Minimal length maximal green sequences

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Introduction

Maximal green sequences appear in many areas including

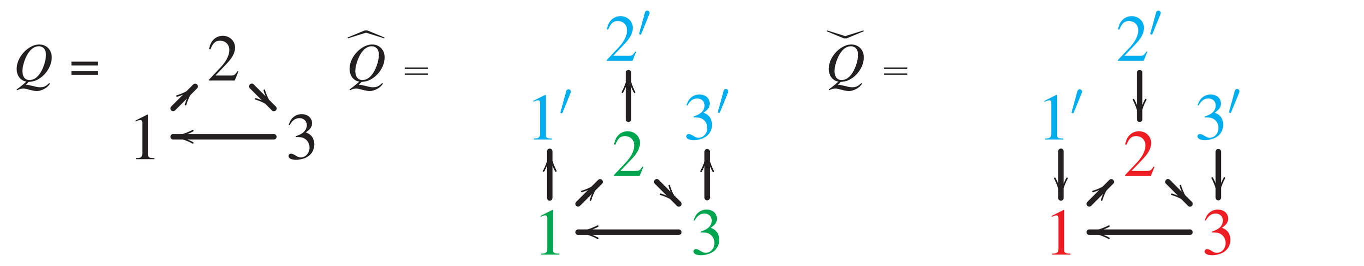
i) representation theory (Keller 2011, Brüstle–Dupont–Pérotin 2013),

ii) cluster algebras (Gross–Hacking–Keel–Kontsevich 2014),

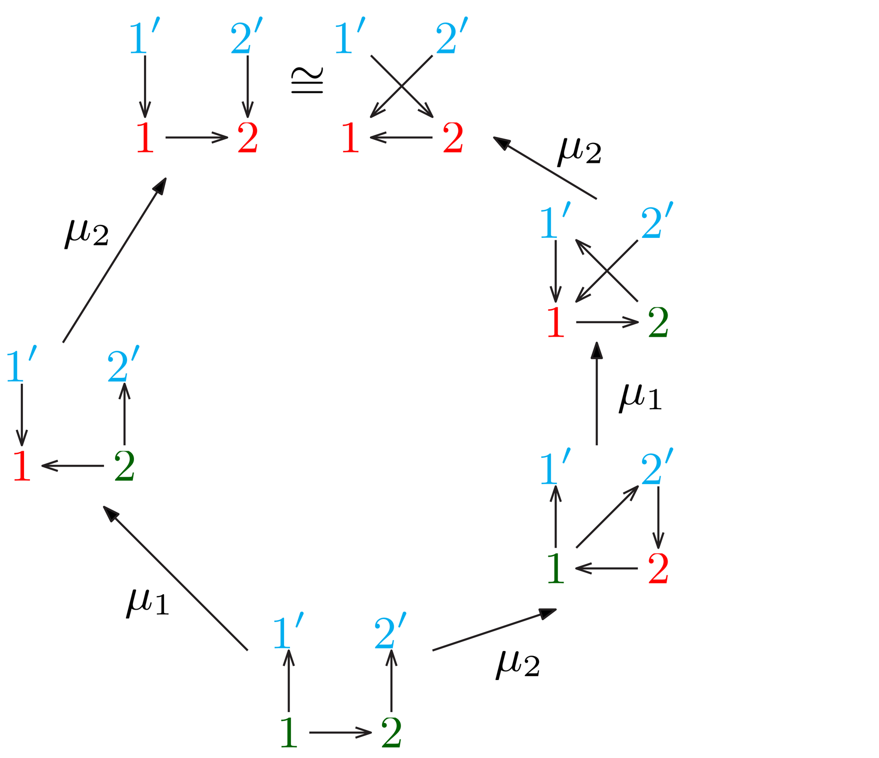
iii) BPS spectra in string theory (Alim–Cecotti–Cordova–Espahbodi–Rastogi–Vafa 2013)

