Exchange Graphs of Polygonal Subdivisions

Al Garver
(joint with Thomas McConville)

Quivers and Bipartite Graphs, Notre Dame's London Global Gateway

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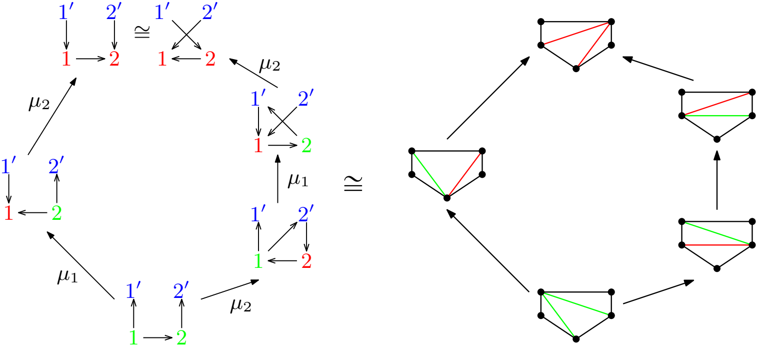
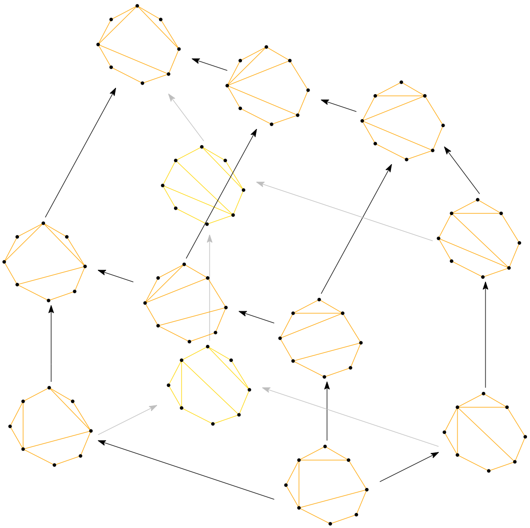
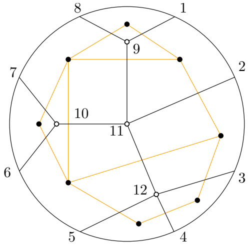


oriented exchange graph
(Keller,
Brüstle–Dupont–Pérotin)

exchange graph of triangulations
(Fomin–Shapiro–Thurston,
Fomin–Thurston)

Goal: Obtain a natural notion of flipping arcs in polygonal subdivisions.

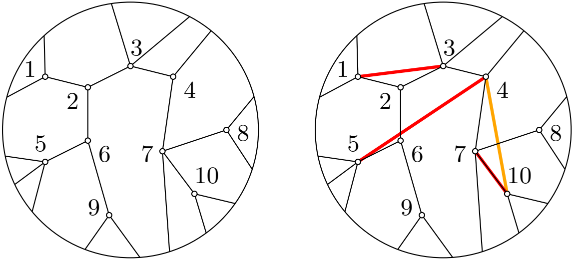




Oriented flip graphs

Fix T a tree embedded in a disk with exactly its leaves on the boundary and whose interior vertices have degree at least 3.

T =



A **segment** $s = (v_0, \ldots, v_t) = [v_0, v_t]$ with $t \geqslant 1$ is a sequence of interior vertices of T that takes a "sharp" turn at each v_i. In particular, interior vertices of T are not segments.

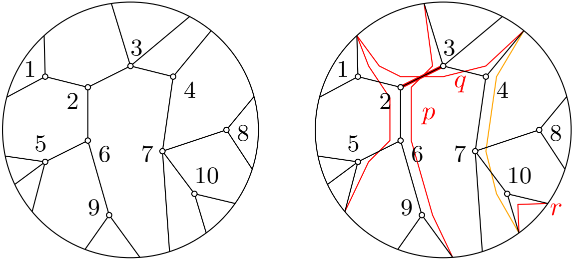
Example

The sequences $(1, 2, 3)$, $(5, 6, 2, 3, 4)$, and $(7, 10)$ are segments. The sequence $(4, 7, 10)$ is not a segment.

