Noncrossing Tree Partitions & Tiling Algebras

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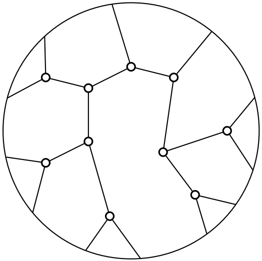
Fix T a tree embedded in a disk with exactly its leaves on the boundary and whose interior vertices have degree at least 3.

$T =$ 

$\mathrm{Facets}(\Delta^{NC}(T)) \xrightarrow{\sim} \mathrm{NCP}(T)$, $\quad \varphi: \mathrm{Facets}(\Delta^{NC}(T)) \to \mathrm{Bic}(T)$, $\quad \eta: \mathrm{Bic}(T) \to \mathrm{Facets}(\Delta^{NC}(T))$, $\quad \mathrm{NCP}(T) \to \mathrm{Bic}(T)$

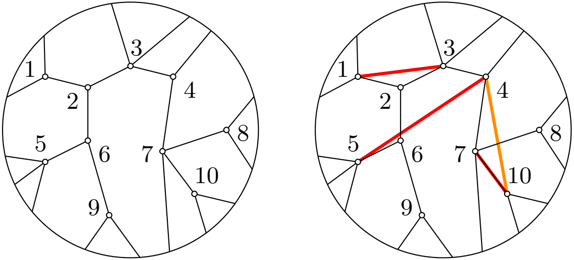
$\mathrm{torsf}(\Lambda_T) \xrightarrow{\sim} \text{2-smc}(\Lambda_T)$, $\quad \varphi: \mathrm{torsf}(\Lambda_T) \to \mathrm{Bic}(\Lambda_T)$, $\quad \eta: \mathrm{Bic}(\Lambda_T) \to \mathrm{torsf}(\Lambda_T)$, $\quad \text{2-smc}(\Lambda_T) \to \mathrm{Bic}(\Lambda_T)$

combinatorics of T — representation theory of Λ_T

Goal: Understand the combinatorics and representation theory related to noncrossing tree partitions.

Noncrossing Tree Partitions

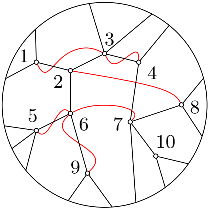
Fix T a tree embedded in a disk with exactly its leaves on the boundary and whose interior vertices have degree at least 3.

$T =$ 

A **segment** $s = (v_0, \ldots, v_t) = [v_0, v_t]$ with $t \geqslant 1$ is a sequence of interior vertices of T that takes a "sharp" turn at each v_i. In particular, interior vertices of T are not segments.

Example

The sequences $(1, 2, 3)$, $(5, 6, 2, 3, 4)$, and $(7, 10)$ are segments. The sequence $(4, 7, 10)$ is not a segment.



$\longleftrightarrow$ $\mathbf{B} = \{\{1,3,4\},\{2,8\},\{5,6,7,9\},\{10\}\}$

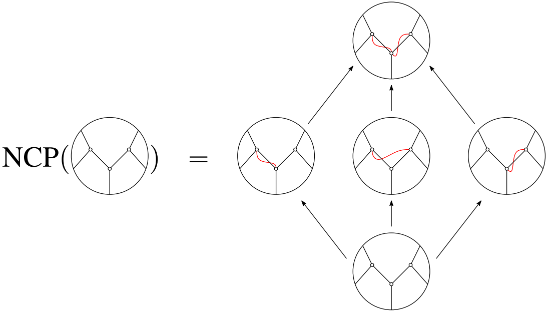
A **noncrossing tree partition** $\mathbf{B} = (B_1, \ldots, B_k)$ of T is a set partition of the interior vertices of T where

- vertices in B_i can be connected by pairwise nonoverlapping **red admissible curves** (i.e. curves whose endoints define segments of T and leave their endpoints to the right) and
- red admissible curves connecting vertices of B_i do not cross those connecting vertices of B_j for $i \neq j$.

Let $\mathrm{NCP}(T)$ denote the poset of noncrossing tree partitions ordered by refinement.

Given $\mathbf{B} \in \mathrm{NCP}(T)$, let $\mathrm{Seg}(\mathbf{B})$ be the segments of T defined by $\mathbf{B}$ (for example, $\mathrm{Seg}(\mathbf{B}) = \{[1,3],[3,4],[2,8],[5,6],[6,7],[6,9]\}$).

Noncrossing tree partitions generalize the classical noncrossing set partitions.



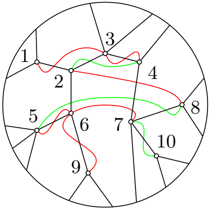
Proposition (G.–McConville)

If the interior vertices of T have degree exactly 3, then $\#NCP(T) = \frac{1}{n+1}\binom{2n}{n}$ *where* $n = \#(\text{interior vertices of } T)$.

Theorem (G.–McConville)

The poset $NCP(T)$ *is a lattice.*

One also has a notion of **Kreweras complement** on noncrossing tree partitions.



