcal{F} \in \overrightarrow{FG}(T)$, and let $s_1, \ldots, s_k$ be a set of segments for which there exists flips $\mathcal{F} \xrightarrow{s_i} \mathcal{F}'$ for each i. Let $\mathbf{B} = \{B_1, \ldots, B_l\}$ be the noncrossing tree partition with segments $\mathrm{Seg}(\mathbf{B}) = \{s_1, \ldots, s_k\}$. Let T_i denote the contracted tree T_{B_i}. Then

$$[\mathcal{F}, \bigvee \mathcal{F}'] \cong \overrightarrow{FG}(T_1) \times \cdots \times \overrightarrow{FG}(T_l),$$

where the join is taken over $\mathcal{F}'$ for which $\mathcal{F} \xrightarrow{s_i} \mathcal{F}'$ for some s_i (see Figure 15).

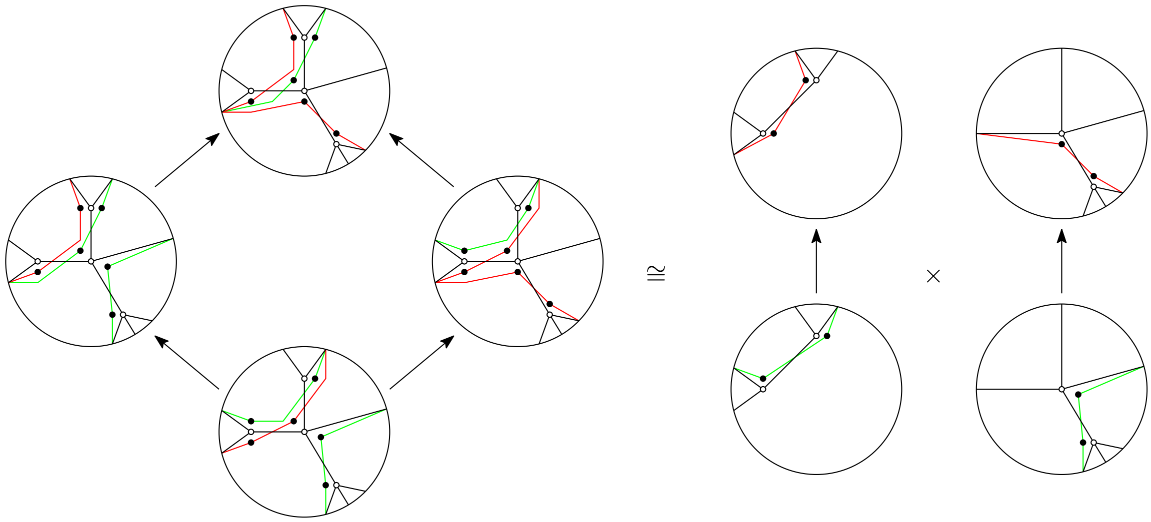


FIGURE 15. An example of the isomorphism appearing in Proposition 5.15 where $\mathcal{F}$ is a facet from the oriented flip graph in Figure 7.

Proof. Let X be the biclosed set $\pi^{\uparrow}(\phi(\mathcal{F}))$. Then $\{s_1, \ldots, s_k\}$ is the set of segments for which $X \cup \{s_i\}$ is biclosed, and $\eta(X \cup \{s_i\}) = \mathcal{F}'$ where $\mathcal{F}'$ is the facet obtained by flipping $\mathcal{F}$ at s_i. Set $Y = X \cup \overline{\{s_1, \ldots, s_k\}}$.

Let $\mathbf{B} = \{B_1, \ldots, B_l\}$ be the noncrossing tree partition with $\mathrm{Seg}(\mathbf{B}) = \{s_1, \ldots, s_k\}$, and let T_i be the contracted tree T_{B_i}. By Proposition 4.3, the interval $[X, Y]$ is isomorphic to $\mathrm{Bic}(T_1) \times \cdots \times \mathrm{Bic}(T_l)$.

As usual, we let Θ denote the lattice congruence that identifies $\overrightarrow{FG}(T)$ with $\mathrm{Bic}(T)/\Theta$. We let Θ_i denote the corresponding lattice congruence on $\mathrm{Bic}(T_i)$. By Lemma 2.3, the quotient interval $[X, Y]/\Theta$ is isomorphic to $[\mathcal{F}, \bigvee \mathcal{F}']$. Hence, we prove

$$[X, Y]/\Theta \cong \mathrm{Bic}(T_1)/\Theta_1 \times \cdots \times \mathrm{Bic}(T_l)/\Theta_l.$$

Given a segment s supported by T_i, we let C_s^i (resp., K_s^i) denote the intersection $C_s \cap \mathrm{Seg}(T_i)$ (resp., $K_s \cap \mathrm{Seg}(T_i)$), and we define maps $\pi_{\downarrow}^i$ and $\pi_i^{\uparrow}$ by the congruence Θ_i. Explicitly, we have

$$\pi_{\downarrow}^i(Z) = \{s \in \mathrm{Seg}(T_i) : \ C_s^i \subseteq Z\} \text{ and}$$
$$\pi_i^{\uparrow}(Z) = \{s \in \mathrm{Seg}(T_i) : \ K_s^i \cap Z \neq \varnothing\}.$$

Let $Z, Z' \in [X, Y]$. Then $Z = X \cup \bigcup_{i=1}^{l} Z_i$ and $Z' = X \cup \bigcup_{i=1}^{l} Z_i'$ for some (unique) $Z_i, Z_i' \in \mathrm{Bic}(T_i)$. We prove that $Z \equiv Z' \mod \Theta$ if and only if $Z_i \equiv Z_i' \mod \Theta_i$ for all i.

