The canonical join complex for biclosed sets

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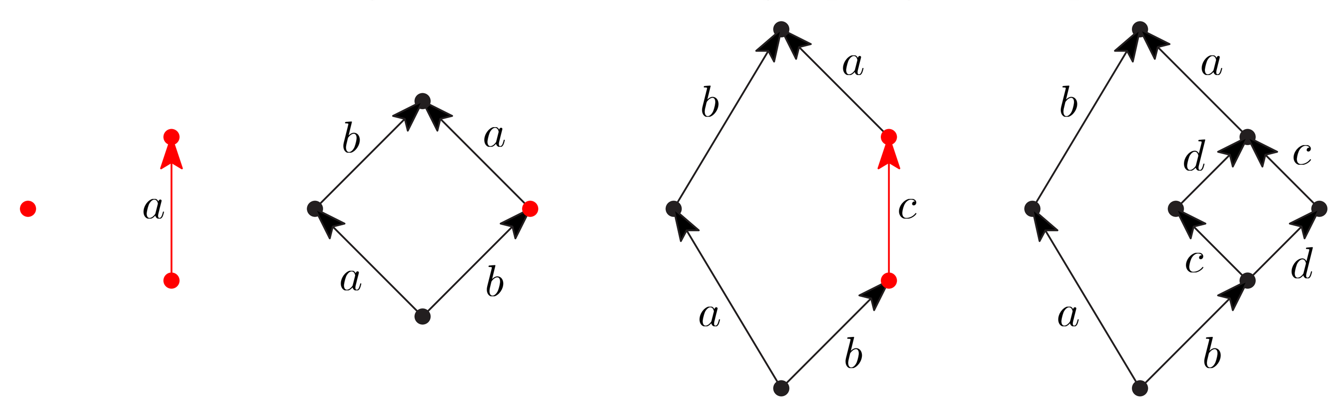
Goal: combinatorially describe the canonical join complexes and shard intersection orders of a family of congruence-uniform lattices.

Congruence-uniform lattices

A finite lattice L is **congruence-uniform** if it is constructible from the one element lattice by interval doublings [Day, 1979].



Congruence-uniformity is equivalent to a function $\lambda : \mathrm{Cov}(L) \to P$ with certain properties. Say λ is a **CU-labeling** of L.
Any element $x \in L$ has a **canonical join representation**. That is, there exists $A \subseteq \mathrm{JI}(L)$ such that $x = \bigvee A$ and where

- $\bigvee A' < \bigvee A$ for any $A' \subsetneq A$, and
- if $B \subseteq \mathrm{JI}(L)$ satisfies $x = \bigvee B$ and $\bigvee B' < \bigvee B$ for any $B' \subsetneq B$, then for any $a \in A$ there exists $b \in B$ such that $a \leq b$.

Lemma: There is a bijection $f : \mathrm{JI}(L) \to \lambda(\mathrm{Cov}(L))$ where $j \mapsto \lambda(j_*, j)$.
The canonical join representation of any $x \in L$ is given by $\bigvee f^{-1}(\lambda_\downarrow(x))$ where $\lambda_\downarrow(x) := \{\lambda(y,x) : y \lessdot x\}$.
The **canonical join complex** of L, denoted $\Delta^{CJ}(L)$, is the abstract simplicial complex whose vertex set is $\mathrm{JI}(L)$ and whose faces are sets of join-irreducibles whose join is a canonical join representation of some element of L.