Quiver mutation

Q – a **2-acyclic** quiver (i.e., Q has no loops or 2-cycles). Add **frozen vertices** to Q.



framed quiver **coframed quiver**



Mutate $\widehat{Q}$ at any non-frozen vertex k to obtain a quiver $\mu_k(\widehat{Q})$. The quiver $\mu_k(\widehat{Q})$ is obtained from $\widehat{Q}$ by

(i) inserting new arrow $i \to j$ for each 2-path $i \to k \to j$ in $\widehat{Q}$

(ii) reversing arrows incident to k

(iii) delete any 2-cycles

- Any non-frozen vertex of a quiver mutation-equivalent to $\widehat{Q}$ is either green or red (Derksen–Weyman–Zelevinsky 2010)

Combinatorial properties of maximal green sequences

A **maximal green sequence** of Q is a sequence $\mathbf{i} = (i_1, \ldots, i_k)$ of non-frozen vertices of $\widehat{Q}$ where

(i) for all $j \in [k]$ vertex i_j is **green** in $\mu_{i_{j-1}} \circ \cdots \circ \mu_{i_1}(\widehat{Q})$ and

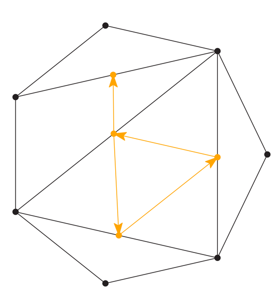
(ii) all vertices in $\mu_{i_k} \circ \cdots \circ \mu_{i_1}(\widehat{Q})$ are **red**.

$\ell_{\text{min}}(Q) :=$ length of shortest maximal green sequence of Q

$\ell_{\text{max}}(Q) :=$ length of longest maximal green sequence of Q

- If Q is Dynkin, $\ell_{\text{min}}(Q) = |Q_0|$ and $\ell_{\text{max}}(Q) = |\Phi^+(Q)|$ where $|Q_0| = n$. (Brüstle–Dupont–Pérotin 2013)
- If Q is acyclic, $\ell_{\text{min}}(Q) = |Q_0|$. (Brüstle–Dupont–Pérotin 2013)
- If Q is **mutation type** $\mathbb{A}$, then $\ell_{\text{min}}(Q) = |Q_0| + |\{$3-cycles of $Q\}|$ (Cormier–Dillery–Resh–Serhiyenko–Whelan 2015)

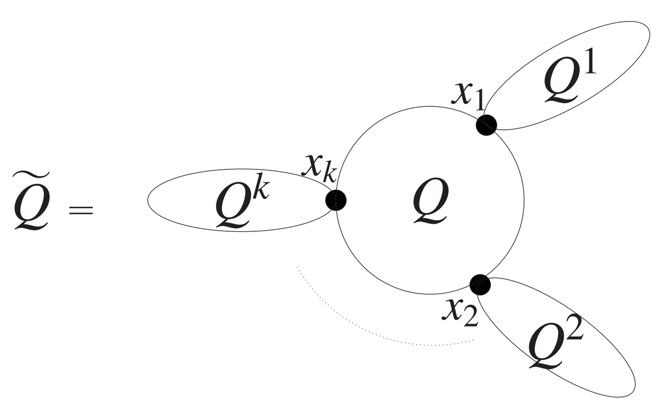
Quivers of mutation type $\mathbb{A}_n$ are adjacency quivers of triangulations of convex $(n+3)$-gons.



Minimal length maximal green sequences

Let $\widetilde{Q}$ be a quiver composed of full connected subquivers $Q, Q^1, Q^2, \ldots, Q^k$ satisfying the following:

- $Q_0^i \cap Q_0 = \{x_i\}$.
- $Q_0^i \cap Q_0^j = \begin{cases} \{x_i\} & \text{if } x_i = x_j \\ \varnothing & \text{otherwise} \end{cases}$.
- if $\alpha \in \widetilde{Q}_1$ has an endpoint in $Q_0^i \backslash \{x_i\}$ then the other is in Q_0^i.
- for every i the quiver Q^i is of mutation type $\mathbb{A}$.



