Lattice Properties of Oriented Exchange Graphs

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(joint with Thomas McConville)

Positive Grassmannians: Applications to integrable systems and super Yang-Mills scattering amplitudes

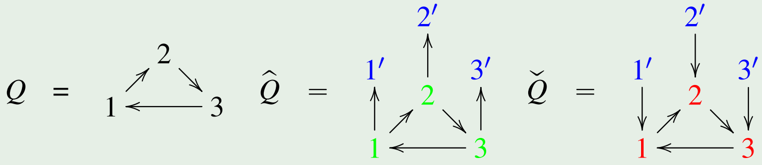
July 28, 2015

Outline

1. Oriented exchange graphs
2. Torsion classes
3. Biclosed subcategories
4. Application: maximal green sequences

Let Q be a finite, connected quiver without loops or 2-cycles whose vertices are $[n] := \{1, 2, \ldots, n\}$.

Example



Oriented exchange graphs

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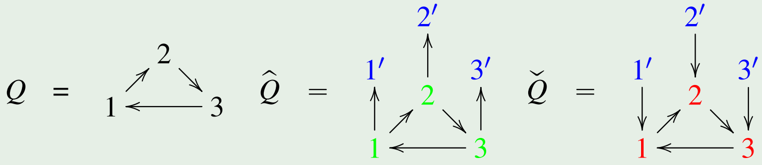
Definition

Given a quiver Q, the **framed quiver** (resp. **coframed quiver**) of Q, denoted $\widehat{Q}$ (resp. $\check{Q}$), is formed by

(i) adding a **frozen vertex** i' for each vertex i in Q

(ii) adding an arrow $i \to i'$ (resp. $i \leftarrow i'$) for each vertex i in Q.

Example



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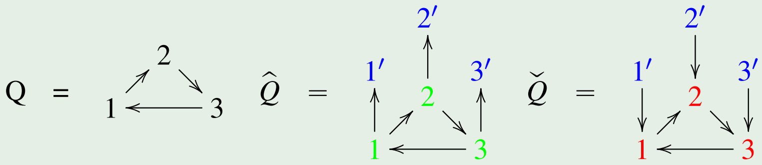

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A **maximal green sequence** of Q is a sequence $\mathbf{i} = (i_1, \ldots, i_k)$ of mutable vertices of $\widehat{Q}$ where

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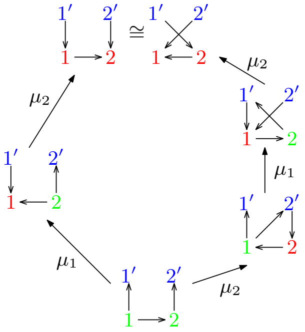
Oriented exchange graphs

Definition (Brüstle-Dupont-Pérotin)

The **oriented exchange graph** of Q, denoted $\overrightarrow{EG}(\widehat{Q})$, is the directed graph with vertices the elements of $\mathrm{Mut}(\widehat{Q})$ and edges $\overline{Q}_1 \longrightarrow \mu_k \overline{Q}_1$ if and only if k is green in $\overline{Q}_1$.



The oriented exchange graph of $Q = 1 \to 2$ has maximal green sequences $(1, 2)$ and $(2, 1, 2)$.

Torsion classes

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Our work is based on ideas developed in [Brüstle-Yang 2014] and [Reading 2006].

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Let Q be mutation-equivalent to a Dynkin quiver. Then $\overrightarrow{EG}(\widehat{Q}) \cong tors(\Lambda)$ where $\Lambda = \Bbbk Q/I$ is the cluster-tilted (or Jacobian) algebra associated to Q.

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$$\Gamma(\Lambda\text{-mod}) \quad = \quad \begin{array}{ccccc} & & \Bbbk \xrightarrow{1} \Bbbk & & \\ & \nearrow & & \searrow & \\ 0 \xrightarrow{0} \Bbbk & & & & \Bbbk \xrightarrow{0} 0. \end{array}$$

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We use $\Gamma(\Lambda\text{-mod})$ to describe the torsion classes of Λ.