Biclosed sets of segments of a tree

Let T be a tree embedded in a disk. A path $[v_0, v_n] = (v_0, \ldots, v_n)$ in T is a **segment** if, for each $0 \leq j \leq n-2$, $[v_{j+1}, v_{j+2}]$ is immediately clockwise or counterclockwise from $[v_j, v_{j+1}]$ with respect to v_{j+1}.
Let $\mathrm{Seg}(T)$ denote the set of segments of T. A subset $B \subset \mathrm{Seg}(T)$ is **closed** if for all composable $s_1, s_2 \in B$, $s_1 \circ s_2 \in B$. The set B is **biclosed** if both B and $B^c := \mathrm{Seg}(T) \backslash B$ are closed.

$$\mathrm{Bic}(T) := \{\text{biclosed subsets of } \mathrm{Seg}(T)\}$$

Theorem: [Garver–McConville, 2016] The poset $\mathrm{Bic}(T)$ is a congruence-uniform lattice.

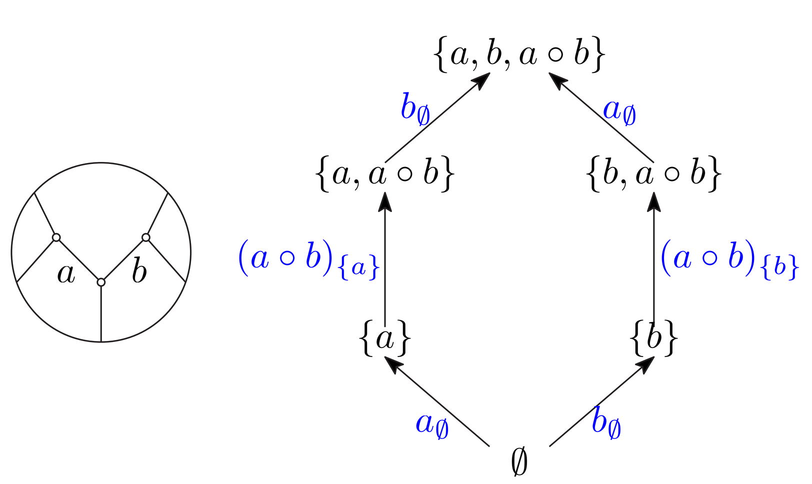


Figure: A tree T and the poset $\mathrm{Bic}(T)$ along with the CU-labeling.

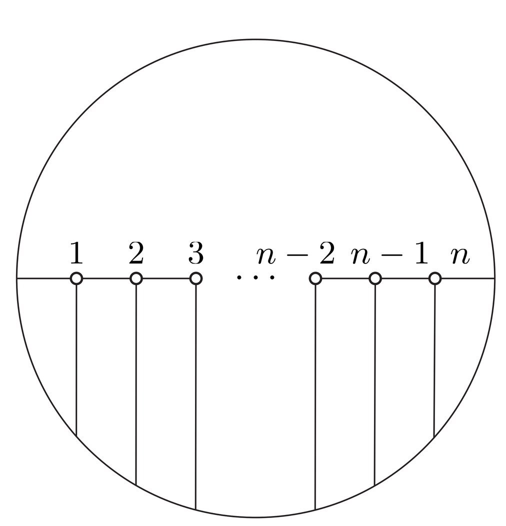


Figure: The poset $\mathrm{Bic}(T)$ is isomorphic to the weak order on $\mathfrak{S}_n$.

CU-labeling of $\mathrm{Bic}(T)$

Define its poset $\mathcal{S}_T$ of **labels** whose elements are of the form $s_\mathcal{D} := (s, \{s_1, s_2, \ldots, s_m\})$ where $s = (v_0, v_1, \ldots, v_{m+1})$ and where:

- each s_i is a **split** of s (i.e., there exists $t \in \mathrm{Seg}(T)$ such that $s = s_i \circ t$),
- $s \neq s_i \circ s_j$ for any $i, j \in \{1, \ldots, m\}$.

Proposition: The map $\lambda : \mathrm{Cov}(\mathrm{Bic}(T)) \to \mathcal{S}_T$ given by $\lambda(B, B \sqcup \{s\}) = s_\mathcal{D}$ is a CU-labeling.