Oriented flip graphs

An **arc** $p = (v_0, \ldots, v_t)$ with $t \geqslant 1$ is a sequence of vertices with the same turning condition as a segment, but v_0 and v_t are leaves of T.

$T =$



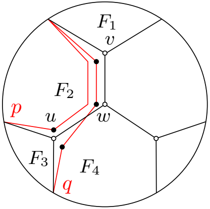
If two arcs cross, there exists a segment of T where the two agree.

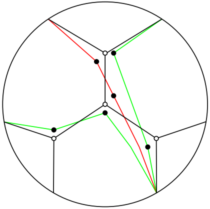
Example

The arcs p and q cross along $[2, 3]$. The orange subgraph of T is *not* an arc. The arc r is a **boundary arc**.

Oriented flip graphs

A corner (v, F) of T is a non-leaf vertex v of T and a **face** F of T. An arc p is **marked** at corner (v, F) if p contains (v, F) and any other arc q containing (v, F) has fewer faces of T on the same side of it as p.





Example

Arc p is marked at (u, F_2). Arc q is marked at (v, F_2) and (u, F_4).

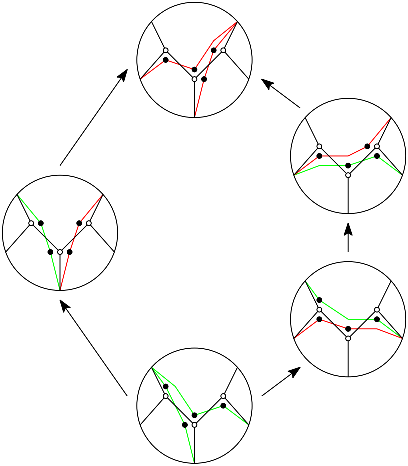
Proposition

Let $\mathcal{F}$ be a maximal collection of noncrossing, non-boundary arcs on T. If $p \in \mathcal{F}$, it is marked at exactly 2 corners.

Oriented flip graphs

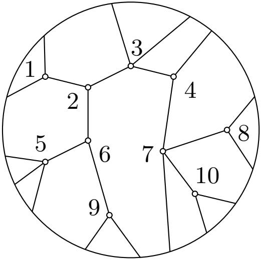
The **oriented flip graph** of T is the directed graph whose vertices are maximal collections of noncrossing arcs of T without any boundary arcs and whose edges are **flips** between such collections.

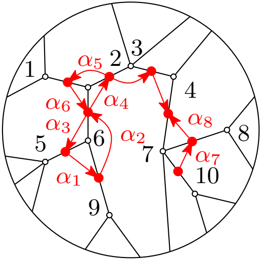
$\overrightarrow{FG}(T) =$



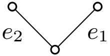
Tiling algebras

Let $\Bbbk = \overline{\Bbbk}$. A tree T defines a finite dimensional $\Bbbk$-algebra, denoted $\Lambda_T = \Bbbk Q_T / I_T$, called a **tiling algebra** (Coelho Simoes–Parsons '16). The elements of $\Bbbk Q_T$ are $\Bbbk$-linear combinations of paths in Q_T.

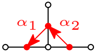
$T \quad =$ 

$\longmapsto \quad Q_T \quad =$ 

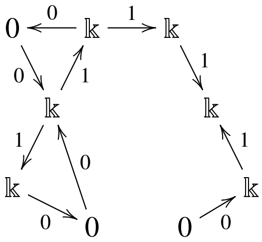
$\{\text{vertices of } Q_T\} \quad = \quad \{e : \text{ where } e \text{ is an interior edge of } T\}$

$\{\text{arrows of } Q_T\} \quad = \quad \Big\{ e_1 \xrightarrow{\alpha} e_2 :$ where e_1 and e_2 form a **corner** of T 

$\Big\}$

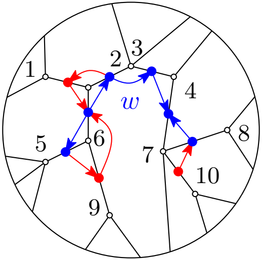
$I_T \quad = \quad \langle \alpha_1\alpha_2 :$ 

$\rangle = \langle \alpha_2\alpha_1, \alpha_3\alpha_2, \alpha_1\alpha_3, \alpha_5\alpha_4, \alpha_6\alpha_5, \alpha_4\alpha_6, \alpha_8\alpha_7 \rangle$