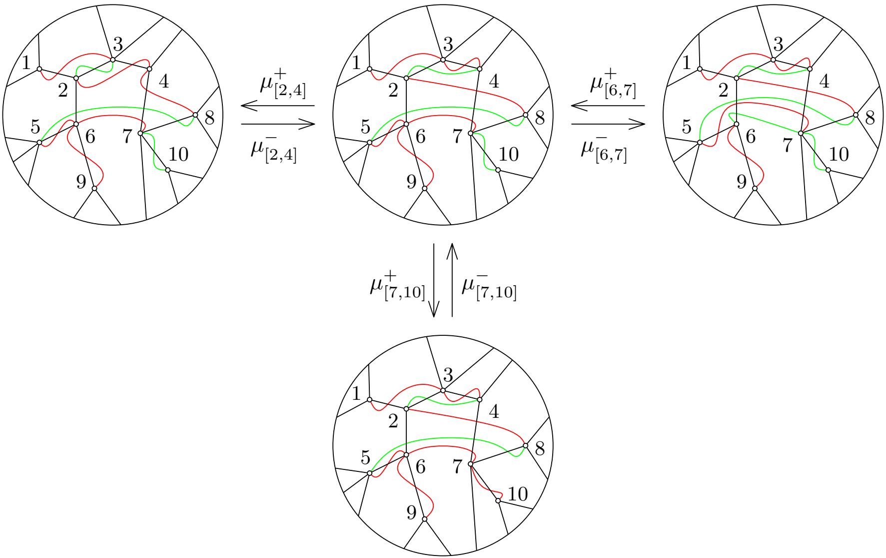
$\longleftrightarrow$ $\mathrm{Kr}(\mathbf{B}) = \left\{ \begin{array}{c} \{1\}, \{2,4\}, \{3\}, \{5,8\}, \\ \{6\}, \{7,10\}, \{9\} \end{array} \right\}$

The Kreweras complement of **B** is the unique noncrossing tree partition of T such that when drawn using **green admissible curves** one obtains a noncrossing tree on the interior vertices of T.

Theorem (G.–McConville)

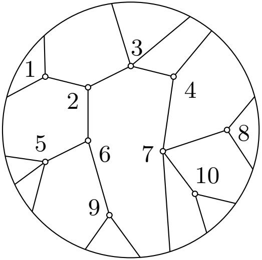
The map $Kr : NCP(T) \longrightarrow NCP(T)$ is a bijection.

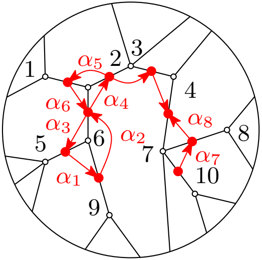
Given a noncrossing tree partition and its Kreweras complement $(\mathbf{B}, \mathrm{Kr}(\mathbf{B}))$, one can obtain all other such pairs by local moves.



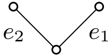
Representation theory of Λ_T

Let $\Bbbk = \overline{\Bbbk}$. A tree T defines a finite dimensional $\Bbbk$-algebra, denoted $\Lambda_T = \Bbbk Q_T / I_T$, called a **tiling algebra** (Parsons-Simoes '16). The elements of $\Bbbk Q_T$ are $\Bbbk$-linear combinations of paths in Q_T.

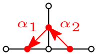
$T =$ 

$\longmapsto$ $Q_T =$ 

$\{\text{vertices of } Q_T\} = \{e : \text{ where } e \text{ is an interior edge of } T\}$

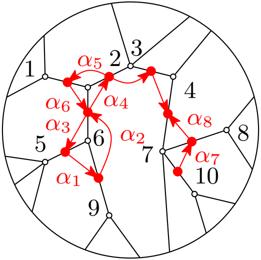
$\{\text{arrows of } Q_T\} = \left\{ e_1 \xrightarrow{\alpha} e_2 : \right.$ where e_1 and e_2 form a **corner** of T 

$\left.\right\}$

$I_T = \langle \alpha_1\alpha_2 :$ 

$\rangle = \langle \alpha_2\alpha_1, \alpha_3\alpha_2, \alpha_1\alpha_3, \alpha_5\alpha_4, \alpha_6\alpha_5, \alpha_4\alpha_6, \alpha_8\alpha_7 \rangle$

$Q_T \quad =$

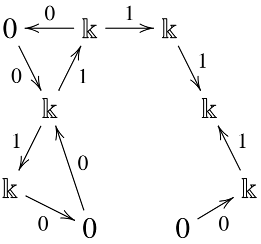


Proposition (Butler–Ringel ‘87)

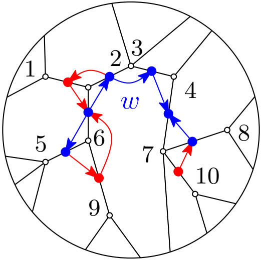
The algebra Λ_T is a ***string algebra*** *since*

- *each vertex of Q_T has at most two arrows starting at it and at most two arrows ending at it and*
- *for each arrow α_1 of Q_T there are at most two arrows α_2, α_3 such that $\alpha_1\alpha_2 \notin I_T$ and $\alpha_3\alpha_1 \notin I_T$.*

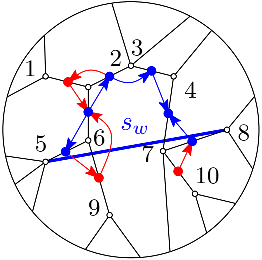
Thus the indecomposable Λ_T-modules are given by ***string modules*** *$M(w)$ (i.e. representations of Q_T supported on connected subgraphs that obey the relations from I_T).*



$M(w)$
string module



w
string



s_w
segment

Proposition (G.–McConville)

The indecomposable Λ_T-modules are indexed by the segments of T.