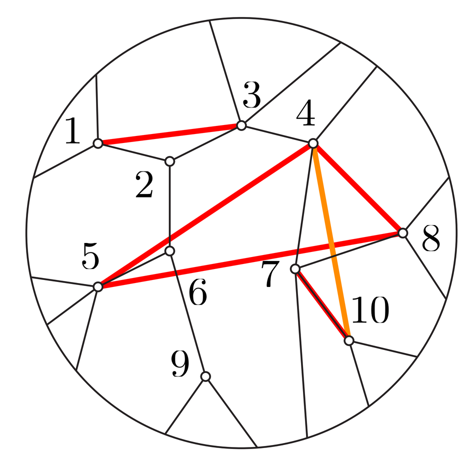


Figure: The segments $[1,3]$, $[4,8]$, $[5,4]$, $[5,8]$, $[7,10]$. Here $[5,8] = [5,4] \circ [4,8]$. The path $[4,10]$ is not a segment.

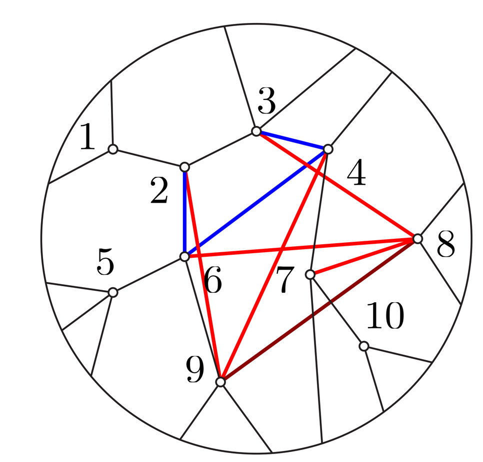


Figure: The join irreducible biclosed set $J([9,8]_{\{[9,2],[9,4],[6,8],[3,8],[7,8]\}})$.

Given $s_\mathcal{D} \in \mathcal{S}_T$, define

$$J(s_\mathcal{D}) := \{s\} \sqcup \mathcal{D} \sqcup \bigcup_{t \in \mathcal{D}} S(t)$$

where $S(t) \subset \mathrm{Seg}(T)$ is the set of all splits s' of t satisfying the following:
i) segment s' is not a split of s, and
ii) segment s' is not composable with any segment in $\mathcal{D}$ (see Figure 4).

Proposition: The set $J(s_\mathcal{D})$ satisfies $\lambda_\downarrow(J(s_\mathcal{D})) = \{s_\mathcal{D}\}$. Moreover, any biclosed set B with $s_\mathcal{D} \in \lambda_\downarrow(B)$ satisfies $J(s_\mathcal{D}) \leq B$, and the reverse containment holds if and only if $\lambda_\downarrow(B) = \{s_\mathcal{D}\}$. Consequently, the map $J(-) : \mathcal{S}_T \to \mathrm{JI}(\mathrm{Bic}(T))$ is a bijection.