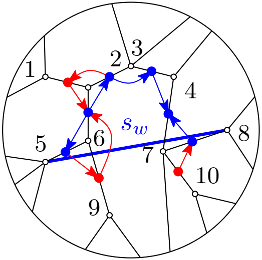
$M(w)$

string module



w

string



s_w

segment

Proposition (Coelho Simoes–Parsons '16)

The algebra Λ_T *is a* ***gentle algebra****. Thus the indecomposable* Λ_T*-modules are given by* ***string modules*** $M(w)$ *(Wald–Waschbuesch '83).*

Proposition (G.–McConville)

The indecomposable Λ_T*-modules are indexed by the segments of* T*.*

Examples of tiling algebras include

- **Jacobian algebras** of type $\mathbb{A}$ (Caldero–Chapoton–Schiffler, Derksen–Weyman–Zelevinsky)
- ***m*-cluster-tilted algebras** of type $\mathbb{A}$ (Baur–Marsh)
- **surface algebras** where the surface is a disk (David-Roesler–Schiffler)

Theorem (G.–McConville)

The oriented flip graph $\overrightarrow{FG}(T)$ is the Hasse diagram of the lattice torsf(Λ_T) of ***torsion-free classes*** *of Λ_T, ordered by inclusion. Thus $\overrightarrow{FG}(T)$ is connected.*

torsf(Λ) := torsion-free classes of Λ, ordered by inclusion

Let Λ be a finite dimensional $\Bbbk$-algebra. Then a full, additive subcategory $\mathcal{F}$ of Λ-mod is a **torsion-free class** of Λ if it is

a) **extension closed** : if $X, Y \in \mathcal{F}$ and one has an exact sequence $0 \to X \to Z \to Y \to 0$, then $Z \in \mathcal{F}$,

b) **submodule closed** : if $X \in \mathcal{F}$ and one has an injection $Z \hookrightarrow X$, then $Z \in \mathcal{F}$.

Theorem (G.–McConville)

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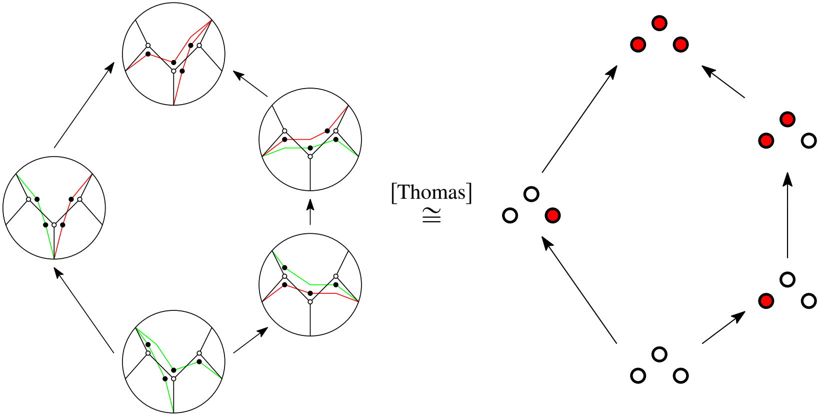


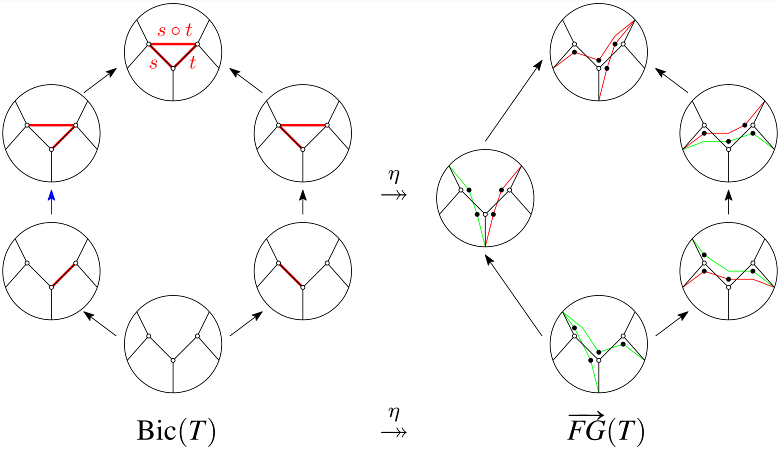
$$\mathcal{F} \longmapsto \bigvee_{\text{red } s_w \in \mathcal{F}} \mathrm{Sub}(M(w))$$

$$\mathrm{Sub}(X) := \{Z \in \Lambda_T\text{-mod} : \; Z \hookrightarrow X^{\oplus m} \text{ for some } m \in \mathbb{N}\}$$

