Minimal length maximal green sequences

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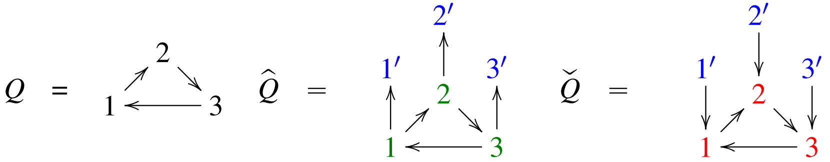
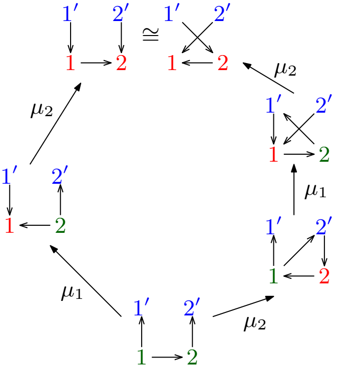
- Maximal green sequences
- Main result and application
- Techniques for proving these

Q – a **2-acyclic** quiver (i.e., Q has no loops or 2-cycles).

Add **frozen vertices** to Q.



framed quiver **coframed quiver**



We **mutate** $\widehat{Q}$ at any non-frozen vertex k to obtain a quiver $\mu_k(\widehat{Q})$. The quiver $\mu_k(\widehat{Q})$ is obtained from $\widehat{Q}$ by

(i) inserting new arrow $i \to j$ for each 2-path $i \to k \to j$ in $\widehat{Q}$

(ii) reversing all arrows incident to k

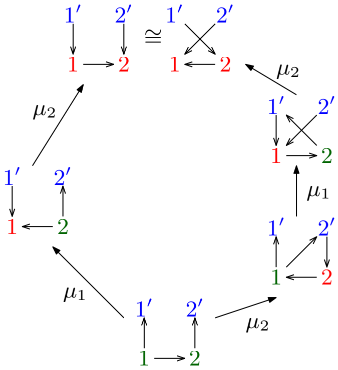
(ii) delete any 2-cycles

Definition (Keller, 2011)

A **maximal green sequence** of Q is a sequence $\mathbf{i} = (i_1, \ldots, i_k)$ of non-frozen vertices of $\widehat{Q}$ where

(i) for all $j \in [k]$ vertex i_j is **green** in $\mu_{i_{j-1}} \circ \cdots \circ \mu_{i_1}(\widehat{Q})$ and

(ii) all vertices in $\mu_{i_k} \circ \cdots \circ \mu_{i_1}(\widehat{Q})$ are **red**.

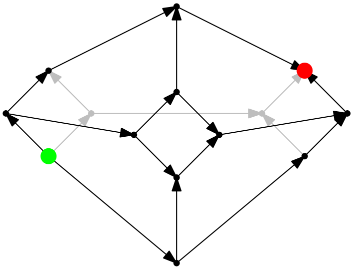


(1,2) and (2,1,2) are the only maximal green sequences of $Q = 1 \rightarrow 2$

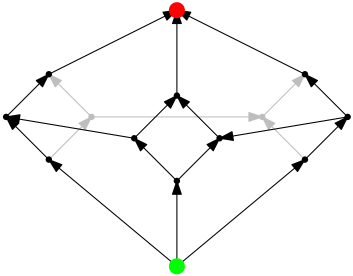
$\leftarrow$ the **oriented exchange graph** of $Q = 1 \rightarrow 2$

Maximal green sequences can be identified with

- finite length maximal chains in the poset of functorially finite torsion classes of the **Jacobian algebra** of Q, [Brüstle–Yang, 2014]
- certain sequences of reachable chambers in the **consistent scattering diagram** of Q [Gross–Hacking–Keel–Kontsevich, 2014]



$Q = 1 \to 2 \to 3$



Q oriented 3-cycle

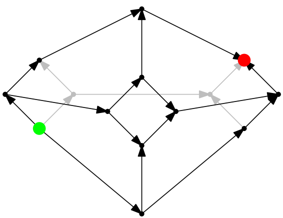
Goal: Understand combinatorial properties of the maximal green sequences of Q.