Canonical join complex of $\mathrm{Bic}(T)$

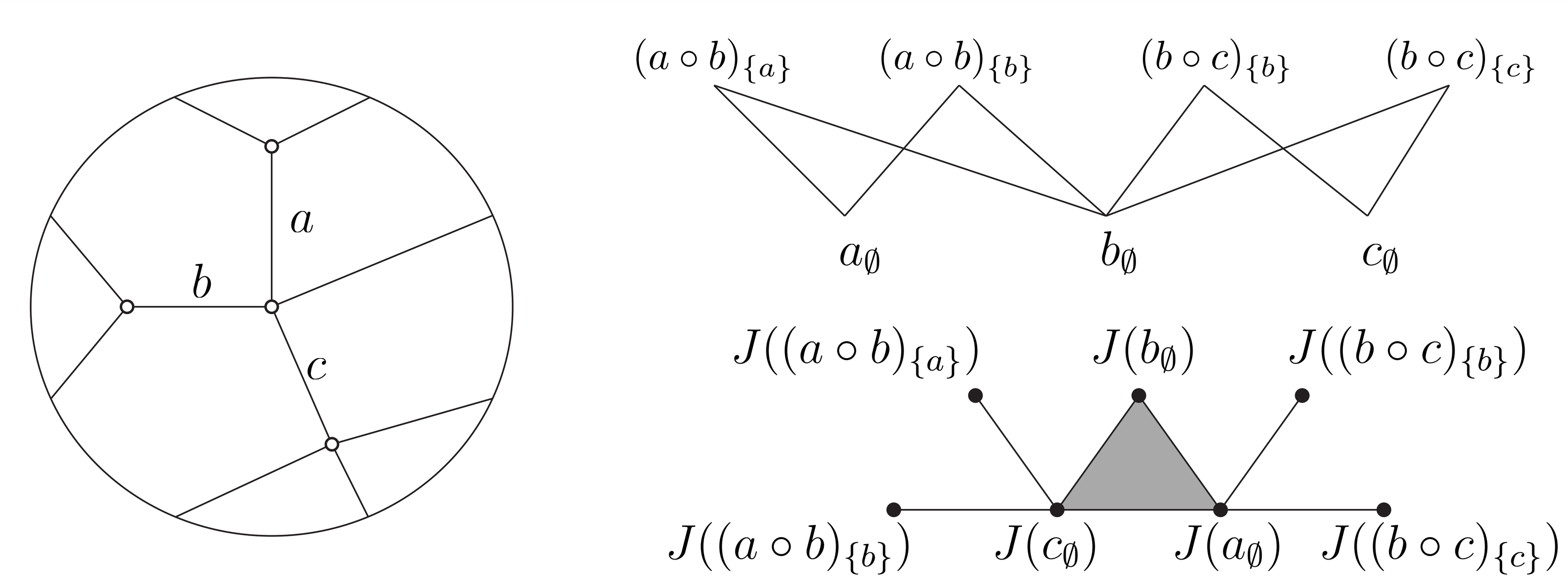


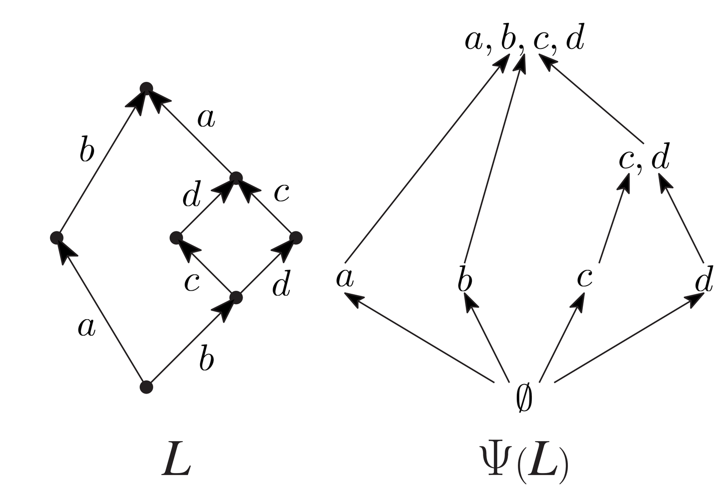
Figure: A tree T, its poset of labels $\mathcal{S}_T$, and its canonical join complex $\Delta^{CJ}(\mathrm{Bic}(T))$. The shortest segments of T are labeled a, b, and c.

Theorem: A collection $\{J(s^1_{\mathcal{D}^1}), \ldots, J(s^k_{\mathcal{D}^k})\} \subset \mathrm{JI}(\mathrm{Bic}(T))$ is a face of $\Delta^{CJ}(\mathrm{Bic}(T))$ if and only if labels $s^i_{\mathcal{D}^i}$ and $s^j_{\mathcal{D}^j}$ satisfy the following:

1) segments s^i and s^j are distinct,
2) neither s^i nor s^j is expressible as a composition of at least two segments in $J(s^i_{\mathcal{D}^i}) \cup J(s^j_{\mathcal{D}^j})$, and
3) neither $J(s^i_{\mathcal{D}^i}) \leq J(s^j_{\mathcal{D}^j})$ nor $J(s^j_{\mathcal{D}^j}) \leq J(s^i_{\mathcal{D}^i})$

for any distinct $i, j \in \{1, \ldots, k\}$.

