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Minimal Length Maximal Green Sequences

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Abstract. Maximal green sequences are important objects in representation theory, cluster algebras, and string theory. It is an open problem to determine what lengths are achieved by maximal green sequences of a quiver. We use the combinatorics of surface triangulations to address this problem. Our main result is a formula for the length of minimal length maximal green sequences of quivers defined by triangulations of an annulus or a punctured disk.

Résumé. Les suites vertes maximales sont des objets importants dans la théorie des représentation, les algèbres amassées et la théorie des cordes. C'est un problème ouvert que de déterminer les longueurs que peuvent prendre les suites vertes maximales d'un carquois. Nous utilisons la combinatoire des triangulations de surface pour étudier ce problème. Notre résultat principal est une formule pour la longueur minimale des suites vertes maximales de carquois définis par des triangulations d'un anneau ou d'un disque perforé.

Keywords: quiver mutation, maximal green sequence, triangulated surface

1 Introduction

A maximal green sequence is a distinguished sequence of local transformations, known as **mutations**, of a given **quiver** (i.e., directed graph). Maximal green sequences were introduced by Keller in [13] in order to obtain combinatorial formulas for the refined Donaldson-Thomas invariants of Kontsevich and Soibelman [14]. They are also important in string theory [1], representation theory [3, 4], and cluster algebras [10].

Recently, there have been many developments on the combinatorics of maximal green sequences (for example, see [11, 16, 15] and references therein). Our goal is to add to the known combinatorics by developing a numerical invariant of the set of maximal green sequences of Q: the length of minimal length maximal green sequences.

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This invariant is natural from the perspective of cluster algebras. Fixing a quiver Q induces an orientation of the edges of the corresponding exchange graph, and the maximal green sequences of Q are in natural bijection with finite length maximal directed paths in the resulting oriented exchange graph (see [3]). Examples of oriented exchange graphs include the Hasse diagrams of Tamari lattices and Cambrian lattices of type $\mathbb{A}$, $\mathbb{D}$, and $\mathbb{E}$ [17]. In these examples the minimal length maximal green sequences always have length equal to the number of vertices of Q, but in general the minimal length of a maximal green sequence is not known. Thus understanding the length of minimal length maximal green sequences provides new information about oriented exchange graphs.

Our main result (Theorem 5) is a formula for this minimal length when Q is of **cluster type** $\mathbb{D}_n$ and **cluster type** $\widetilde{\mathbb{A}}_{n-1}$ (i.e., Q is in the mutation class of a type $\mathbb{D}_n$ or of an affine type $\mathbb{A}_{n-1}$ Dynkin quiver). This number was calculated in cluster type $\mathbb{A}$ in [6].

In Section 2, we review the notions of quiver mutation and maximal green sequences. We also recall some relevant results in cluster type $\mathbb{A}$. We then state our first main result, which allows one to calculate the length of minimal length maximal green sequences of a quiver with finitely many cluster type $\mathbb{A}$ quivers attached to it (see Theorem 2).

In Section 3, we recall how triangulations of Riemann surfaces can be used to model maximal green sequences of an important family of quivers. We use this model to present Theorem 4, which is crucial to proving Theorem 2. We state Theorem 4 in restricted generality in order to formulate it in a way that fits with our exposition.

Finally, in Section 4, we combine Theorem 2 and the combinatorics of surface triangulations to find the length of minimal length maximal green sequences of quivers of cluster type $\mathbb{D}_n$ and cluster type $\widetilde{\mathbb{A}}_{n-1}$.

2 Ice quivers and maximal green sequences

A **quiver** Q is a 4-tuple (Q_0, Q_1, s, t), where $Q_0 = [m] := \{1, 2, \ldots, m\}$ is a set of **vertices**, Q_1 is a set of **arrows**, and two functions $s, t : Q_1 \to Q_0$ defined so that for every $\alpha \in Q_1$, we have $s(\alpha) \xrightarrow{\alpha} t(\alpha)$. An **ice quiver** is a pair (Q, F) with Q a quiver and $F \subset Q_0$ a set of **frozen vertices** where any $i, j \in F$ have no arrows of Q connecting them. By convention, we assume $Q_0 \backslash F = [n]$ and $F = [n+1, m] := \{n+1, n+2, \ldots, m\}$. We refer to elements of $Q_0 \backslash F$ as **mutable vertices**. Any quiver Q is regarded as an ice quiver by setting $F = \varnothing$.