To prove this, we identify $\overrightarrow{FG}(T)$ as a quotient of a lattice of **biclosed sets** of segments of T, denoted $\mathrm{Bic}(T)$.

Definition

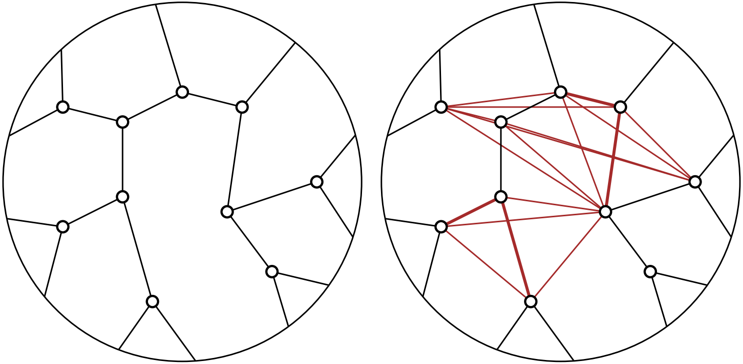
Let $X \subset \mathrm{Seg}(T) := \{\text{segments of } T\}$. The set X is **closed** if for any $s, t \in X$ that satisfy $s \circ t \in \mathrm{Seg}(T)$ one has $s \circ t \in X$. The set X is **biclosed** if, in addition, $\mathrm{Seg}(T) \setminus X$ is closed.



Biclosed Sets

Define a map $\eta : \mathrm{Bic}(T) \to \overrightarrow{FG}(T)$ by $\eta(B) = \{p_{(v,F)}\}_{(v,F)}$ a collection of nonboundary arcs where

- for any interior vertex u of $p_{(v,F)}$, the arc $p_{(v,F)}$ oriented from v to u turns left at u if and only if $[v, u] \in B$.



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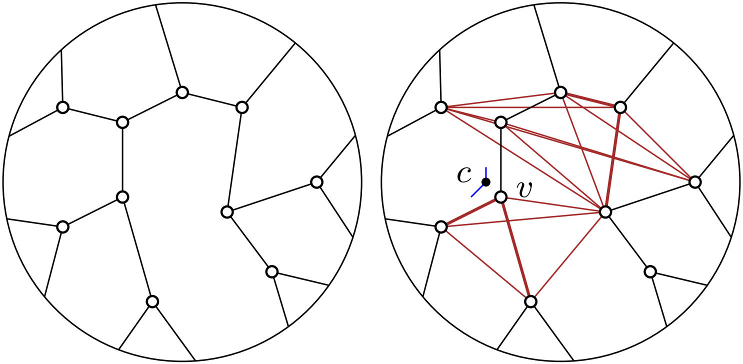

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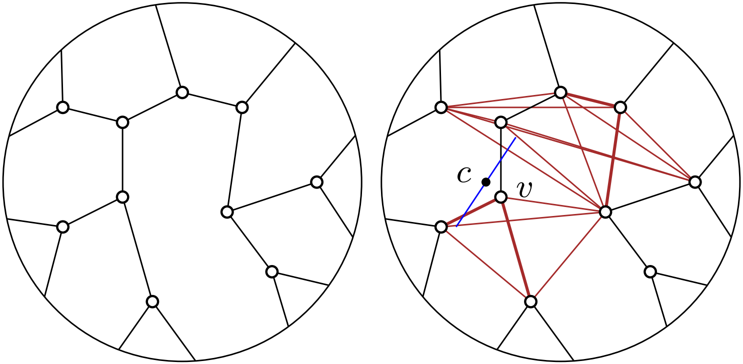