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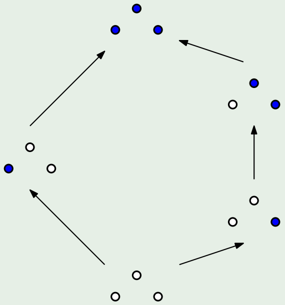

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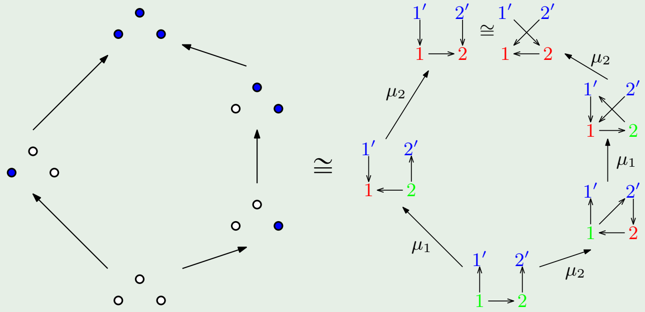

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The partially ordered set $\operatorname{tors}(\Lambda)$ is a **lattice** (i.e. any two torsion classes $\mathcal{T}_1, \mathcal{T}_2 \in \operatorname{tors}(\Lambda)$ have a **join** (resp. **meet**), denoted $\mathcal{T}_1 \vee \mathcal{T}_2$ (resp. $\mathcal{T}_1 \wedge \mathcal{T}_2$)).

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i) $\mathcal{T}_1 \wedge \mathcal{T}_2 = \mathcal{T}_1 \cap \mathcal{T}_2$,

ii) $\mathcal{T}_1 \vee \mathcal{T}_2 = \mathcal{F}ilt(\mathcal{T}_1, \mathcal{T}_2)$ *where* $\mathcal{F}ilt(\mathcal{T}_1, \mathcal{T}_2)$ *consists of objects* X *with a filtration* $0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$ *such that* X_j/X_{j-1} *belongs to* $\mathcal{T}_1$ *or* $\mathcal{T}_2$. *[G.–McConville]*

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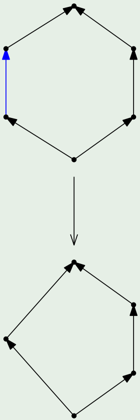
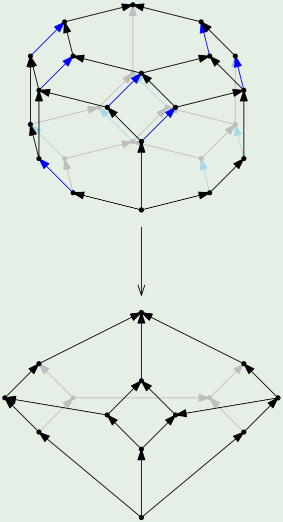
Goal: Realize $\text{tors}(\Lambda)$ as a **quotient** of a lattice with nice properties so that $\text{tors}(\Lambda)$ will inherit these nice properties.

Torsion classes

Example

A lattice quotient map $\pi_{\downarrow} : L \to L/{\sim}$ is a surjective map of lattices.





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A full, additive subcategory $\mathcal{B}$ of Λ-mod is **biclosed** if

a) $\mathcal{B} = \mathrm{add}(\oplus_{i=1}^{k} X_i)$ for some set of indecomposables $\{X_i\}_{i=1}^{k}$ (here $\mathrm{add}(\oplus_{i=1}^{k} X_i)$ consists of objects $\oplus_{i=1}^{k} X_i^{m_i}$ where $m_i \geqslant 0$), Now we assume that Q is mutation-equivalent to $1 \to 2 \to \cdots \to n$ or is an oriented cycle.