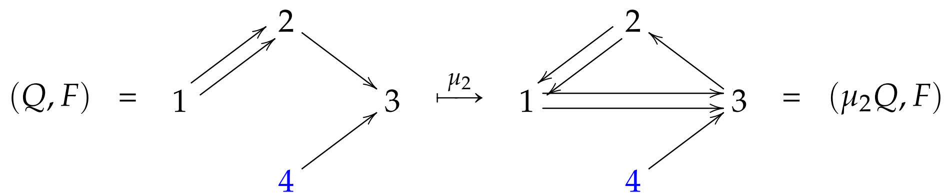
If an ice quiver (Q, F) is **2-acyclic** (i.e., Q has no loops or 2-cycles), we can define a local transformation of it called **mutation**. The **mutation** of (Q, F) at a mutable vertex k, denoted μ_k, produces a new ice quiver $(\mu_k Q, F)$ by the three step process:

(1) For every 2-path $i \to k \to j$ in Q, adjoin a new arrow $i \to j$.

(2) Reverse the direction of all arrows incident to k in Q.

(3) Remove any 2-cycles, and remove any arrows that connect two frozen vertices.

From now on, we will only work with 2-acyclic quivers. We show an example of mutation below with the mutable (respectively, frozen) vertices in black (respectively, blue).



Let $\mathrm{Mut}((Q,F))$ denote the collection of ice quivers obtainable from (Q,F) by finitely many mutations where such ice quivers are considered up to an isomorphism of quivers that fixes the frozen vertices. We refer to $\mathrm{Mut}((Q,F))$ as the **mutation class** of (Q,F). The following description of the mutation class of cluster type $\mathbb{A}_n$ quivers will be useful.

Lemma 1. *[5, Prop. 2.4] A connected quiver Q with n vertices is of cluster type $\mathbb{A}_n$ if and only if Q satisfies the following:*

- *i) All non-trivial cycles in the underlying graph of Q are oriented and of length 3.*
- *ii) Any vertex has degree at most 4.*
- *iii) If a vertex has degree 4, then two of its adjacent arrows belong to one 3-cycle, and the other two belong to another 3-cycle.*
- *iv) If a vertex has degree 3, then two of its adjacent arrows belong to a 3-cycle, and the third arrow does not belong to any 3-cycle.*

The **framed quiver** of Q is the ice quiver $\widehat{Q}$ where $\widehat{Q}_0 := Q_0 \sqcup [n+1, 2n]$, $F = [n+1, 2n]$, and $\widehat{Q}_1 := Q_1 \sqcup \{i \to n+i : i \in [n]\}$. A mutable vertex i of some quiver $\overline{Q} \in \mathrm{Mut}(\widehat{Q})$ is **green** (respectively, **red**) if there are no arrows in $\overline{Q}$ of the form $i \leftarrow n+j$ (respectively, $i \to n+j$) for some $j \in [n]$. Sign-coherence of **c**-vectors [7, Theorem 1.7] implies that any mutable vertex of a quiver $\overline{Q} \in \mathrm{Mut}(\widehat{Q})$ is either green or red.

Definition 1 ([13])**.** *A* ***maximal green sequence*** *of Q is a sequence $\boldsymbol{i} = (i_1, \ldots, i_k)$ of mutable vertices of $\widehat{Q}$ where*

- *i) for all $j \in [k]$ vertex $i_j \in [n]$ is green in $\mu_{i_{j-1}} \circ \cdots \circ \mu_{i_1}(\widehat{Q})$, and*
- *ii) each mutable vertex $i \in [n]$ of $\mu_{\boldsymbol{i}}(\widehat{Q})$ is red where $\mu_{\boldsymbol{i}} := \mu_{i_k} \circ \cdots \circ \mu_{i_1}$.*

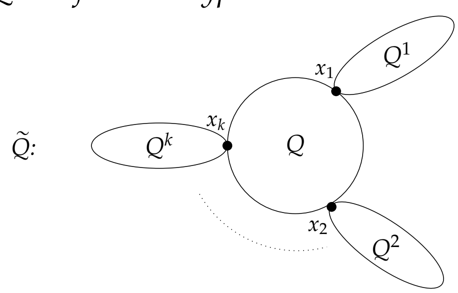
It is an open problem to determine what positive integers can be realized as lengths of maximal green sequences of a quiver Q. In [3, Lemma 2.20], it is shown that if Q is acyclic, then Q has a maximal green sequence of length $\#Q_0$ and there are no shorter maximal green sequences of Q. The following theorem is the first to address this open problem for an infinite family of quivers, many of which are not acyclic.