Theorem (G.–McConville–Serhiyenko 2017)

$$\ell_{\text{min}}(\widetilde{Q}) = \ell_{\text{min}}(Q) - k + \sum_{i=1}^{k} (|Q_0^i| + |\{\text{3-cycles in } Q^i\}|)$$

Types $\mathbb{D}$ and $\widetilde{\mathbb{A}}$

The theorem applies to quivers of mutation types $\mathbb{D}$ (Vatne 2008) and $\widetilde{\mathbb{A}}$ (Bastian 2009). There are four families of mutation type $\mathbb{D}$ quivers.

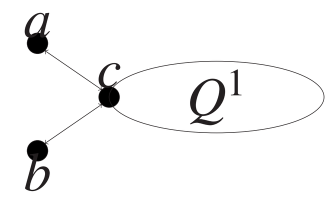


Figure: Type I quivers

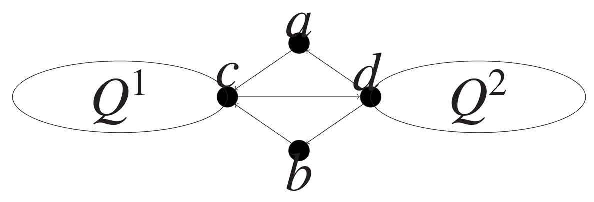


Figure: Type II quivers

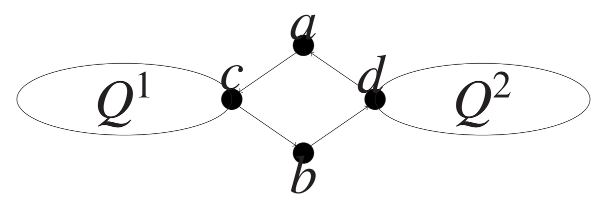


Figure: Type III quivers

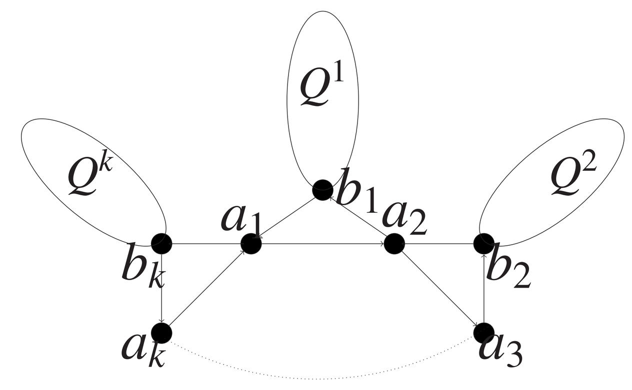


Figure: Type IV quivers

Mutation type $\widetilde{\mathbb{A}}$ and Type IV quivers have the same underlying graphs.

Corollary (G.–McConville–Serhiyenko 2017)

i) $\ell_{\text{min}} = n + |\{$3-cycles in $\widetilde{Q}\}|$ ($\widetilde{Q}$ is Type I or type $\widetilde{\mathbb{A}}_{n-1}$)