Conjecture (“No Gap Conjecture” Brüstle–Dupont–Pérotin, 2013)

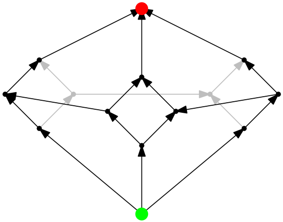
For each $\ell_{min}(Q) \leqslant k \leqslant \ell_{max}(Q)$, there exists a maximal green sequence of length k where

$$\begin{aligned} \ell_{min}(Q) &:= \text{length of shortest maximal green sequence of } Q \\ \ell_{max}(Q) &:= \text{length of longest maximal green sequence of } Q \end{aligned}$$

- True in mutation type $\mathbb{A}$ [G.–McConville, 2015]
- True for tame hereditary algebras [Hermes–Igusa, 2016]



$Q = 1 \rightarrow 2 \rightarrow 3$

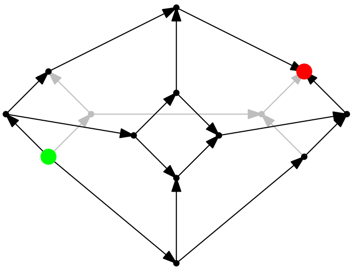


Q oriented 3-cycle

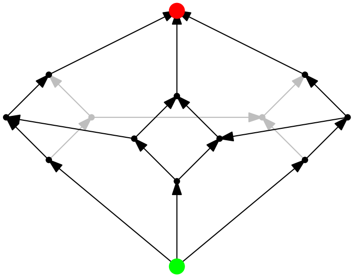
$\ell_{\min}(Q) :=$ length of shortest maximal green sequence of Q

$\ell_{\max}(Q) :=$ length of longest maximal green sequence of Q.

- If Q is Dynkin, $\ell_{\min}(Q) = |Q_0|$ and $\ell_{\max}(Q) = |\Phi^+(Q)|$. [Brüstle–Dupont–Pérotin, 2013]
- If Q is acyclic, $\ell_{\min}(Q) = |Q_0|$. [Brüstle–Dupont–Pérotin, 2013]
- If Q is mutation type $\mathbb{A}$, then $\ell_{\min}(Q) = |Q_0| + |\{\text{3-cycles of } Q\}|$ [Cormier–Dillery–Resh–Serhiyenko–Whelan, 2015]



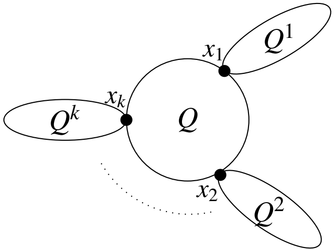
$Q = 1 \to 2 \to 3$



Q oriented 3-cycle

Let $\widetilde{Q}$ be a quiver composed of full connected subquivers $Q, Q^1, Q^2, \ldots, Q^k$, such that all of the following conditions hold.

- $Q^i_0 \cap Q_0 = \{x_i\}$.
- $Q^i_0 \cap Q^j_0 = \begin{cases} \{x_i\} & \text{if } x_i = x_j \\ \varnothing & \text{otherwise} \end{cases}$.
- if $\alpha \in \widetilde{Q}_1$ has an endpoint in $Q^i_0 \setminus \{x_i\}$ then the other is in Q^i_0.
- for every i the quiver Q^i is of mutation type $\mathbb{A}$.

$\widetilde{Q} =$ 

Theorem (G.–McConville–Serhiyenko, 2017)

$\ell_{min}(\widetilde{Q}) = \ell_{min}(Q) - k + \sum_{i=1}^{k}(|Q^i_0| + |\{\text{3-cycles in } Q^i\}|)$

The theorem applies to quivers of mutation types $\mathbb{D}$ [Vatne, 2008] and $\widetilde{\mathbb{A}}$ [Bastian, 2009]. There are four families mutation type $\mathbb{D}$ quivers.

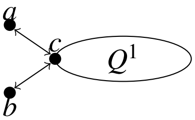


Figure: Type I quivers

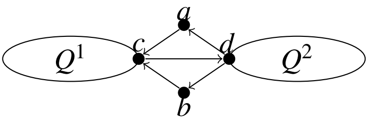


Figure: Type II quivers

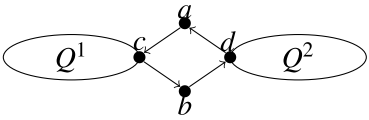


Figure: Type III quivers

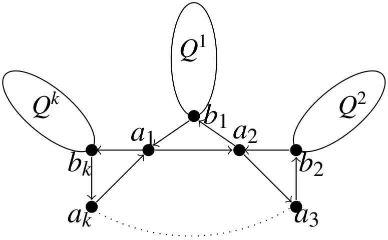


Figure: Type IV quivers

Mutation type $\widetilde{\mathbb{A}}$ and Type IV quivers have the same underlying graphs.