Theorem 1. *[6] The length of a minimal length maximal green sequence of a cluster type $\mathbb{A}_n$ quiver Q is $n + \#\{$3-cycles of $Q\}$.*

Our first main result (Theorem 2) shows that the problem of finding the minimal length of maximal green sequences of a quiver reduces to solving this problem for quivers Q without any cluster type $\mathbb{A}$ quivers attached to Q as in the following definition. The proof of Theorem 2 uses properties of the **consistent scattering diagram** associated with quiver Q, which are proven in [12]. We refrain from discussing scattering diagrams here. Instead, we emphasize the applications of Theorem 2 in this paper.

Definition 2. *Given a quiver Q let $\widetilde{Q}$ be a quiver composed of full connected subquivers $Q, Q^1, Q^2, \ldots, Q^k$, such that all of the following conditions hold:*

- $Q_0^i \cap Q_0 = \{x_i\}$,
- $Q_0^i \cap Q_0^j = \begin{cases} \{x_i\} & \text{if } x_i = x_j \\ \varnothing & \text{otherwise} \end{cases}$,
- *for every arrow in $\widetilde{Q}$, whenever one of the endpoints belongs to $Q_0^i \backslash \{x_i\}$ then the other endpoint belongs to Q_0^i,*
- *for every i the quiver Q^i is of cluster type $\mathbb{A}$.*



Theorem 2. *Let $\widetilde{Q}$ be a quiver as in Definition 2. Then the minimal length of a maximal green sequence of $\widetilde{Q}$ is $l_{min} + l_{min}^1 + l_{min}^2 + \cdots + l_{min}^k - k$ where l_{min}^i (respectively, l_{min}) is the minimal length of a maximal green sequence for Q^i (respectively, Q).*

3 Surface triangulations and shear coordinates

Let $\mathbf{S}$ denote an oriented Riemann surface, and let $\mathbf{M} \subset \mathbf{S}$ be a finite subset of $\mathbf{S}$ where we require that for each component $\mathbf{B}$ of $\partial\mathbf{S}$ we have $\mathbf{B} \cap \mathbf{M} \neq \varnothing$. We call the elements of $\mathbf{M}$ **marked points** and the elements of $\mathbf{M} \backslash (\mathbf{M} \cap \partial\mathbf{S})$ **punctures**. We call the pair $(\mathbf{S}, \mathbf{M})$ a **marked surface**[4]. We henceforth fix a marked surface $(\mathbf{S}, \mathbf{M})$.

We define an **arc** on $\mathbf{S}$ to be a curve γ in $\mathbf{S}$ such that

[4]We require that $(\mathbf{S}, \mathbf{M})$ is not a sphere with one, two, or three punctures; a disc with one, two, or three marked points on the boundary; or a punctured disc with one marked point on the boundary.

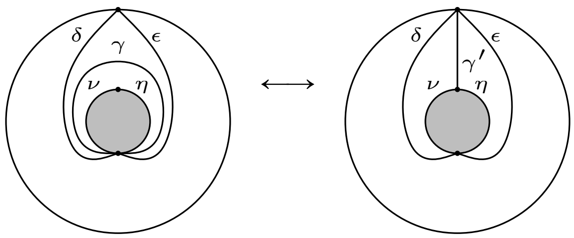


Figure 1: A flip connecting two triangulations of an annulus.

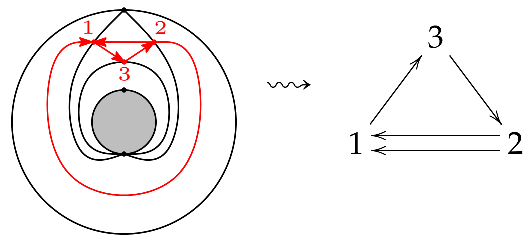


Figure 2: The (cluster type $\widetilde{\mathbb{A}}_2$) quiver $Q_{\mathbf{T}}$ defined by a triangulation $\mathbf{T}$.