Suppose $Z \equiv Z' \mod \Theta$, and fix $i \in \{1, \ldots, l\}$. To prove that $Z_i \equiv Z_i' \mod \Theta_i$, it suffices to show that $\pi_i^{\uparrow}(Z_i) = \pi^{\uparrow}(Z) \cap \mathrm{Seg}(T_i)$. If $s \in \pi_i^{\uparrow}(Z_i)$, then $K_s^i \cap Z_i$ is nonempty. But this implies $K_s \cap Z$ is nonempty, so $s \in \pi^{\uparrow}(Z) \cap \mathrm{Seg}(T_i)$. Conversely, if $s \in \pi^{\uparrow}(Z) \cap \mathrm{Seg}(T_i)$, then $K_s \cap Z$ is nonempty. Since $\pi^{\uparrow}(X) = X$, we deduce that $K_s \cap \bigcup_{j=1}^{l} Z_j$ is nonempty. But $K_s \cap Z_j = \varnothing$ whenever $j \neq i$ since blocks B_i and B_j are noncrossing. Hence, $s \in \pi_i^{\uparrow}(Z_i)$, as desired.

Now assume $Z_i \equiv Z_i' \mod \Theta_i$ for all i. Since $\pi_i^{\uparrow}(Z_i) \subseteq \pi^{\uparrow}(Z)$ and $\pi^{\uparrow}$ is idempotent, we have

$$\begin{aligned}\pi^{\uparrow}(Z) &= \pi^{\uparrow}(X \cup \bigcup_{i=1}^{l} \pi_i^{\uparrow}(Z_i)) \\ &= \pi^{\uparrow}(X \cup \bigcup_{i=1}^{l} \pi_i^{\uparrow}(Z_i')) \\ &= \pi^{\uparrow}(Z').\end{aligned}$$

Therefore, $Z \equiv Z' \mod \Theta$. □

Theorem 5.16. The map $\rho \circ \psi^{-1} : \Psi(\overrightarrow{FG}(T)) \to \mathrm{NCP}(T)$ is a Kreweras-equivariant isomorphism of posets.

Proof. Let $\mathcal{F}$ be an element of $\overrightarrow{FG}(T)$, and let

$$\mathcal{F}' = \bigwedge\{\mathcal{F}'' : \mathcal{F}'' \to \mathcal{F}\}.$$

Let $S = \mathrm{Seg}(\rho(\mathcal{F}))$. By Lemma 2.11, $\mathcal{F}$ is equal to

$$\bigvee\{\mathcal{F}'' : \mathcal{F}' \xrightarrow{s} \mathcal{F}'',\ s \in S\}.$$

Let $\rho(\mathcal{F}) = \{B_1, \dots, B_l\}$, and let T_i be the contracted tree T_{B_i}. By Proposition 5.15, the interval $[\mathcal{F}', \mathcal{F}]$ is isomorphic to $\overrightarrow{FG}(T_1) \times \cdots \times \overrightarrow{FG}(T_l)$.

The set $\psi(\mathcal{F})$ is defined to be the set of labels s such that there exists a covering relation $\mathcal{F}^{(1)} \xrightarrow{s} \mathcal{F}^{(2)}$ where $\mathcal{F}' \leqslant \mathcal{F}^{(1)} \lessdot \mathcal{F}^{(2)} \leqslant \mathcal{F}$. Hence,

$$\psi(\mathcal{F}) = \bigsqcup_{i=1}^{l} \mathrm{Seg}(T_i) = \overline{\mathrm{Seg}(\rho(\mathcal{F}))}.$$

From this description, it is clear that ψ is a bijection. Hence, the inverse ψ^{-1} exists, and the composite map $\rho \circ \psi^{-1}$ is a bijection. Since the Kreweras complement is defined for both $\Psi(\overrightarrow{FG}(T))$ and $\mathrm{NCP}(T)$ via the bijections ρ and ψ, the Kreweras-equivariance is immediate. If $\mathcal{F}_1, \mathcal{F}_2 \in \overrightarrow{FG}(T)$ satisfy $\psi(\mathcal{F}_1) \subseteq \psi(\mathcal{F}_2)$, then the corresponding noncrossing tree partitions are ordered by refinement. Conversely, it is clear that if $\rho(\mathcal{F}_1) \leqslant \rho(\mathcal{F}_2)$, then any segment in $\psi(\mathcal{F}_1)$ is contained in $\psi(\mathcal{F}_2)$. Hence, the bijection $\rho \circ \psi^{-1}$ is an isomorphism of posets. ☐

6. Polygonal subdivisions

In this section, we show how oriented flip graphs can be equivalently described using certain decompositions of a convex polygon $P \subseteq \mathbb{R}^2$ into smaller convex polygons called polygonal subdivisions. The notion of a flip between two facets of the reduced noncrossing complex will translate into a type of *flip* between polygonal subdivisions of P. After that, we show that the polygonal subdivision corresponding to the top element of an oriented flip graph is obtained by *rotating* the arcs in the polygonal subdivision corresponding to the bottom element. We show that oriented exchange graphs of quivers that are mutation-equivalent to type $\mathbb{A}$ Dynkin quivers are examples of oriented flip graphs. Lastly, we show that the Stokes posets of quadrangulations are also examples of oriented flip graphs.