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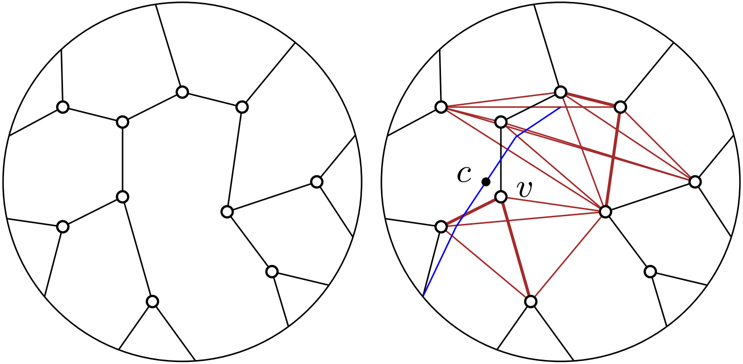

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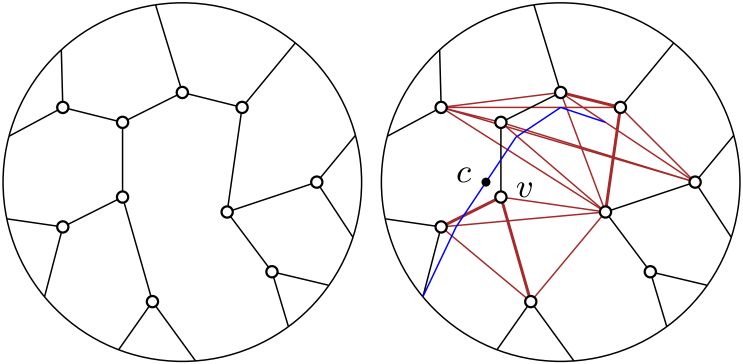

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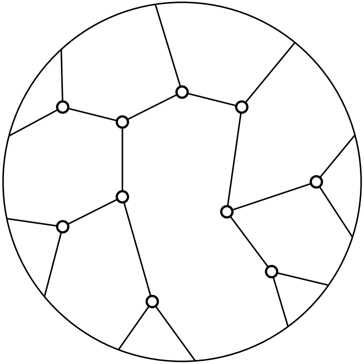

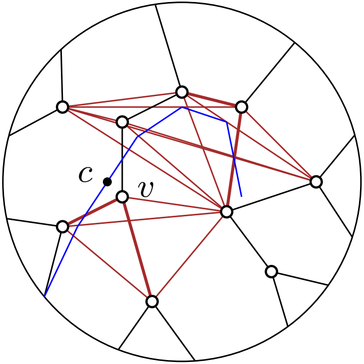

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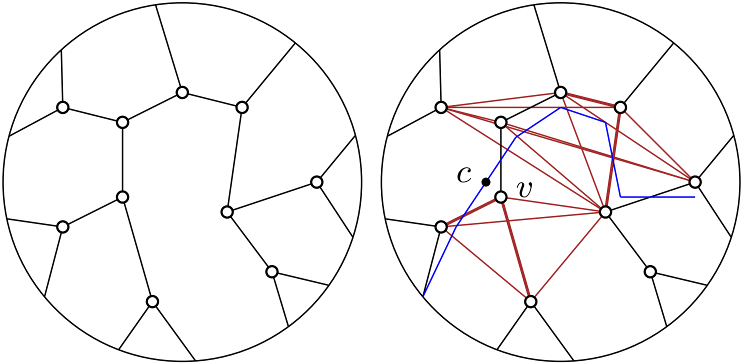

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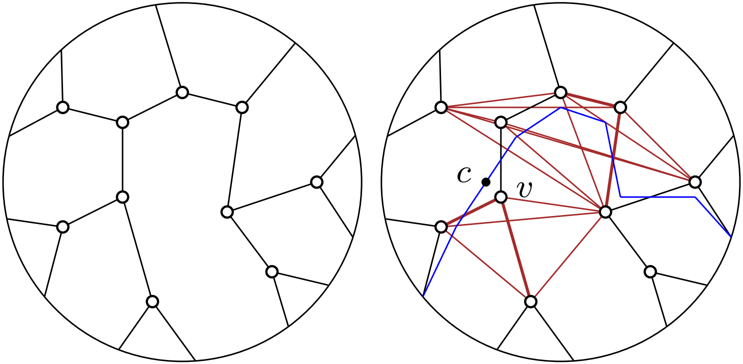