- its endpoints are marked points;
- γ does not intersect itself, except that its endpoints may coincide;
- except for the endpoints, γ is disjoint from $\mathbf{M}$ and from the boundary of $\mathbf{S}$;
- γ does not cut out an unpunctured monogon or and unpunctured digon. (In other words, γ is not contractible into $\mathbf{M}$ or onto the boundary of $\mathbf{S}$.)

An arc γ is considered up to isotopy relative to the endpoints of γ. We say two arcs γ_1 and γ_2 on $\mathbf{S}$ are **compatible** if they are isotopic relative to their endpoints to curves that are nonintersecting except possibly at their endpoints. A **triangulation** of $(\mathbf{S}, \mathbf{M})$, denoted $\mathbf{T}$, is defined to be a maximal collection of pairwise compatible arcs.

One moves between triangulations by local moves called **flips**. Define the **flip** of an arc $\gamma \in \mathbf{T}$ as the unique arc $\gamma' \neq \gamma$ that produces a triangulation $\mathbf{T}' = (\mathbf{T} \backslash \{\gamma\}) \sqcup \{\gamma'\}$ (see Figure 1). If $\mathbf{M}$ contains punctures, there exist triangulations containing **self-folded triangles** (e.g., the region of $\mathbf{S}$ bounded by γ_3 and γ_4 in Figure 3 is a self-folded triangle). We refer to the arc γ_3 (respectively, γ_4) in Figure 3 as a **loop** (respectively, a **radius**).

As the flip of a radius is not defined, **tagged arcs** were introduced in [8] to obtain such a notion. A **tagged arc** $\iota(\gamma)$ is obtained from an arc γ that does not cut out a once-punctured monogon and "tagging" its ends either as **plain** or **notched** so that:

- an end of γ lying on the boundary of $\mathbf{S}$ is tagged plain; and
- both ends of a loop have the same tagging.

We use the symbol $\bowtie$ to indicate that an end of an arc is notched. We say two tagged arcs $\iota(\gamma_1)$ and $\iota(\gamma_2)$ are **compatible** if the following hold:

- Their underlying arcs γ_1 and γ_2 are the same, and the tagged arcs $\iota(\gamma_1)$ and $\iota(\gamma_2)$ have the same tagging at exactly one endpoint.
- Their underlying arcs γ_1 and γ_2 are distinct and compatible, and any common endpoints of $\iota(\gamma_1)$ and $\iota(\gamma_2)$ have the same tagging.

A **tagged triangulation** of $(\mathbf{S}, \mathbf{M})$ is a maximal collection of pairwise compatible tagged arcs. It follows from the construction that any arc in a tagged triangulation can be flipped. For example, see Figure 3 for an example of how to flip the radius γ_4.

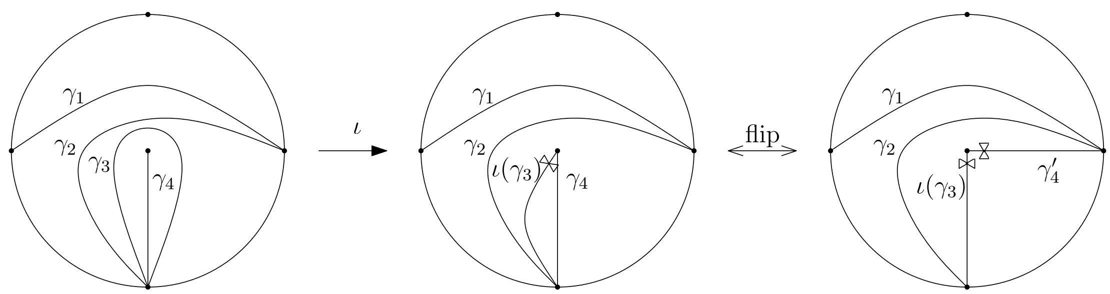


Figure 3: The map identifying a triangulation of a punctured disk as a tagged triangulation of a punctured disk and the flip of the tagged arc γ_4 from that tagged triangulation.