A **polygonal subdivision** $\mathcal{P} = \{P_i\}_{i \in [\ell]}$ of a polygon P is a family of distinct polygons $P_1, \dots, P_\ell$ such that

- $\bigcup_{i=1}^{l} P_i = P$,
- $P_i \cap P_j$ is a face of P_i and P_j for all i, j, and
- every vertex of P_i is a vertex of P for all i.

Equivalently, we can define a polygonal subdivision of P to be a collection of pairwise noncrossing **diagonals** of P (i.e., curves in $\mathbb{R}^2$ connecting two vertices of P) up to endpoint fixing isotopy.

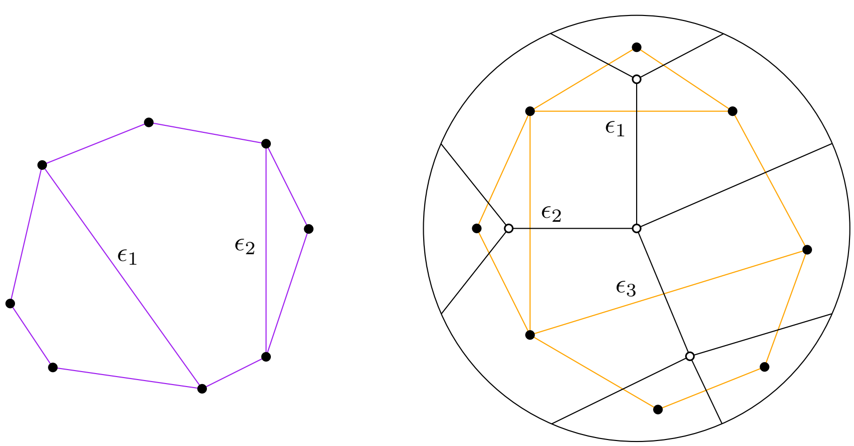


Figure 16. Two examples of polygonal subdivisions where the latter is drawn with its corresponding tree.

Remark 6.1. Trees and polygonal subdivisions are dual. Given any tree T embedded in D^2, it defines a polygonal subdivision $\mathcal{P}$ as follows. Let P be a polygon with vertex set $\{v_F : F \text{ is a face of } T\}$ and where v_{F_1} is connected to v_{F_2} by a diagonal if and only if there is an edge of T that is incident to both F_1 and F_2. Using the data of the embedding of T, the resulting collection of polygons $\mathcal{P}$ is a polygonal subdivision of P. It is straightforward to verify that this construction can be reversed. We show an example of this duality in Figure 16.

Remark 6.2. Given a polygonal subdivision $\mathcal{P} = \{P_i\}_{i\in[\ell]}$ of a polygon P, there is a natural bound quiver $(Q_\mathcal{P}, I_\mathcal{P})$ that we associate to $\mathcal{P}$. Define $Q_\mathcal{P}$ to be the quiver (see Section 6.1) whose vertices are in bijection with diagonals of $\mathcal{P}$ belonging to two distinct polygons $P_i, P_j \in \mathcal{P}$ and whose arrows are exactly those of the form $\epsilon_1 \xrightarrow{\alpha} \epsilon_2$ satisfying:

i) ϵ_1 and ϵ_2 share a vertex of P,

ii) ϵ_2 is clockwise from ϵ_1 about their shared vertex.

The admissible ideal $I_\mathcal{P}$ is, by definition, generated by the relations $\alpha\beta$ where $\alpha : \epsilon_2 \longrightarrow \epsilon_3$, $\beta : \epsilon_1 \longrightarrow \epsilon_2$, and ϵ_1, ϵ_2, and ϵ_3 all belong to a common polygon $P_i \in \mathcal{P}$. One can define a finite dimensional $\Bbbk$-algebra $\Lambda_\mathcal{P} := \Bbbk Q_\mathcal{P}/I_\mathcal{P}$. These algebras were introduced in [32] where they were called **tiling algebras**.

When $\mathcal{P} = \{P_i\}_{i\in[\ell]}$ is a **triangulation** of a polygon P (i.e., each polygon P_i is a triangle), the definition of the algebra $\Lambda_\mathcal{P}$ agrees with the definition of the **Jacobian algebra** [12] associated to the triangulation. Moreover, the triangulations of P are exactly those polygonal subdivisions whose corresponding tree has only degree 3 interior vertices. When $\mathcal{P} = \{P_i\}_{i\in[\ell]}$ is an **$(m+2)$-angulation** of P where $m \geqslant 1$ (i.e., each polygon P_i is an $(m+2)$-gon), the algebra $\Lambda_\mathcal{P}$ is an **m-cluster-tilted algebra** of type $\mathbb{A}$ as was shown in [24].

Additionally, the class of tiling algebras also contains the **surface algebras** when the surface is the disk. These algebras were introduced in [10] and studied further in [9] and [1].

Example 6.3. If $\mathcal{P}$ is left polygonal subdivision in Figure 16, then $Q_\mathcal{P}$ is the quiver with two vertices ϵ_1 and ϵ_2 and no arrows and $I_\mathcal{P}$ is the zero ideal. In this case, $\Lambda_\mathcal{P} = \Bbbk\epsilon_1 \oplus \Bbbk\epsilon_2$. If $\mathcal{P}$ is the right polygonal subdivision in Figure 16, then $Q_\mathcal{P} = \epsilon_1 \xrightarrow{\beta} \epsilon_2 \xrightarrow{\alpha} \epsilon_3$ and $I_\mathcal{P} = \langle\alpha\beta\rangle$. In this case, $\Lambda_\mathcal{P} = \Bbbk\epsilon_1 \oplus \Bbbk\epsilon_2 \oplus \Bbbk\epsilon_3 \oplus \Bbbk\alpha \oplus \Bbbk\beta$.