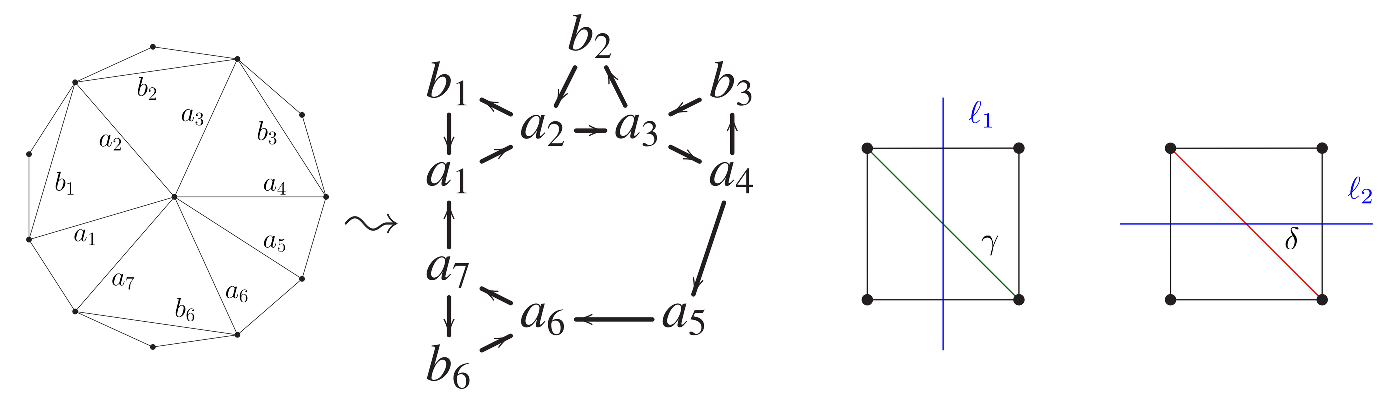
ii) $\ell_{\text{min}} = n + 1 + |\{$3-cycles in $Q^1\}| + |\{$3-cycles in $Q^2\}|$ ($\widetilde{Q}$ is Type II)

iii) $\ell_{\text{min}} = n + 2 + |\{$3-cycles in $\widetilde{Q}\}|$ ($\widetilde{Q}$ is Type III)

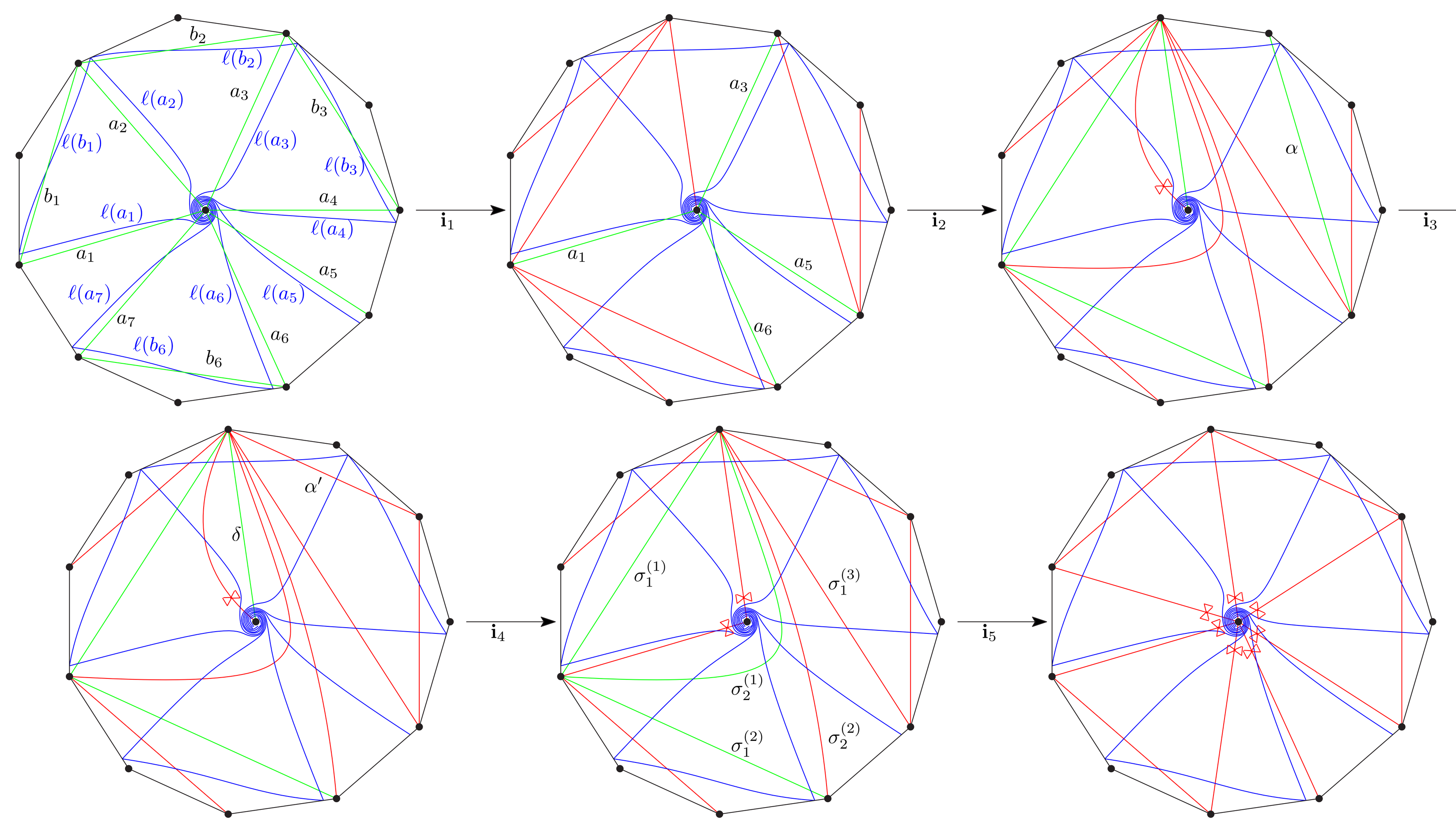
iv) $\ell_{\text{min}} = n + k - 2 + |\{a_i : \deg(a_i) = 4\}| + \sum_{i=1}^{k} |\{$3-cycles in $Q^i\}|$ ($\widetilde{Q}$ is Type IV)