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Biclosed subcategories

Example ($Q = 1 \xrightarrow{\alpha} 2$)

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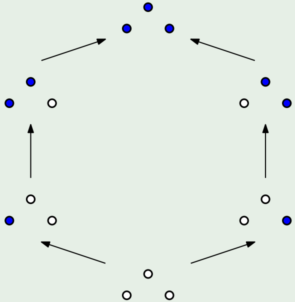

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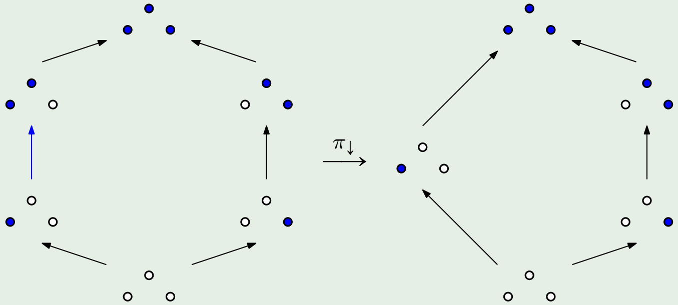

Theorem (G.– McConville)

Let $\mathcal{B} = add(\bigoplus_{i=1}^{k} X_i) \in \mathcal{BIC}(Q)$ *and let*

$$\pi_\downarrow(\mathcal{B}) := add(\bigoplus_{j=1}^{\ell} X_{i_j} : X_{i_j} \twoheadrightarrow Y \implies Y \in \mathcal{B}).$$

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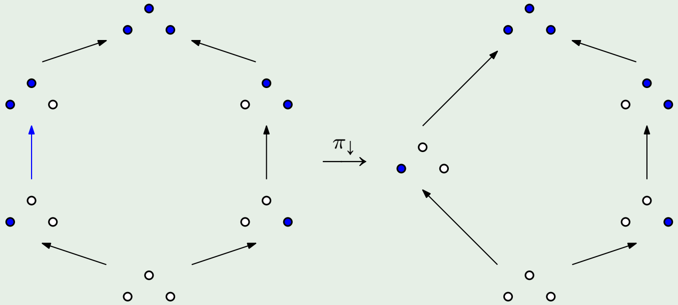

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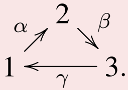
Then $\pi_\downarrow : \mathcal{BIC}(Q) \to tors(\Lambda)$ is a lattice quotient map.

Application: maximal green sequences

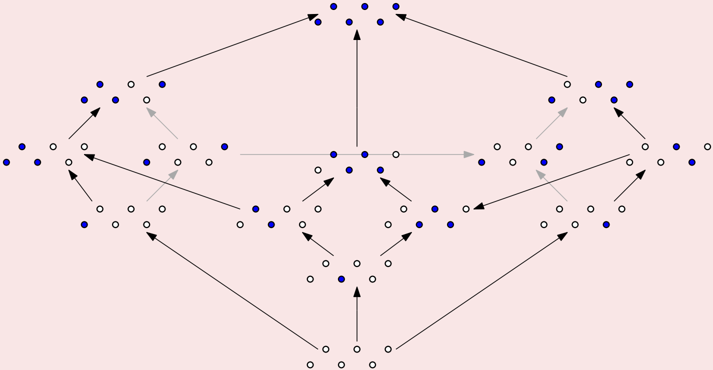
Example

Let

$Q(3)$:=



Then $\Lambda = \Bbbk Q(3)/\langle \beta\alpha, \gamma\beta, \alpha\gamma \rangle$.



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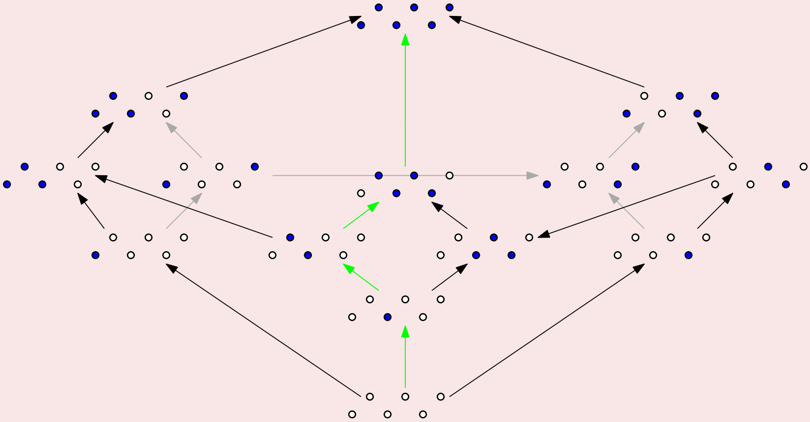
The maximal green sequences of $Q(3)$ are connected by **polygonal flips**.