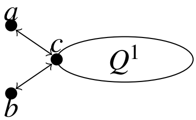
Corollary (G.–McConville–Serhiyenko, 2017)

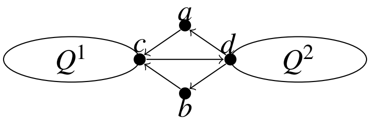
i) $\ell_{min} = n + |\{\text{3-cycles in } \widetilde{Q}\}|$ ($\widetilde{Q}$ *is of Type I or of type* $\widetilde{\mathbb{A}}_{n-1}$)

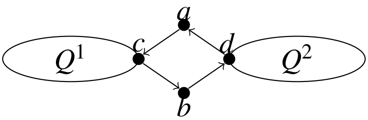
ii) $\ell_{min} = n + 1 + |\{\text{3-cycles in } Q^1\}| + |\{\text{3-cycles in } Q^2\}|$ ($\widetilde{Q}$ *is of Type II*)

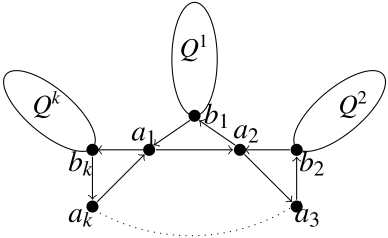
iii) $\ell_{min} = n + 2 + |\{\text{3-cycles in } \widetilde{Q}\}|$ ($\widetilde{Q}$ *is of Type III*)

iv) $\ell_{min} = n + k - 2 + |\{a_i : deg(a_i) = 4\}| + \sum_{i=1}^{k} |\{\text{3-cycles in } Q^i\}|$.









Question

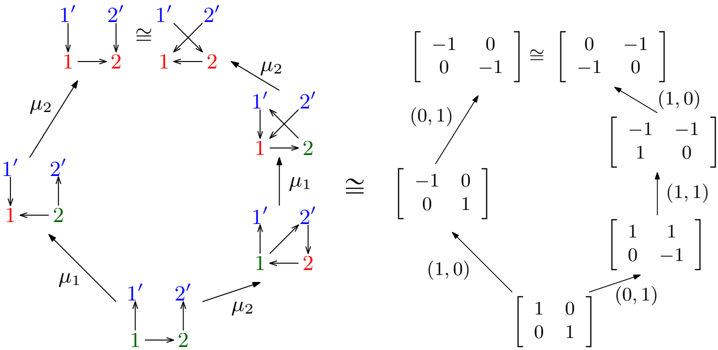
If Q^1 and Q^2 have derived-equivalent cluster-tilted algebras $\Bbbk Q^1/I^1$ and $\Bbbk Q^2/I^2$, is $\ell_{min}(Q^1) = \ell_{min}(Q^2)$?

- (mutation type $\mathbb{A}$) If $\Bbbk Q^1/I^1$ and $\Bbbk Q^2/I^2$ are derived-equivalent if and only if $|Q_0^1| = |Q_0^2|$ and $|\{\text{3-cycles of } Q^1\}| = |\{\text{3-cycles of } Q^2\}|$. [Buan–Vatne, 2007]
- (mutation type $\widetilde{\mathbb{A}}$) If $\Bbbk Q^1/I^1$ and $\Bbbk Q^2/I^2$ are derived-equivalent, then $|Q_0^1| = |Q_0^2|$ and $|\{\text{3-cycles of } Q^1\}| = |\{\text{3-cycles of } Q^2\}|$. [Bastian, 2009]
- (mutation type $\mathbb{D}$) There are six conjectural derived equivalence classes. A quiver can be put into one of these forms using mutations that preserve $\ell_{\min}(Q)$. [Bastian–Holm–Ladkani, 2010]

To prove the theorem:

- construct a maximal green sequence of length $\ell_{\min}(Q) - k + \sum_{i=1}^{k}(|Q_0^i| + |\{\text{3-cycles in } Q^i\}|)$
- show there are no shorter maximal green sequences (*)

To address *, one uses the **c-vectors** of Q. These record the arrows between non-frozen vertices and frozen vertices.



We identify maximal green sequences with their sequences of **c**-vectors.

$$\mathbf{i} = (i_1, \dots, i_k) \longleftrightarrow \mathbf{c}(\mathbf{i}) = (\mathbf{c}_1, \dots, \mathbf{c}_k)$$

The following was essentially proved by [Muller, 2015].