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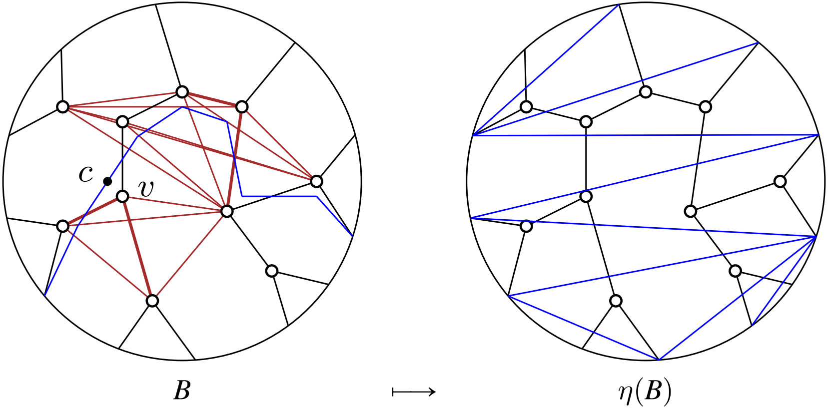
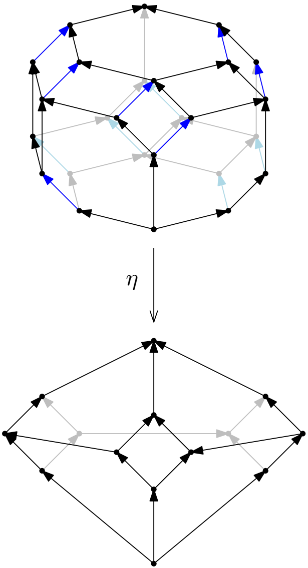
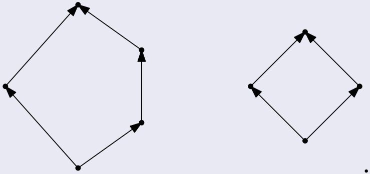


The map $\eta : \mathrm{Bic}(T) \to \overrightarrow{FG}(T)$ is a lattice quotient map.



Theorem (G.–McConville)

*The lattice $\overrightarrow{FG}(T)$ is a polygonal lattice whose **polygons** are of the form*



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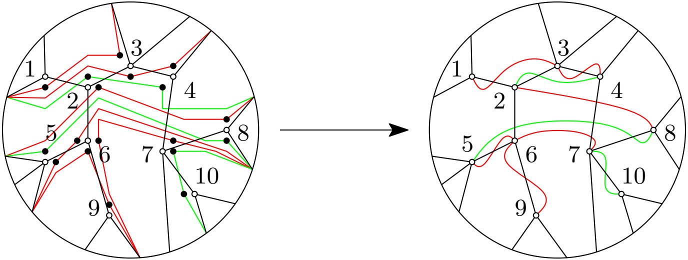
Corollary

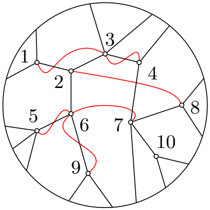
Assume that every interior vertex of T has degree exactly 3. Then the set of lengths of the maximal green sequences of Q_T is of the form $\{\ell_{min}, \ell_{min}+1, \ldots, \ell_{max}-1, \ell_{max}\}$ for some $\ell_{min}, \ell_{max} \in \mathbb{N}$.

Noncrossing tree partitions

Goal: Understand the flip operation more explicitly.



- $\mathbf{B} = \{\{1, 3, 4\}, \{2, 8\}, \{5, 6, 7, 9\}, \{10\}\}$ a **noncrossing tree partition**
- $\mathrm{Kr}(\mathbf{B}) = \{\{1\}, \{2, 4\}, \{3\}, \{5, 8\}, \{6\}, \{7, 10\}, \{9\}\}$ the **Kreweras complement** of $\mathbf{B}$
- $\mathrm{NCP}(T) :=$ noncrossing tree partitions, ordered by refinement