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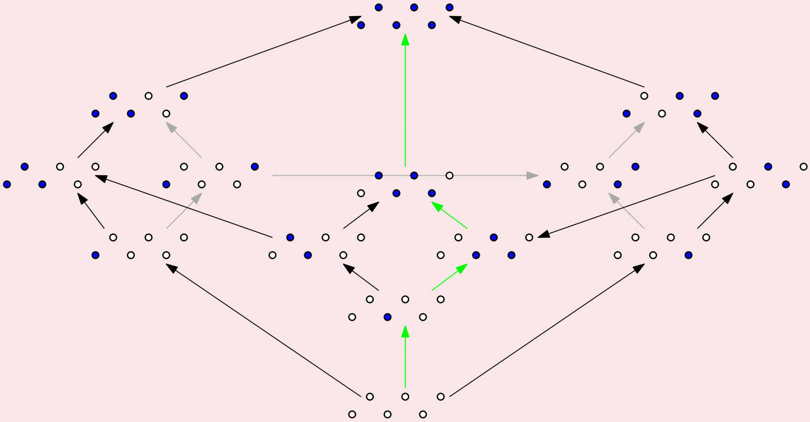

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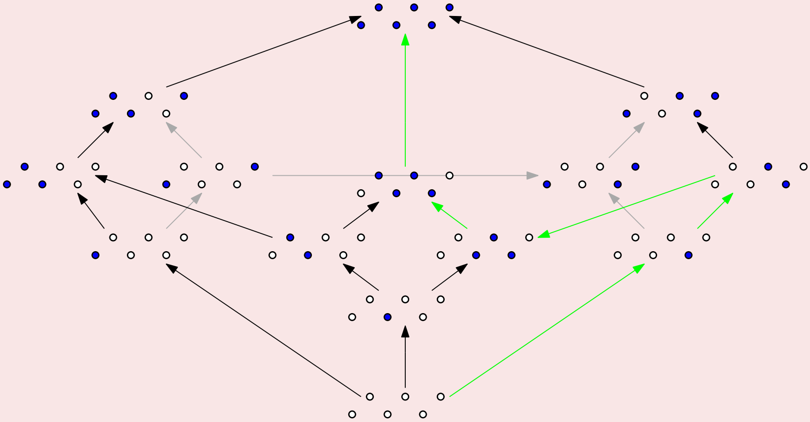

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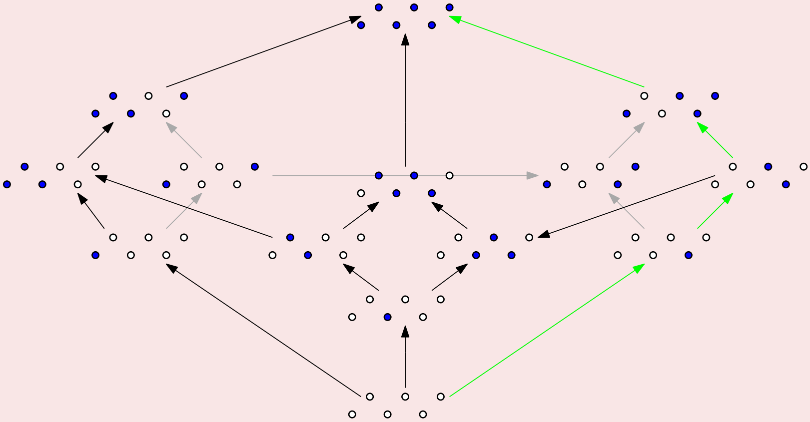