Theorem (G.–McConville–Serhiyenko, 2017)

Let Q be a 2-acyclic quiver and $Q^\dagger$ any full subquiver. There is a map $MGS(Q) \to MGS(Q^\dagger)$ sending $\mathbf{i} \in MGS(Q)$ to $\mathbf{i}^\dagger \in MGS(Q^\dagger)$ where $\mathbf{c}(\mathbf{i}^\dagger) = (\mathbf{c}^{(1)}, \dots, \mathbf{c}^{(\ell)})$ is the subsequence of $\mathbf{c}(\mathbf{i})$ where each $\mathbf{c}^{(j)} = (c_1^{(j)}, \dots, c_n^{(j)})$ satisfies $c_i^{(j)} = 0$ if $i \in Q_0 \backslash Q_0^\dagger$.

(The proof uses properties of the **consistent scattering diagram** of Q.)

$\Longrightarrow$ Let $\mathbf{i} \in \mathrm{MGS}(\widetilde{Q})$.

$$\begin{aligned}
\ell(\mathbf{i}) &\geqslant \ell(\mathbf{i}_{(\widetilde{Q}_0 \backslash Q_0^1) \cup \{x_1\}}) + \ell(\mathbf{i}_{Q_0^1}) - 1 \\
&\geqslant \ell(\mathbf{i}_{(\widetilde{Q}_0 \backslash (Q_0^1 \cup Q_0^2)) \cup \{x_1, x_2\}}) + \ell(\mathbf{i}_{Q_0^1}) + \ell(\mathbf{i}_{Q_0^2}) - 2 \\
&\vdots \\
&\geqslant \ell_{\min}(Q) - k + \textstyle\sum_{i=1}^k \ell_{\min}(Q^i)
\end{aligned}$$

To prove the corollary, use the Theorem to reduce to calculating the $\ell_{\min}(Q)$. We focus on the mutation type $\mathbb{D}$ case.

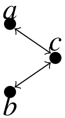


Figure: $\ell_{\min}(Q) = 3$

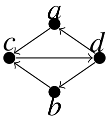


Figure: $\ell_{\min}(Q) = 4 + 1$

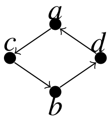


Figure: $\ell_{\min}(Q) = 4 + 2$

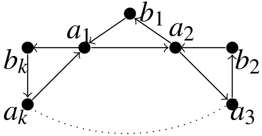
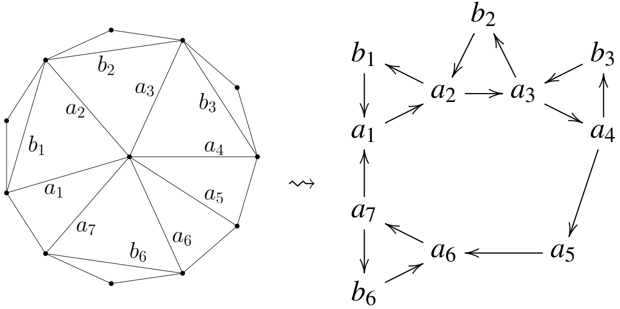


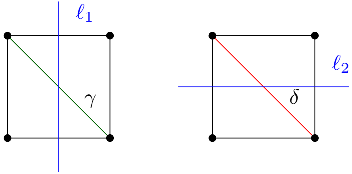
Figure: $\ell_{\min}(Q) \overset{??}{=} n + k - 2 + |\{a_i : \deg(a_i) = 4\}|$

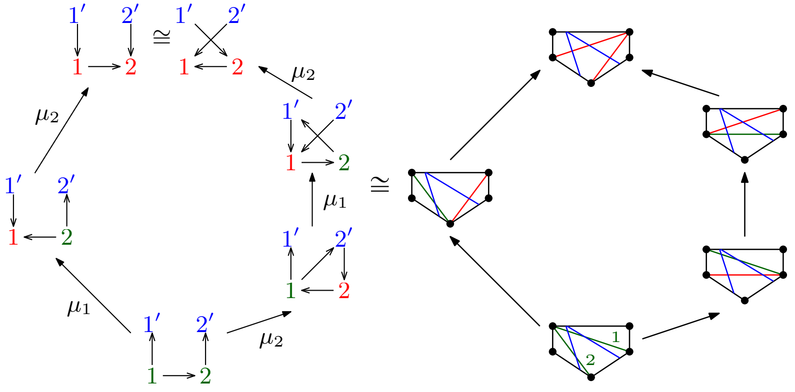
Show that the reduced Type IV quivers have
$\ell_{\min}(Q)=n+k-2+|\{a_i : \deg(a_i)=4\}|$. (not quite easy)

- These quivers arise from triangulations of a punctured disk.