Shard intersection order of $\mathrm{Bic}(T)$



Figure

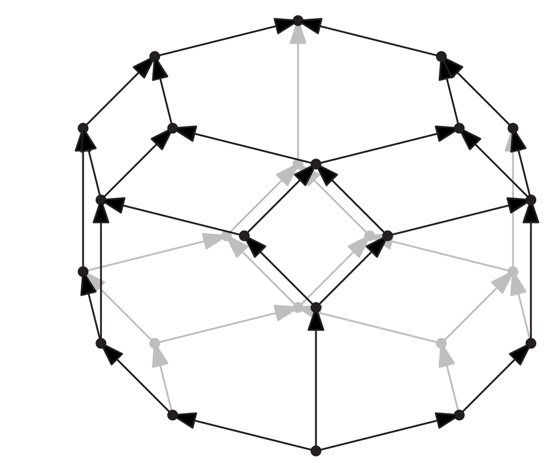


Figure: Bic( )

- L a congruence-uniform lattice,
- $\lambda : \mathrm{Cov}(L) \to P$ a CU-labeling,
- $x \in L$ and covers $y_1, \ldots y_k \in L$,

The **shard intersection order** of L, denoted $\Psi(L)$, is the collection of sets

$$\psi(x) := \left\{\text{labels on covering relations in } \left[\bigwedge_{i=1}^{k} y_i, x\right]\right\}$$

partially ordered by inclusion [Reading, 2016].
Theorem: Given a biclosed set $B \in \mathrm{Bic}(T)$, $\psi(B)$ is the set of all labels of the form $(s^{i_1} \circ \cdots \circ s^{i_\ell})_\mathcal{D}$ with $s^{i_j}_{\mathcal{D}_{i_j}} \in \lambda_\downarrow(B)$ where $\mathcal{D}$ is any set of segments that satisfies the following properties:
(i) $|\mathcal{D}| = |\{\text{breaks of } s^{i_1} \circ \cdots \circ s^{i_\ell}\}|$,
(ii) each segment $t \in \mathcal{D}$ is a split of $s^{i_1} \circ \cdots \circ s^{i_\ell}$,
(iii) no two distinct splits $t_1, t_2 \in \mathcal{D}$ appear in the same break of $s^{i_1} \circ \cdots \circ s^{i_\ell}$, and
(iv) whenever $t \in \mathcal{D}$ is a non-faultline split of $s^{i_1} \circ \cdots \circ s^{i_\ell}$, we have that $t = s^{i_1} \circ \cdots \circ s^{i_{j-1}} \circ t_j$ for some $j = 1, \ldots, \ell$ and some $t_j \in \mathcal{D}_{i_j}$ or $t = t_j \circ s^{i_{j+1}} \circ \cdots \circ s^{i_\ell}$ for some $j = 1, \ldots, \ell$ and some $t_j \in \mathcal{D}_{i_j}$. In the former case if $j = 1$, we mean $t = t_1$, and in the latter case, if $j = \ell$, we mean $t = t_\ell$.