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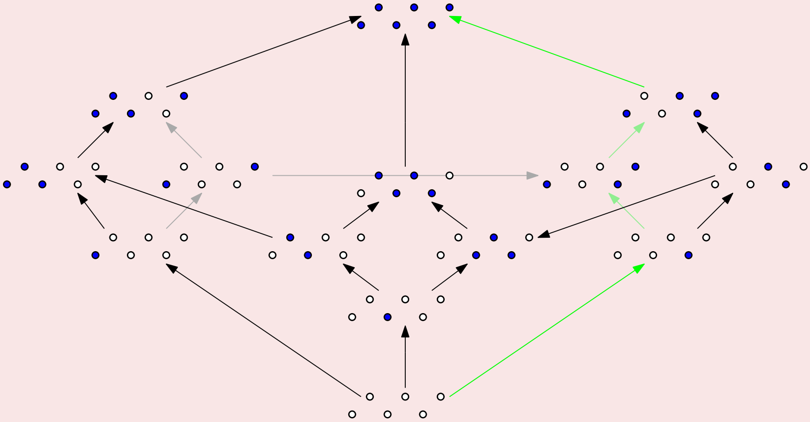

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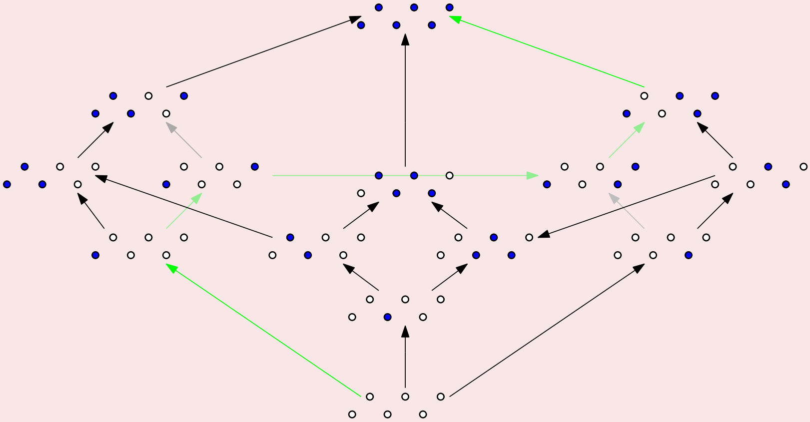

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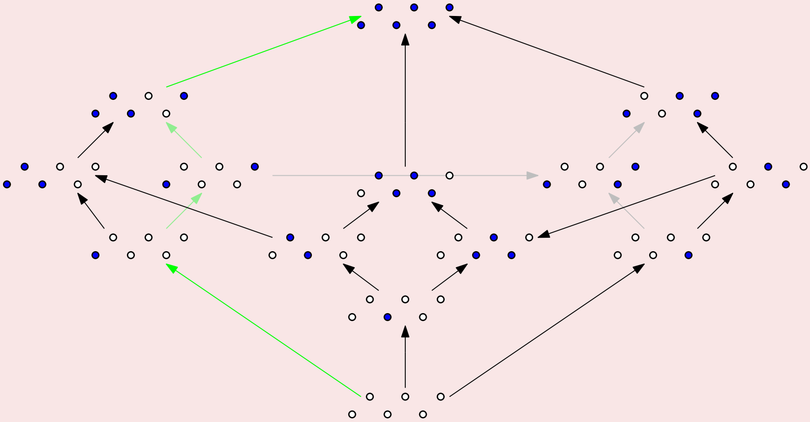

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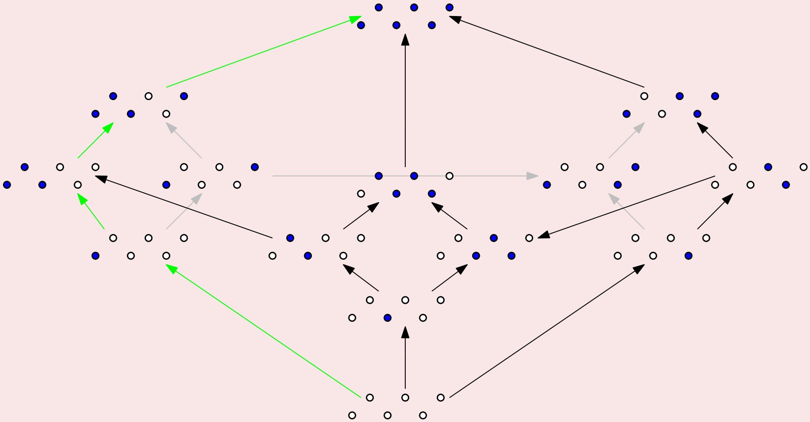