Remark 6.4. In [32], it is shown that tiling algebras can be defined without reference to a combinatorial model such as a polygonal subdivision, and any tiling algebra so defined naturally gives rise to a polygonal subdivision. More generally, it is shown in [31] that any gentle algebra gives rise to a certain finite graph. If the algebra is a tiling algebra, then its polygonal subdivision from [32] is exactly the finite graph associated to it in [31].

Now let T be a tree embedded in D^2. Using Remark 6.1, let $\mathcal{P}_T$ be the polygonal subdivision of the polygon P_T defined by T, and let $\{v_F : F \text{ is a face of } T\}$ be the set of vertices of the polygon P_T. There is an obvious bijection between elements of $\{v_F : F \text{ is a face of } T\}$ and the set of boundary vertices of T given by sending v_F to the counterclockwise most leaf of T in face F. Using this bijection and the fact that any arc of T is completely determined by the leaves of T it connects, we obtain the following.

Proposition 6.5. Let T be a tree embedded in D^2. The map sending each arc in a facet $\mathcal{F} \in \widetilde{\Delta}^{NC}(T)$ to its corresponding diagonal of P_T defines a polygonal subdivision $\mathcal{P}(\mathcal{F})$ of P_T. This map defines an injection from the facets of $\widetilde{\Delta}^{NC}(T)$ to the set of polygonal subdivisions of P_T.

Example 6.6. By Proposition 6.5, we can identify the vertices of $\overrightarrow{FG}(T)$ with a certain subset of the polygonal subdivisions of P_T. In Figure 17, we show the oriented flip graph from Figure 7 with its vertices represented by the corresponding polygonal subdivisions of P_T.

Next, we show how the polygonal subdivisions corresponding to the top and bottom elements of an oriented flip graph compare to each other. Note that there is a natural cyclic action on the diagonals of the polygon P_T. If α is a diagonal of P_T, we define the **rotation** of α, denoted $\varrho(\alpha)$, to be the diagonal of P whose endpoints are the vertices of P immediately clockwise from the endpoints of α (see Figure 18). If $\mathcal{P}(\mathcal{F})$ is a polygonal subdivision of P_T, we let $\varrho(\mathcal{P}(\mathcal{F}))$ denote the polygonal subdivision of P_T obtained by applying ϱ to each diagonal in $\mathcal{P}(\mathcal{F})$.

Theorem 6.7. Let T be a tree embedded in D^2. Then the bottom element (resp., top element) of $\overrightarrow{FG}(T)$ corresponds to the polygonal subdivision $\mathcal{P}_T$ (resp., $\varrho(\mathcal{P}_T)$).

Proof. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the facets of $\widetilde{\Delta}^{NC}(T)$ corresponding to the bottom and top elements $\overrightarrow{FG}(T)$, respectively. Using Proposition 6.5, we let $\mathcal{P}(\mathcal{F}_1)$ and $\mathcal{P}(\mathcal{F}_2)$ be the corresponding polygonal subdivisions of P_T. It is clear that $\mathcal{F}_2 = \eta(\text{Seg}(T))$ and $\mathcal{F}_1 = \eta(\varnothing)$.