Simple-minded collections

Noncrossing tree partitions arise as configurations of objects in derived categories.

$$\begin{array}{ccc} \Lambda & \rightsquigarrow & \mathcal{D}(\Lambda) \text{ the } \textbf{derived category} \text{ of } \Lambda \\ & & \text{(Grothendieck, Verdier '60s)} \\ \text{a ring} & & \text{a triangulated category} \end{array}$$

Objects of $\mathcal{D}(\Lambda)$ are cochain complexes of Λ-modules (i.e.

$$X \quad = \quad \cdots \xrightarrow{d_X^{-2}} X^{-1} \xrightarrow{d_X^{-1}} X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} X^2 \xrightarrow{d_X^2} \cdots$$

that satisfies $d_X^{i+1} \circ d_X^i = 0$ for each $i \in \mathbb{Z}$) defined up to cohomology. The category $\mathcal{D}(\Lambda)$ also has a **shift functor** $[1] : \mathcal{D}(\Lambda) \to \mathcal{D}(\Lambda)$ where

$$X[1] \quad = \quad \cdots \xrightarrow{-d_X^{-1}} X^0 \xrightarrow{-d_X^0} X^1 \xrightarrow{-d_X^1} X^2 \xrightarrow{-d_X^2} X^3 \xrightarrow{-d_X^3} \cdots .$$

Question: When do two rings Λ and Γ have equivalent derived categories (as triangulated categories)?

Theorem (Rickard '89)

*Two rings Λ and Γ are **derived equivalent** if and only if there is a **tilting complex** $T \in \mathcal{D}(\Lambda)$ such that $\Gamma \cong End_{\mathcal{D}(\Lambda)}(T)$ (as rings).*

Theorem (Rickard '02)

*If Λ is a finite dimensional symmetric $\mathbb{k}$-algebra, then any **simple-minded collection** $\{X_1, \ldots, X_n\}$ defines a tilting complex $T = \bigoplus_{i=1}^{n} X_i$.*

Other examples of simple-minded collections are given by

- a complete set of nonisomorphic simple modules regarded as elements of $\mathcal{D}^b(\Lambda\text{-mod})$ and
- **spherical collections** (Seidel-Thomas) in algebraic geometry.

Definition (Koenig-Yang '13)

A collection $\{X_1, \ldots, X_n\}$ of objects of $\mathcal{D}^b(\Lambda\text{-mod})$ is **simple-minded** if the following hold for any $i, j \in [n]$:

i) $\mathrm{Hom}_{\mathcal{D}^b(\Lambda\text{-mod})}(X_i, X_j[k]) = 0$ for any $k < 0$,

ii) $\mathrm{Hom}_{\mathcal{D}^b(\Lambda\text{-mod})}(X_i, X_j) = \begin{cases} \Bbbk & : \text{ if } i = j \\ 0 & : \text{ otherwise,} \end{cases}$

iii) the smallest triangulated category containing $X_1, \ldots, X_n$ and closed under taking summands of objects is $\mathcal{D}^b(\Lambda\text{-mod})$.

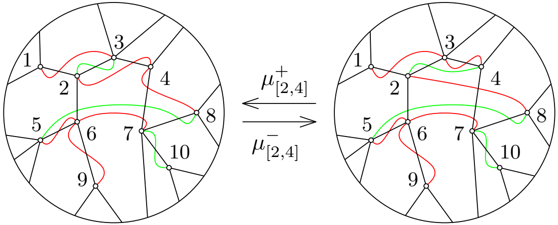
Let 2-smc(Λ_T) denote the set simple-minded collections $\{X_1, \ldots, X_n\}$ where $H^k(X_i) =$ for any $i \in \{1, \ldots, n\}$ and any $k \neq 0, -1$.

Theorem (G.–McConville)

The map $\{(\boldsymbol{B}, Kr(\boldsymbol{B}))\}_{\boldsymbol{B} \in NCP(T)} \to$ 2-*smc*(Λ_T) *given by*

$$(\boldsymbol{B}, Kr(\boldsymbol{B})) \xrightarrow{\theta} \begin{array}{l} \{M(u)[1] : s_u \in Seg(B) \text{ where } B \in \boldsymbol{B}\} \sqcup \\ \{M(v) : s_v \in Seg(B') \text{ where } B' \in Kr(\boldsymbol{B})\} \end{array}$$

is a bijection. Furthermore, this map is compatible with ***mutations*** *(as introduced by Koenig–Yang '13).*



Thanks!