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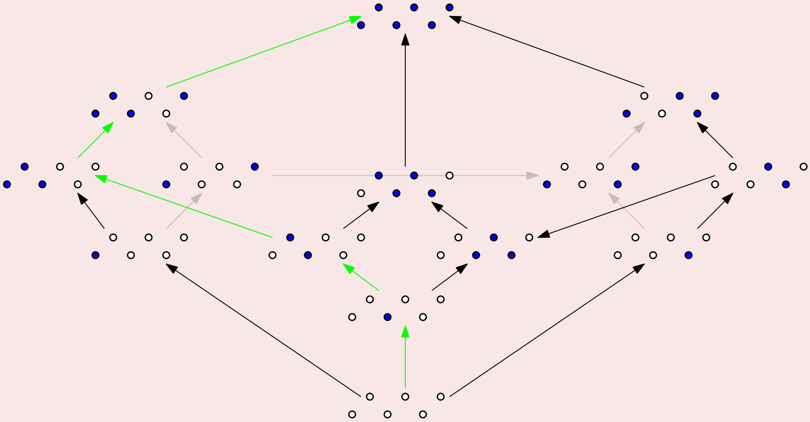

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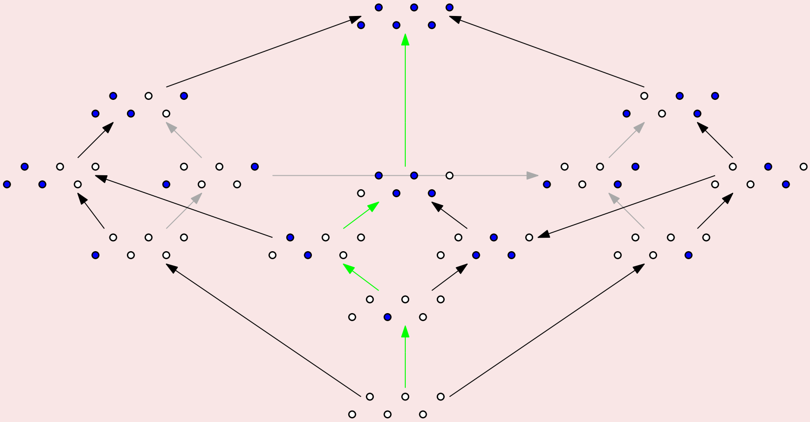

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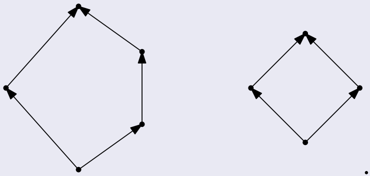
*If Q is mutation-equivalent to $1 \to 2 \to \cdots \to n$ or if Q is an oriented cycle, $\overrightarrow{EG}(\widehat{Q})$ is a **polygonal lattice** whose **polygons** are of the form*



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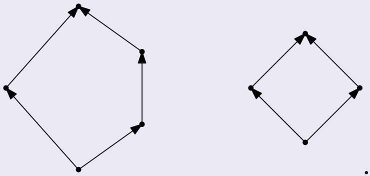

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Corollary (Conjectured by Brüstle-Dupont-Pérotin for any quiver Q)

If Q is mutation-equivalent to $1 \to 2 \to \cdots \to n$ or if Q is an oriented cycle, the set of lengths of the maximal green sequences of Q is of the form $\{\ell_{min}, \ell_{min}+1, \ldots, \ell_{max}-1, \ell_{max}\}$ where

$$\ell_{min} := \textit{length of the shortest maximal green sequence of } Q,$$

$$\ell_{max} := \textit{length of the longest maximal green sequence of } Q.$$

Thanks!