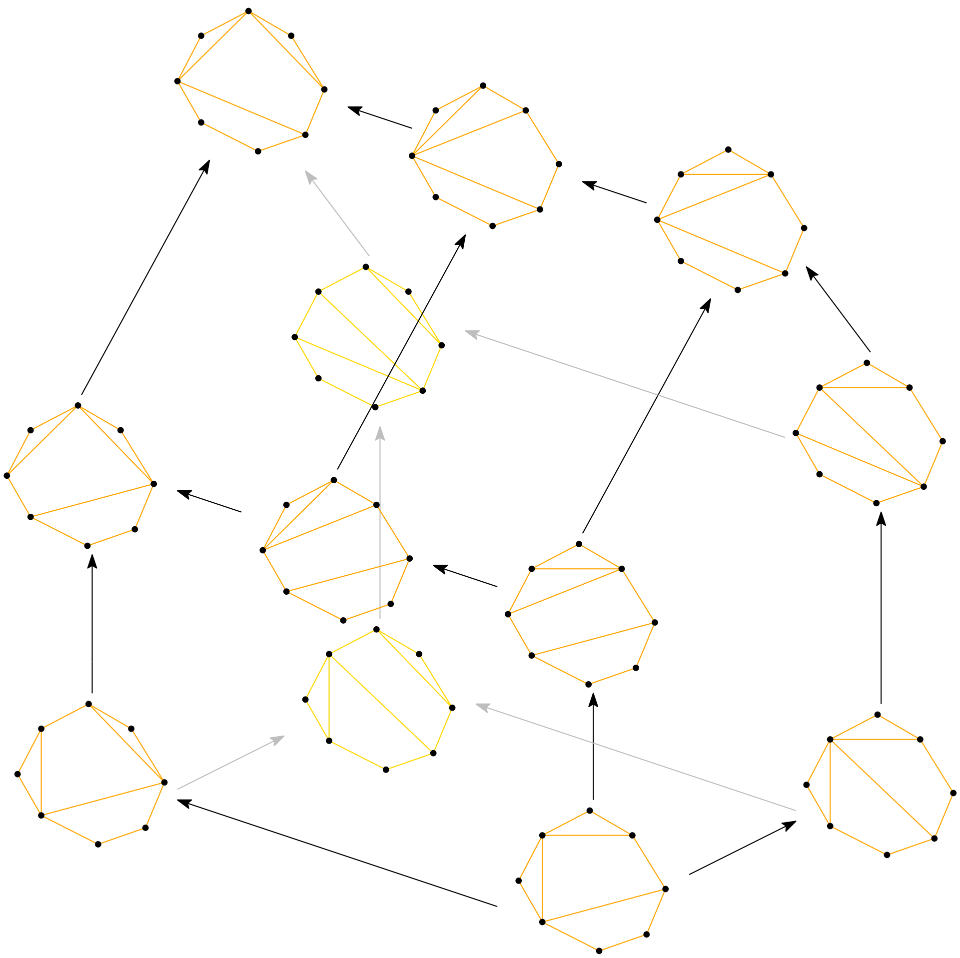


Figure 17

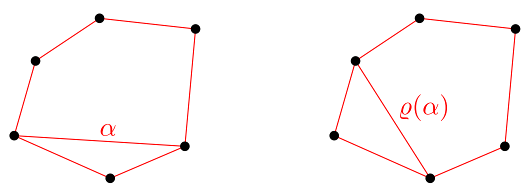


Figure 18. The effect of ϱ on a diagonal α.

Let $p_{(v,F)}$ be any arc of T that appears in $\eta(\mathrm{Seg}(T))$ (resp., $\eta(\varnothing)$). Let u be any interior vertex of T that appears in $p_{(v,F)}$, and orient the arc $p_{(v,F)}$ from v to u. By the definition of η, the arc $p_{(v,F)}$ must turn left (resp., right) at u.

Next, let $e = (v_1, v_2)$ be an edge of T whose endpoints are internal vertices of T, and let F and G be the two faces of T that are incident to e and satisfy (v_1, F) (resp., (v_2, G)) is immediately clockwise from (v_1, G) (resp., (v_2, F)). Define $p := p_{(v_1,G)} \in \eta(\varnothing)$ and $q := p_{(v_1,F)} \in \eta(\mathrm{Seg}(T))$ and let $\alpha_p \in \mathcal{P}(\mathcal{F}_1)$ and $\alpha_q \in \mathcal{P}(\mathcal{F}_2)$ be the diagonals corresponding to p and q, respectively. If we write $p = (u_1, \ldots, u_k, v_1, v_2, u_{k+1}, \ldots, u_r)$ and $q = (w_1, \ldots, w_\ell, v_1, v_2, w_{\ell+1}, \ldots, w_s)$, then the argument in the previous paragraph implies that the corners contained in q are $(w_2, F), \ldots, (w_\ell, F), (v_1, F), (v_2, G), (w_{\ell+1}, G), \ldots, (w_{s-1}, G)$ and the corners contained in p are $(u_2, G), \ldots, (u_k, G), (v_1, G), (v_2, F), (u_{k+1}, F), \ldots, (u_{r-1}, F)$. Thus we have that $\alpha_q = \varrho(\alpha_p)$ and $\mathcal{P}(\mathcal{F}_1) = \mathcal{P}_T$. The desired result follows. $\square$

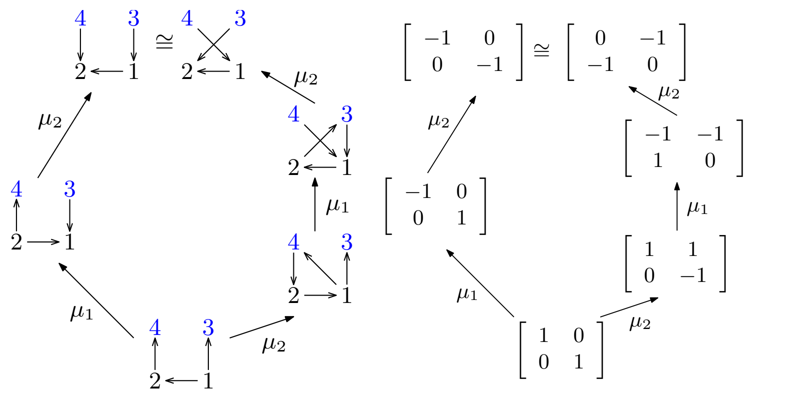


FIGURE 19. The oriented exchange graph of of $Q = 2 \leftarrow 1$ and the corresponding **c**-matrices.

6.1. **Oriented exchange graphs.** Oriented flip graphs generalize a certain subclass of oriented exchange graphs of quivers, which are important objects in representation theory of finite dimensional algebras. We recall the definition of oriented exchange graphs here.

A **quiver** Q is a directed graph. In other words, Q is a 4-tuple (Q_0, Q_1, s, t), where $Q_0 = [m] := \{1, 2, \ldots, m\}$ is a set of **vertices**, Q_1 is a set of **arrows**, and two functions $s, t : Q_1 \to Q_0$ defined so that for every $\alpha \in Q_1$, we have $s(\alpha) \xrightarrow{\alpha} t(\alpha)$. An **ice quiver** is a pair (Q, F) with Q a quiver and $F \subseteq Q_0$ a set of **frozen vertices** with the restriction that any $i, j \in F$ have no arrows of Q connecting them. By convention, we assume $Q_0 \backslash F = [n]$ and $F = [n+1, m] := \{n+1, n+2, \ldots, m\}$. Any quiver Q is regarded as an ice quiver by setting $Q = (Q, \varnothing)$.