Each triangulation $\mathbf{T}$ of $\mathbf{S}$ defines a **signed adjacency quiver** $Q_{\mathbf{T}}$ by associating vertices to arcs and arrows based on oriented adjacencies (see Figure 2). More precisely, given a triangulation $\mathbf{T}$ consider a map $\pi : \mathbf{T} \to \mathbf{T}$ on the set of arcs defined as follows. If γ is a radius of a self-folded triangle, then let $\pi(\gamma)$ be the corresponding loop, otherwise let $\pi(\gamma) = \gamma$. Then the quiver $Q_{\mathbf{T}}$ consists of vertices i_γ for every $\gamma \in \mathbf{T}$ and arrows $i_\gamma \to i_{\gamma'}$ for every non self-folded triangle with sides $\pi(\gamma)$ and $\pi(\gamma')$ such that $\pi(\gamma)$ follows $\pi(\gamma')$ in the clockwise order. Finally, remove any 2-cycles from $Q_{\mathbf{T}}$ to produce a 2-acyclic quiver[5]. The following theorem shows that flips are compatible with mutations.

Theorem 3. *[8] Given a tagged triangulation $\mathbf{T}$ and a tagged arc $\gamma \in \mathbf{T}$, let $\mathbf{T}'$ be obtained from $\mathbf{T}$ by flipping γ. Then $\mu_{i_\gamma}(Q_{\mathbf{T}}) = Q_{\mathbf{T}'}$.*

We now show how shear coordinates provide a way to describe maximal green sequences geometrically. Recall that a fixed orientation $\mathcal{O}$ of a surface $\mathbf{S}$ induces an orientation $\mathcal{O}$ on each component of $\partial\mathbf{S}$ such that the surface $\mathbf{S}$ lies to the right of every component. If γ is a tagged arc in $\mathbf{S}$, the **elementary lamination** ℓ_γ is a curve that runs along γ within a small neighborhood of it. If γ has an endpoint M on the boundary $\partial\mathbf{S}$, then ℓ_γ begins at a point $M' \notin \mathbf{M}$ on $\partial\mathbf{S}$ located near M in the direction of $\mathcal{O}$, and proceeds along γ. If γ has an endpoint at a puncture p, then ℓ_γ spirals into p: clockwise if γ is notched at p, and counterclockwise if it is tagged plain (e.g., see Figure 4). A set of curves L is a **lamination** if it consists of elementary laminations arising from some pairwise compatible tagged arcs.

Definition 3. *Let L be a lamination, and let $\mathbf{T}$ be a triangulation. For each arc $\gamma \in \mathbf{T}$, that is not a radius of a self-folded triangle, the* **shear coordinate**[6] *of L with respect to $\mathbf{T}$, denoted by $b_\gamma(\mathbf{T}, L)$, is defined as a sum of contributions from all intersections of curves in L with γ. Specifically, such an intersection contributes $+1$ (respectively, -1) to $b_\gamma(\mathbf{T}, L)$ if the corresponding segment of*

[5] For the definition of $Q_{\mathbf{T}}$ where $\mathbf{T}$ is any tagged triangulation, we refer the reader to [8].

[6] For the definition of shear coordinates where $\mathbf{T}$ is any tagged triangulation, we refer the reader to [9].

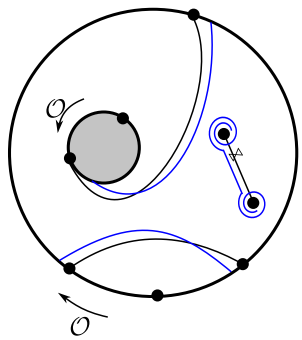


Figure 4: Lamination corresponding to the given set of arcs.

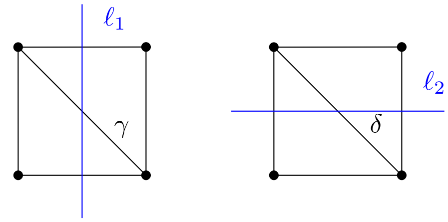


Figure 5: The shear coordinate of γ with respect to ℓ_1 is $+1$, and the shear coordinate of δ with respect to ℓ_2 is -1.

a curve in L cuts through the quadrilateral surrounding γ as shown in Figure 5 on the left (respectively, right).