When Q is defined by a triangulation, one keeps track of red and green by adding a **lamination** to the triangulation. [Fomin–Thurston, 2012]





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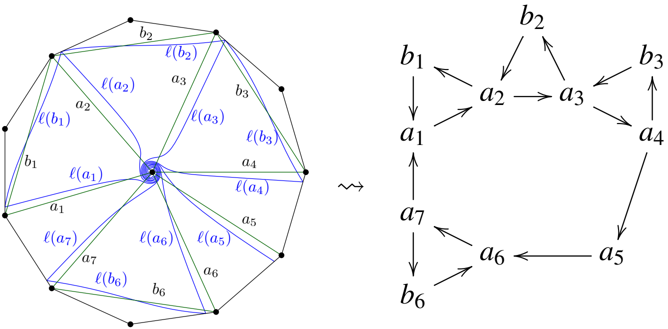
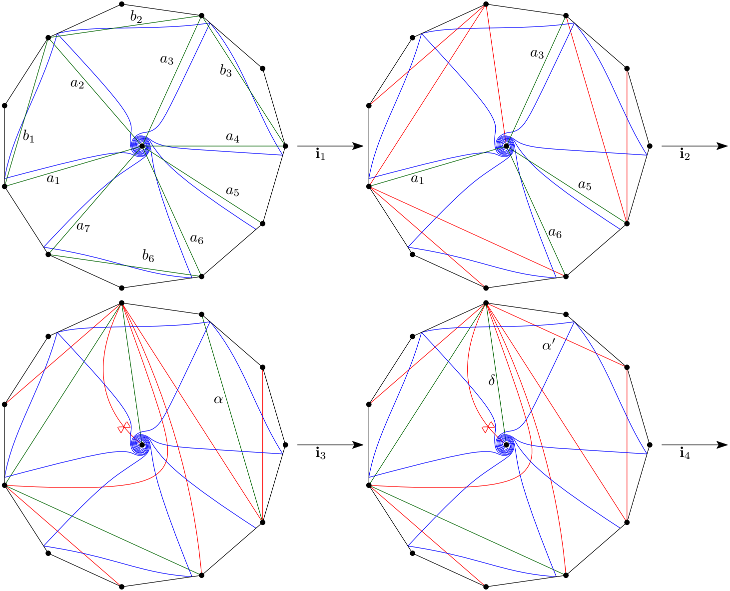


- We construct a maximal green sequence $\mathbf{i} = \mathbf{i}_1 \circ \mathbf{i}_2 \circ \mathbf{i}_3 \circ \mathbf{i}_4 \circ \mathbf{i}_5$ of the desired length. (*)


b_2
a_3
b_3
a_2
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b_6
$\mathbf{i}_1$
a_3
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α
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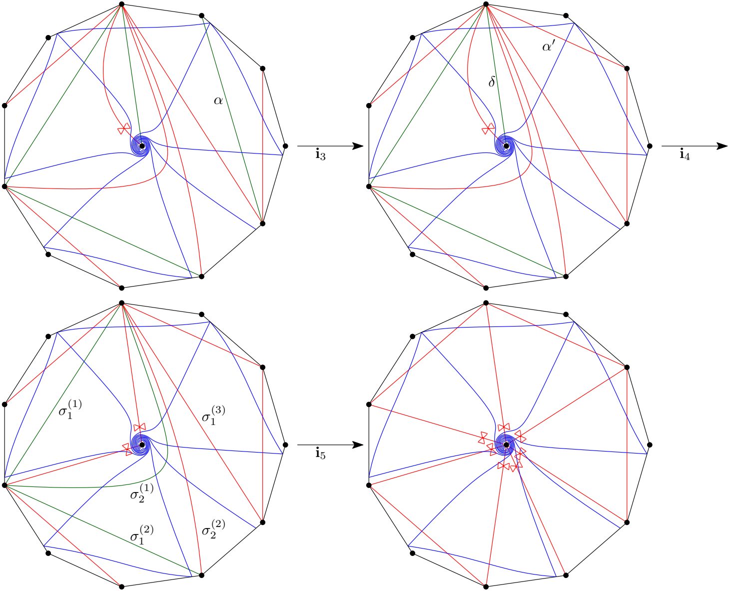

α
$\mathbf{i}_3$
δ
α'
$\mathbf{i}_4$
$\sigma_1^{(1)}$
$\sigma_1^{(3)}$
$\sigma_2^{(1)}$
$\sigma_1^{(2)}$
$\sigma_2^{(2)}$
$\mathbf{i}_5$

Thanks!