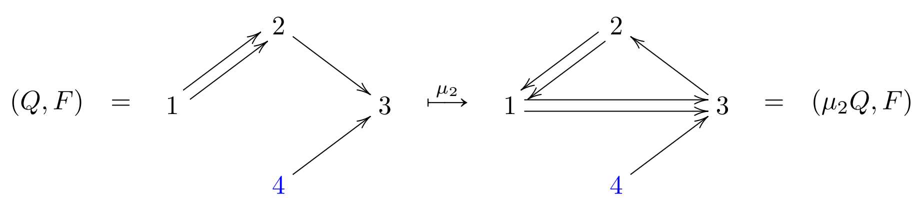
If a given ice quiver (Q, F) has no loops or 2-cycles, we can define a local transformation of (Q, F) called **mutation**. The **mutation** of an ice quiver (Q, F) at a nonfrozen vertex k, denoted μ_k, produces a new ice quiver $(\mu_k Q, F)$ by the three step process:

(1) For every 2-path $i \to k \to j$ in Q, adjoin a new arrow $i \to j$.

(2) Delete any 2-cycles created during the first steps.

(3) Remove a maximal collection of disjoint 2-cycles in the resulting quiver as well as all of the arrows between two frozen vertices.

We show an example of mutation below with the nonfrozen (resp., frozen) vertices in black (resp., blue).



Let $\mathrm{Mut}((Q, F))$ denote the collection of ice quivers obtainable from (Q, F) by finitely many mutations where such ice quivers are considered up to an isomorphism of quivers that fixes the frozen vertices. We refer to $\mathrm{Mut}((Q, F))$ as the **mutation-class** of Q.

Given a quiver Q, we define its **framed** quiver to be the ice quiver $\widehat{Q}$ where $\widehat{Q}_0 := Q_0 \sqcup [n+1, 2n]$, $F = [n+1, 2n]$, and $\widehat{Q}_1 := Q_1 \sqcup \{i \to n+i : i \in [n]\}$. We define the **exchange graph** of $\widehat{Q}$, denoted $EG(\widehat{Q})$, to be the graph whose vertices are elements of $\mathrm{Mut}(\widehat{Q})$ and two vertices are connected by an edge if the corresponding quivers differ by a single mutation.

The exchange graph of $\widehat{Q}$ has natural acyclic orientation using the notion of **c**-vectors. We refer to this directed graph as the **oriented exchange graph** of Q, denoted $\overrightarrow{EG}(\widehat{Q})$. Given $\widehat{Q}$, we say that $C = C_R$ is a **c-matrix** of Q if there exists $R \in EG(\widehat{Q})$ such that $C = (b_{ij})_{i \in [n], j \in [n]}$ is the $n \times n$ matrix where $b_{ij} := \#\{i \xrightarrow{\alpha} j + n \in Q_1\} - \#\{j + n \xrightarrow{\alpha} i \in Q_1\}$. We let $\mathbf{c}\text{-mat}(Q) := \{C_R : R \in EG(\widehat{Q})\}$. A row vector of a **c**-matrix, denoted $\mathbf{c}_i$, is known as a **c-vector**. Since a vertex of $EG(\widehat{Q})$ is defined up to an isomorphism of quivers that fixes the frozen vertices, a **c**-matrix C is defined up to a permutation of its rows.

The celebrated theorem of Derksen, Weyman, and Zelevinsky [13, Theorem 1.7], known as sign-coherence of **c**-vectors, states that for any $R \in EG(\widehat{Q})$ and $i \in [n]$ the **c**-vector $\mathbf{c}_i$ is a nonzero element of $\mathbb{Z}^n_{\geqslant 0}$ or $\mathbb{Z}^n_{\leqslant 0}$. If $\mathbf{c}_i \in \mathbb{Z}^n_{\geqslant 0}$ (resp., $\mathbf{c}_i \in \mathbb{Z}^n_{\leqslant 0}$) we say it is **positive** (resp., **negative**). It turns out that for any quiver Q one has $\mathbf{c}\text{-vec}(Q) := \{\mathbf{c}\text{-vectors of } Q\} = \mathbf{c}\text{-vec}(Q)^+ \sqcup -\mathbf{c}\text{-vec}(Q)^+$ where $\mathbf{c}\text{-vec}(Q)^+ := \{\text{positive } \mathbf{c}\text{-vectors of } Q\}$.

Definition 6.8. [5] The **oriented exchange graph** of a quiver Q, denoted $\overrightarrow{EG}(\widehat{Q})$, is the directed graph whose underlying unoriented graph is $EG(\widehat{Q})$ with directed edges $(R^1, F) \longrightarrow (\mu_k R^1, F)$ if and only if $\mathbf{c}_k$ is positive in C_{R^1}. In Figure 19, we show $\overrightarrow{EG}(\widehat{Q})$ and we also show all of the **c**-matrices in **c**-mat(Q) where $Q = 2 \leftarrow 1$.

As we will mention in Example 6.10, the flip graph of a tree T with only degree 3 internal vertices is isomorphic to the dual associahedron. By this identification and by Proposition 6.5, we obtain an orientation of the 1-skeleton of the associahedron. This orientation adds the data of a "sign" to the operation of performing a single flip between two triangulations $\mathcal{P}(\mathcal{F}_1), \mathcal{P}(\mathcal{F}_2)$ of P_T. It turns out that this oriented version of flipping between triangulations has been described by Fomin and Thurston (we refer the reader to [16] for more details).