Definition 4. *Fix $\mathbf{T} = \{\gamma_1, \ldots, \gamma_n\}$, $\mathbf{T}'$ a tagged triangulation obtained from $\mathbf{T}$ by a sequence of flips, and $\gamma' \in \mathbf{T}'$. Define the **c-vector** of γ' to be the integer vector $\mathbf{c}(\gamma', \mathbf{T}') := (b_{\gamma'}(\mathbf{T}', \ell_{\gamma_j})) \in \mathbb{Z}^n$. Define γ' to be **green** (respectively, **red**) if $b_{\gamma'}(\mathbf{T}', \ell_{\gamma_j}) \geqslant 0$ (respectively, $b_{\gamma'}(\mathbf{T}', \ell_{\gamma_j}) \leqslant 0$) for each $\gamma_j \in \mathbf{T}$.*

By [9, Proposition 17.3], tagged triangulations and elements of $\mathrm{Mut}(\widehat{Q}_{\mathbf{T}})$ are in bijection. Thus the ice quiver $\overline{Q} \in \mathrm{Mut}(\widehat{Q}_{\mathbf{T}})$ corresponding to $\mathbf{T}'$ has $Q_{\mathbf{T}'}$ as its mutable part and its other arrows are given by the equations $b_{\gamma'}(\mathbf{T}', \ell_{\gamma_j}) = \#\{i_{\gamma'} \to n + j\} - \#\{i_{\gamma'} \leftarrow n + j\}$, one for each $\gamma' \in \mathbf{T}'$ and $\gamma_j \in \mathbf{T}$. Moreover, tagged arc $\gamma' \in \mathbf{T}'$ is green (respectively, red) if and only if vertex $i_{\gamma'} \in \overline{Q}_0$ is green (respectively, red). Additionally, [4] implies that a maximal green sequence $\mathbf{i} = (i_1, \ldots, i_k)$ of $Q_{\mathbf{T}}$ is equivalent to the sequence of $\mathbf{c}$-vectors $\mathbf{c}(\mathbf{i}) := (\mathbf{c}(\gamma^{(1)}, \mathbf{T}_1), \ldots, \mathbf{c}(\gamma^{(k)}, \mathbf{T}_k))$ where $\gamma^{(j)}$ is the tagged arc corresponding to vertex $i_j \in (\mu_{i_{j-1}} \cdots \mu_{i_1}(\widehat{Q}_{\mathbf{T}}))_0$ and the mutable part of $\mu_{i_j} \cdots \mu_{i_1}(\widehat{Q}_{\mathbf{T}})$ is $Q_{\mathbf{T}_j}$.

Theorem 4. *Let $Q^\dagger$ be a full subquiver of $Q_{\mathbf{T}}$ whose vertices correspond to $\gamma_{j_1}, \ldots, \gamma_{j_\ell}$ and let $\boldsymbol{i} = (i_1, \ldots, i_k)$ be a maximal green sequence of $Q_{\mathbf{T}}$. Then the sequence $\mathbf{c}^\dagger$ obtained by removing from $\mathbf{c}(\boldsymbol{i})$ every $\mathbf{c}$-vector with nonzero entries at positions other than $j_1, \ldots, j_\ell$ is the sequence of $\mathbf{c}$-vectors of a maximal green sequence of $Q^\dagger$.*

The above is essentially proved in [16], using the consistent scattering diagram of $Q_{\mathbf{T}}$. Moreover, the same technique proves Theorem 4 for a general 2-acyclic quiver Q with the $\mathbf{c}$-vectors of Q appropriately defined. This more general version of Theorem 4 is an ingredient in the proof of Theorem 2.

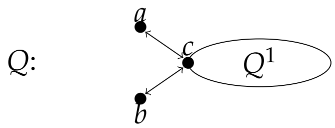


Figure 6: Type I quivers

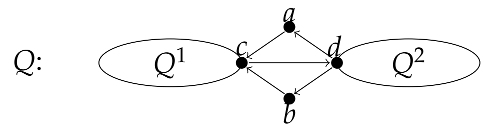


Figure 7: Type II quivers

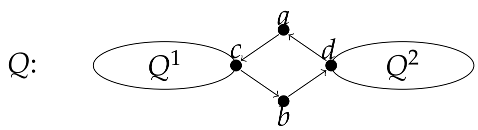


Figure 8: Type III quivers

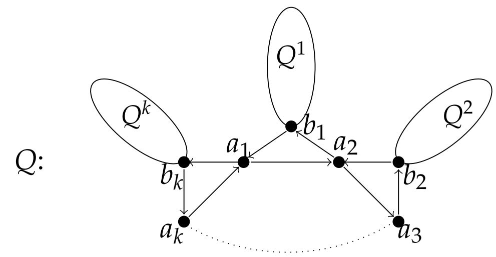


Figure 9: Type IV quivers

4 Quivers of cluster type $\mathbb{D}_n$ and $\widetilde{\mathbb{A}}_{n-1}$

To present our results, we need to recall the classification of cluster type $\mathbb{D}_n$ quivers. By [18, Theorem 3.1], the mutation class of a type $\mathbb{D}_n$ quiver where $n \geqslant 4$ consists of four families of quivers. Before presenting these, we say a vertex c of Q is a **connecting** vertex if it has degree at most 2, and if it has degree 2, it belongs to a 3-cycle of the same quiver.