Idea of proof for Type IV quivers

- Use the Theorem to reduce to calculating the $\ell_{\text{min}}(Q)$.
- Show that the reduced Type IV quivers have $\ell_{\text{min}}(Q) = n + k - 2 + |\{a_i : \deg(a_i) = 4\}|$. (not quite easy)
- These quivers arise from triangulations of a punctured disk.



- One keeps track of red and green by adding a **lamination** to the triangulation. (Fomin–Thurston 2012)
- We construct a maximal green sequence $\mathbf{i} = \mathbf{i}_1 \circ \mathbf{i}_2 \circ \mathbf{i}_3 \circ \mathbf{i}_4 \circ \mathbf{i}_5$ of the desired length using the surface model.



Connection to derived equivalence

Question (G.–McConville–Serhiyenko 2017)

If Q^1 and Q^2 have derived-equivalent Jacobian algebras $\Bbbk Q^1/I^1$ and $\Bbbk Q^2/I^2$, is $\ell_{\text{min}}(Q^1) = \ell_{\text{min}}(Q^2)$?

- (mutation type $\mathbb{A}$) Algebras $\Bbbk Q^1/I^1$ and $\Bbbk Q^2/I^2$ are derived-equivalent if and only if $|Q_0^1| = |Q_0^2|$ and $|\{$3-cycles of $Q^1\}| = |\{$3-cycles of $Q^2\}|$. (Buan–Vatne 2007)
- (mutation type $\widetilde{\mathbb{A}}$) If $\Bbbk Q^1/I^1$ and $\Bbbk Q^2/I^2$ are derived-equivalent, then $|Q_0^1| = |Q_0^2|$ and $|\{$3-cycles of $Q^1\}| = |\{$3-cycles of $Q^2\}|$. (Bastian 2009)
- (mutation type $\mathbb{D}$) There are six conjectural derived equivalence classes. A quiver can be put into one of these forms using mutations that preserve $\ell_{\text{min}}(Q)$. (Bastian–Holm–Ladkani 2010)