Given any triangulation $\mathcal{P}(\mathcal{F}_1)$ of P_T, one adds some additional curves $(L_1, \ldots, L_n)$ to $\mathcal{P}(\mathcal{F}_1)$ (here $n = \#(Q_{\mathcal{P}_T})_0$), called an **elementary lamination** (see [16, Definition 17.2]), and records the **shear coordinates** [16, Definition 12.2] (i.e., integer vectors indicating the number of certain crossings of arcs in $\mathcal{P}(\mathcal{F})$ and the curves $(L_1, \ldots, L_n)$). The elementary lamination is a collection of curves that are slightly deformed versions of the arcs in $\mathcal{P}_T$ and the shear coordinates are the **c**-vectors appearing in the **c**-matrix of the ice quiver corresponding to $\mathcal{P}(\mathcal{F}_1)$. Then there is a directed edge $\mathcal{P}(\mathcal{F}_1) \to \mathcal{P}(\mathcal{F}_2)$ in $\overrightarrow{FG}(T)$ if and only if $\mathcal{P}(\mathcal{F}_2)$ is obtained from $\mathcal{P}(\mathcal{F}_1)$ by performing a single diagonal flip on an arc α in $\mathcal{P}(\mathcal{F}_1)$ and the shear coordinate of α is positive in $\mathcal{P}(\mathcal{F}_1)$. We thus obtain following proposition.

Proposition 6.9. If T is a tree whose internal vertices have degree 3, then $\overrightarrow{FG}(T) \cong \overrightarrow{EG}(\widehat{Q}_{\mathcal{P}_T})$ and this isomorphism commutes with flips and mutations.

Example 6.10. If every internal vertex of T has degree 3, then $\widetilde{\Delta}^{NC}(T)$ is isomorphic to the dual associahedron. By this identification, our notion of performing a flip on a facet of the reduced noncrossing complex of T translates into the well-known operation of performing a **diagonal flip** on the corresponding triangulation (see Figure 20). If every triangle in P_T involves at least one boundary diagonal, then Proposition 6.9 implies that $\overrightarrow{FG}(T)$ is isomorphic to a Cambrian lattice of type A.

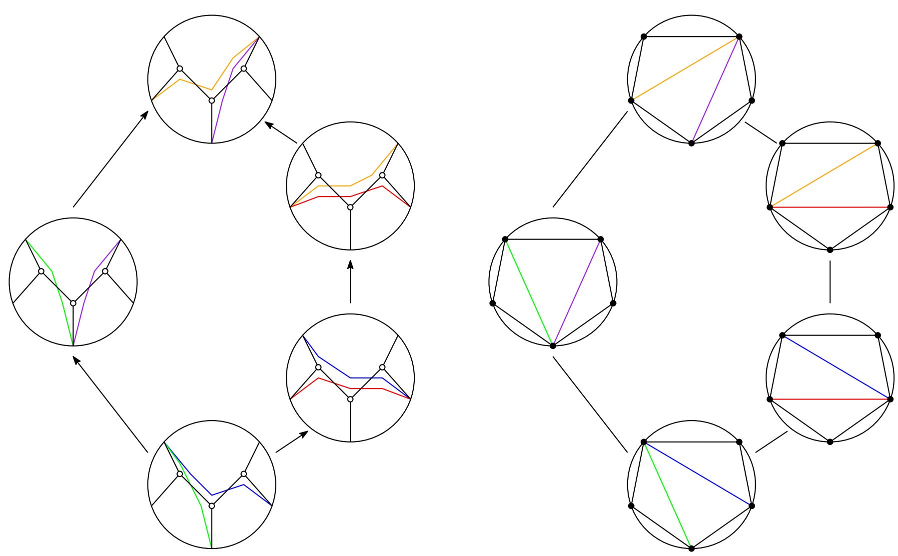


FIGURE 20. An oriented flip graph and the triangulations corresponding to each facet of the reduced noncrossing complex. In this oriented flip graph we only show the nonboundary arcs in each facet.

Remark 6.11. A version of Theorem 6.7 has been established by Brüstle and Qiu (see [6]) for oriented exchange graphs defined by quivers arising from triangulations of **marked surfaces** (see [15] for more details). By identifying a convex polygon with an unpunctured disk, Theorem 6.7 recovers their result in the case where one considers oriented flip graphs of a tree arising from a polygonal subdivision of an unpunctured disk. In their language, ϱ is the **universal tagged rotation** of the marked surface.

6.2. **Stokes poset of quadrangulations.** The Stokes poset defined by Chapoton in [8] is a partial order on a family of quadrangulations which are "compatible" with a given quadrangulation Q. The compatibility condition was defined by Baryshnikov as follows [3]. Let P be a $(2n)$-gon whose vertices lie on a circle. The vertices of P are colored black and white, alternating in color around the circle. Let P' be the same polygon, rotated slightly clockwise. A **quadrangulation** is a polygonal subdivision into quadrilaterals. Fix a quadrangulation Q of P. A quadrangulation Q' of P' is compatible with Q if for each diagonal $q \in Q$ and $q' \in Q'$ such that q and q' intersect, the white endpoint of q' appears clockwise from the white endpoint of q before the black endpoint of q.

Let T be the tree dual to P. We may assume that the leaves of T are the vertices of Q'. If p is a geodesic between two leaves of T that does not take a sharp turn at an interior vertex then it crosses a pair of opposite sides of some quadrilateral in Q. As a result, p cannot be part of a quadrangulation compatible with Q. Let Δ_Q be the simplicial complex on the diagonals of Q' whose facets are quadrangulations compatible with Q. Then Δ_Q is a pure subcomplex of $\Delta^{NC}(T)$ of the same dimension. The complex Δ_Q is thin by Proposition 1.1 of [8]. Since the dual graph of $\Delta^{NC}(T)$ is connected, it follows that Δ_Q and $\Delta^{NC}(T)$ are isomorphic. Moreover, the orientation on the flips of quadrangulations defined in Section 1.3 of [8] coincides with $\overrightarrow{FG}(T)$. Consequently, we deduce the following proposition.