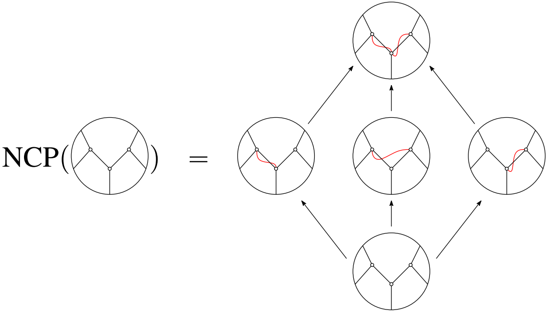
$\longleftrightarrow$ $\mathbf{B} = \{\{1,3,4\},\{2,8\},\{5,6,7,9\},\{10\}\}$

A **noncrossing tree partition** $\mathbf{B} = (B_1, \ldots, B_k)$ of T is a set partition of the interior vertices of T where

- vertices in B_i can be connected by pairwise nonoverlapping **red admissible curves** (i.e. curves whose endoints define segments of T and leave their endpoints to the right) and
- red admissible curves connecting vertices of B_i do not cross those connecting vertices of B_j for $i \neq j$.

Given $\mathbf{B} \in \text{NCP}(T)$, let $\text{Seg}(\mathbf{B})$ be the segments of T defined by $\mathbf{B}$ (for example, $\text{Seg}(\mathbf{B}) = \{[1,3],[3,4],[2,8],[5,6],[6,7],[6,9]\}$).

Noncrossing tree partitions generalize the classical noncrossing set partitions.



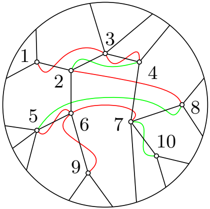
Proposition (G.–McConville)

If the interior vertices of T have degree exactly 3, then $\#NCP(T) = \frac{1}{n+1}\binom{2n}{n}$ where $n = \#(\text{interior vertices of } T)$.

Theorem (G.–McConville)

The poset $NCP(T)$ is a lattice.

One also has a notion of **Kreweras complement** on noncrossing tree partitions.



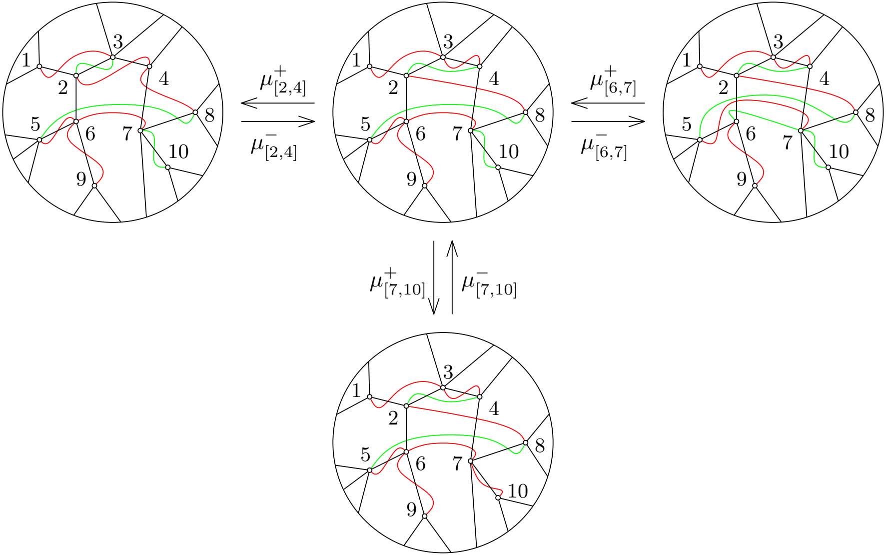
$\longleftrightarrow \quad \mathrm{Kr}(\mathbf{B}) = \left\{ \begin{array}{c} \{1\}, \{2,4\}, \{3\}, \{5,8\}, \\ \{6\}, \{7,10\}, \{9\} \end{array} \right\}$

The Kreweras complement of **B** is the unique noncrossing tree partition of T such that when drawn using **green admissible curves** one obtains a noncrossing tree on the interior vertices of T.

Theorem (G.–McConville)

The map $Kr : NCP(T) \longrightarrow NCP(T)$ *is a bijection.*

Given a noncrossing tree partition and its Kreweras complement $(\mathbf{B}, \mathrm{Kr}(\mathbf{B}))$, one can obtain all other such pairs by local moves.