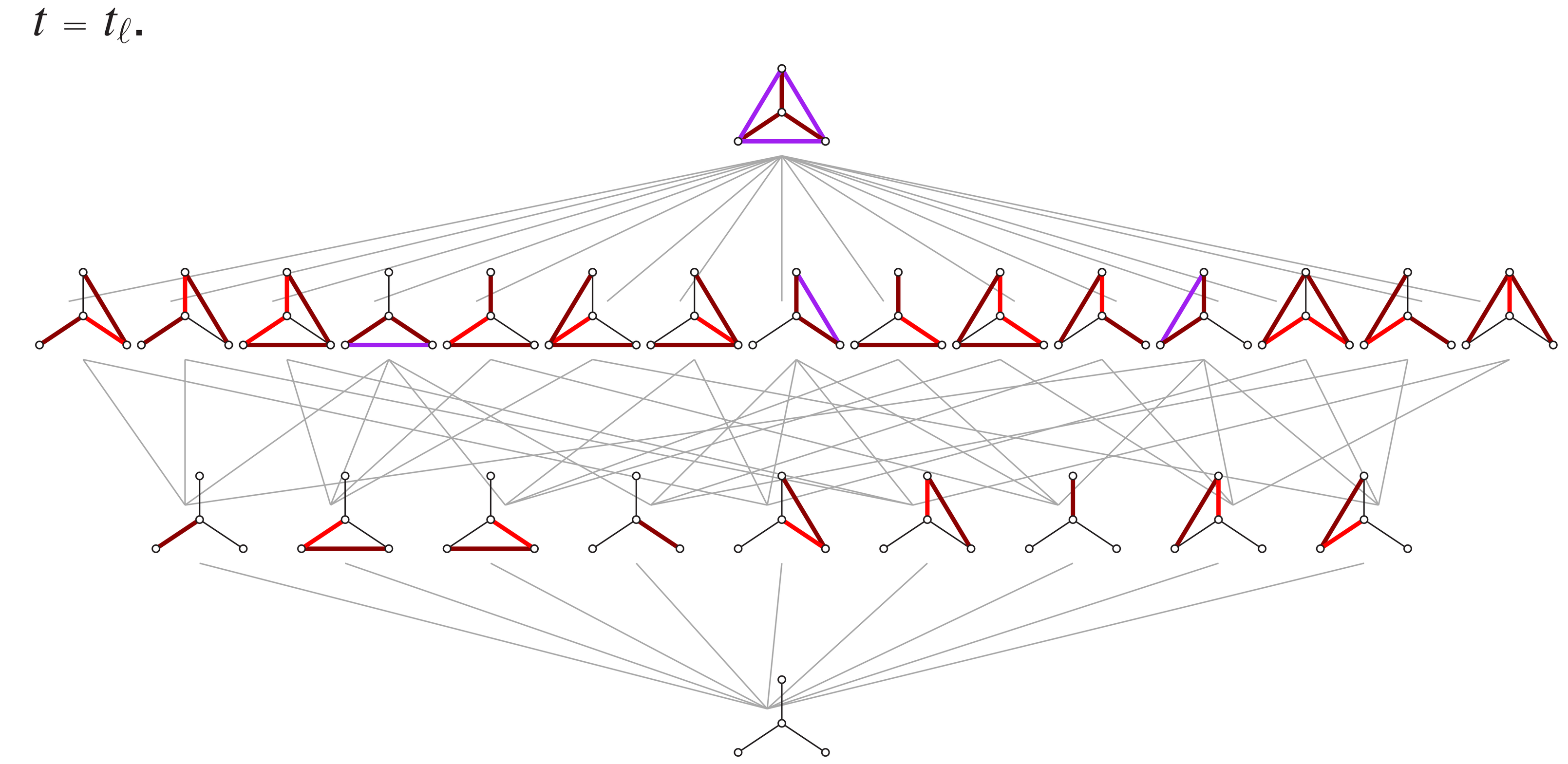


Figure: $\Psi(\mathrm{Bic}($ $))$

Theorem: The shard intersection order $\Psi(\mathrm{Bic}(T))$ is a lattice.

Problem [Reading, 2016]

Determine the set of finite congruence-uniform lattices L with the property that the shard intersection order $\Psi(L)$ is a lattice.

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