Proposition 6.12. If every interior vertex of T has degree 4, then the poset $\overrightarrow{FG}(T)$ is isomorphic to the Stokes poset of quadrangulations compatible with the quadrangulation $\mathcal{P}_T$. In particular, the Stokes poset is a lattice.

The final assertion of Proposition 6.12 was originally conjectured by Chapoton in [8].

References

[1] C. Amiot and Y. Grimeland. Derived invariants for surface algebras. *J. Pure Appl. Algebra*, 220(9):3133–3155, 2016.
[2] E. Barnard. personal communication.
[3] Y. Baryshnikov. On stokes sets. In *New Developments in Singularity Theory*, pages 65–86. Springer, 2001.
[4] A. Björner and M. L. Wachs. Shellable nonpure complexes and posets. II. *Trans. Amer. Math. Soc.*, 349:3945–3976, 1997.
[5] T. Brüstle, G. Dupont, and M. Pérotin. On maximal green sequences. *IMRN*, 2014(16):4547–4586, 2014.
[6] T. Brüstle and Y. Qiu. Tagged mapping class groups: Auslander–Reiten translation. *Math. Z.*, 279(3-4):1103–1120, 2015.
[7] N. Caspard and C. Le Conte Poly-Barbut. Tamari lattices are bounded: a new proof. Technical report, Technical Report TR-2004-03, LACL, Université Paris XII, 2004.
[8] F. Chapoton. Stokes posets and serpent nests. *Discrete Math. Theor. Comput. Sci.*, 18, 2016.
[9] L. David-Roesler. Derived equivalence of surface algebras in genus 0 via graded equivalence. *Algebr. Represent. Theory*, 17(1):1–30, 2014.
[10] L. David-Roesler and R. Schiffler. Algebras from surfaces without punctures. *J. Algebra*, 350(1):218–244, 2012.
[11] A. Day. Congruence normality: the characterization of the doubling class of convex sets. *Algebra Universalis*, 31(3):397–406, 1994.
[12] H. Derksen, J. Weyman, and A. Zelevinsky. Quivers with potentials and their representations I: Mutations. *Selecta Math.*, 14(1):59–119, 2008.
[13] H. Derksen, J. Weyman, and A. Zelevinsky. Quivers with potentials and their representations II: applications to cluster algebras. *J. Amer. Math. Soc.*, 23(3):749–790, 2010.
[14] A. Dermenjian, C. Hohlweg, and V. Pilaud. The facial weak order and its lattice quotients. *arXiv:1602.03158*, 2016.
[15] S. Fomin, M. Shapiro, and D. Thurston. Cluster algebras and triangulated surfaces. part I: Cluster complexes. *Acta Math.*, 201(1):83–146, 2008.
[16] S. Fomin and D. Thurston. Cluster algebras and triangulated surfaces. part II: Lambda lengths. *arXiv:1210.5569*, 2012.
[17] R. Freese, J. Jezek, and J.B. Nation. Free lattices, mathematical surveys and monographs 42. *Amer. Math. Soc., Providence, RI*, 1995.
[18] A. Garver and T. McConville. Lattice properties of oriented exchange graphs and torsion classes. *arXiv:1507.04268*, 2015.
[19] A. Garver and T. McConville. Oriented flip graphs, noncrossing tree partitions, and representation theory of tiling algebras. *preprint available at http://lacim.uqam.ca/~garver/*, 2016.
[20] W. Geyer. *Intervallverdopplung und verwandte Konstruktionen bei Verbänden.* PhD thesis, Fachbereich Mathematik der Technischen Hochschule Darmstadt, 1992.
[21] G. Kreweras. Sur les partitions non croisées d'un cycle. *Discrete Math.*, 1(4):333–350, 1972.
[22] T. McConville. Lattice structure of Grid-Tamari orders. *J. Combin. Theory Ser. A*, 148:27–56, 2017.
[23] H. Mühle. On the lattice property of shard orders. *arXiv preprint arXiv:1708.02104*, 2017.
[24] G. J. Murphy. Derived equivalence classification of m-cluster tilted algebras of type A_n. *J. Algebra*, 323(4):920–965, 2010.
[25] V. Pilaud. Brick polytopes, lattice quotients, and hopf algebras. *arXiv:1505.07665*, 2015.
[26] V. Pons. A lattice on decreasing trees: the metasylvester lattice. *Discrete Math. Theor. Comput. Sci.*, 2015.
[27] N. Reading. Lattice and order properties of the poset of regions in a hyperplane arrangement. *Algebra Universalis*, 50(2):179–205, 2003.
[28] N. Reading. Cambrian lattices. *Adv. Math.*, 2(205):313–353, 2006.
[29] N. Reading. Noncrossing partitions and the shard intersection order. *J. Algebraic Combin.*, 33(4):483–530, 2011.
[30] N. Reading. Lattice theory of the poset of regions. In *Lattice theory: special topics and applications*, volume 2. Birkhäuser/Springer Basel AG, Basel, 2016.
[31] S. Schroll. Trivial extensions of gentle algebras and brauer graph algebras. *J. Algebra*, 444:183–200, 2015.

[32] R. Coelho Simoes and M. J. Parsons. Endomorphism algebras for a class of negative Calabi-Yau categories. *J. Algebra*, 2017.