Definition (Koenig-Yang '13)

A collection $\{X_1, \ldots, X_n\}$ of objects of $\mathcal{D}^b(\Lambda\text{-mod})$ is **simple-minded** if the following hold for any $i, j \in [n]$:

i) $\mathrm{Hom}_{\mathcal{D}^b(\Lambda\text{-mod})}(X_i, X_j[k]) = 0$ for any $k < 0$,

ii) $\mathrm{Hom}_{\mathcal{D}^b(\Lambda\text{-mod})}(X_i, X_j) = \begin{cases} \mathbb{k} & : \text{ if } i = j \\ 0 & : \text{ otherwise,} \end{cases}$

iii) the smallest triangulated category containing $X_1, \ldots, X_n$ and closed under taking summands of objects is $\mathcal{D}^b(\Lambda\text{-mod})$.

Examples of simple-minded collections are given by

- a complete set of nonisomorphic simple modules regarded as elements of $\mathcal{D}^b(\Lambda\text{-mod})$,
- **spherical collections** (Seidel–Thomas) in algebraic geometry.

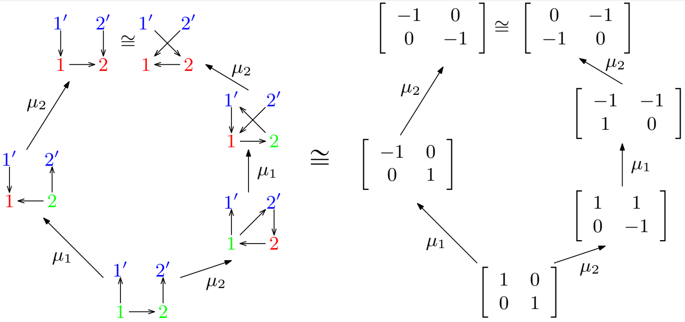
Let 2-smc(Λ) denote the set simple-minded collections $\{X_1, \ldots, X_n\}$ where $H^k(X_i) = 0$ for any $i \in \{1, \ldots, n\}$ and any $k \neq 0, -1$.

Theorem (Brüstle–Yang ’12)

The map $2\text{-}smc(\Lambda) \to \mathbf{c}\text{-}mat(Q)$ *given by*

$$\{X_1, \ldots, X_n\} \mapsto \{\underline{dim}(X_1), \ldots, \underline{dim}(X_n)\}$$

where $\underline{dim} : \mathcal{D}^b(\Lambda) \to \mathbb{Z}^n$ *defined as* $\underline{dim}(X_i) := \sum_{j \in \mathbb{Z}} (-1)^j \underline{dim}(X_i^j)$ *is a bijection.*



$\mathbf{c}$-mat$(1 \to 2)$

Theorem (G.–McConville)

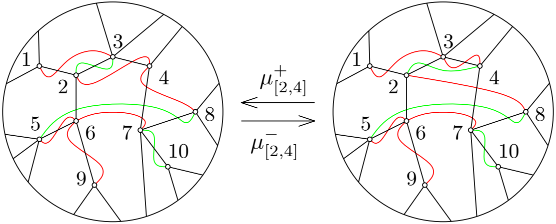
The map $\{(\boldsymbol{B}, Kr(\boldsymbol{B}))\}_{\boldsymbol{B}\in NCP(T)} \to 2\text{-}smc(\Lambda_T)$ *given by*

$$(\boldsymbol{B}, Kr(\boldsymbol{B})) \longmapsto \begin{array}{l} \{M(u)[1] : s_u \in Seg(B) \text{ where } B \in \boldsymbol{B}\} \sqcup \\ \{M(v) : s_v \in Seg(B') \text{ where } B' \in Kr(\boldsymbol{B})\} \end{array}$$

is a bijection. Furthermore, this map is compatible with ***mutations*** *(as introduced by Koenig–Yang '13).*

Corollary

If every internal vertex of T has degree 3, we have a bijection $\{(\boldsymbol{B}, Kr(\boldsymbol{B}))\}_{\boldsymbol{B}\in NCP(T)} \to \boldsymbol{c}\text{-}mat(Q_T)$.



Thanks!