A quiver Q is of **Type I** (see Figure 6) if and only if Q has the following properties:

- it has a full subquiver of the form $a \leftrightarrow c \leftrightarrow b$ (the notation $a \leftrightarrow c$ to indicates that there exists a single arrow in Q connecting a and c),
- the full subquiver Q^1 of Q on the vertices $Q_0 \backslash \{a, b\}$ is of cluster type $\mathbb{A}$, and
- the vertex c is a connecting vertex of Q^1.

A quiver Q is of **Type II** (see Figure 7) if and only if Q has the following properties:

- it has a full subquiver of the form shown in Figure 7 whose vertices are a, b, c, d,
- the subquiver Q' obtained by removing vertices a and b and the arrow $c \to d$ consists of two cluster type $\mathbb{A}$ quivers Q^1 and Q^2, and
- the vertex c (respectively, d) is a connecting vertex of Q^1 (respectively, Q^2).

A quiver Q is of **Type III** (see Figure 8) if and only if Q has the following properties:

- it has a full subquiver as shown in Figure 8 whose vertices are a, b, c, d,
- the full subquiver Q' of Q on the vertices $Q_0 \backslash \{a, b\}$ consists of two connected quivers Q^1 and Q^2, each of which is of cluster type $\mathbb{A}$, and
- the vertex c (respectively, d) is a connecting vertex of Q^1 (respectively, Q^2).

A quiver Q is of **Type IV** (see Figure 8) if and only if Q has the following properties:

- it has a full subquiver R that is an oriented k-cycle where $k \geqslant 3$, $R_0 = \{a_1, a_2, \ldots, a_k\}$, and $R_1 = \{a_i \to a_{i+1} : i \in [k-1]\} \sqcup \{a_k \to a_1\}$,
- for each arrow $\alpha \in R_1$, there may be a vertex $b_i \in Q_0 \backslash R_0$ that is in a 3-cycle $b_i \to a_i \xrightarrow{\alpha} a_{i+1} \to b_i$, which is a full subquiver of Q, but there are no other vertices in $Q_0 \backslash R_0$ that are connected to vertices of R,
- the full subquiver Q' obtained from Q by removing the vertices and arrows of the subquiver R consists of the quivers $\{Q^i\}_{i \in [k]}$ some of which may be empty quivers, and where each quiver Q^i is of cluster type $\mathbb{A}$ and has b_i as a connecting vertex.

The cluster type $\widetilde{\mathbb{A}}_{n-1}$ quivers can be described in a very similar way to the Type IV quivers (see [2]). However, the only oriented cycles in cluster type $\widetilde{\mathbb{A}}_{n-1}$ quivers are 3-cycles. Another important distinction between $\mathbb{D}_n$ and $\widetilde{\mathbb{A}}_{n-1}$ quivers is that $\text{Mut}(\widehat{Q})$ is finite (respectively, infinite) if Q is of cluster type $\mathbb{D}_n$ (respectively, $\widetilde{\mathbb{A}}_{n-1}$).

Theorem 5. *Let l denote the length of a minimal length maximal green sequence of a quiver $\widetilde{Q}$.*

i) If $\widetilde{Q}$ is of Type I or is of cluster type $\widetilde{\mathbb{A}}_{n-1}$, then $l = n + \#\{\text{3-cycles in } \widetilde{Q}\}$.

ii) If $\widetilde{Q}$ is of Type II, then $l = n + 1 + \#\{\text{3-cycles in } Q^1\} + \#\{\text{3-cycles in } Q^2\}$.

iii) If $\widetilde{Q}$ is of Type III, then $l = n + 2 + \#\{\text{3-cycles in } \widetilde{Q}\}$.

iv) If $\widetilde{Q}$ is of Type IV, then $l = n + k - 2 + \#\{a_i : deg(a_i) = 4\} + \sum_{i=1}^{k} \#\{\text{3-cycles in } Q^i\}$.

To prove Theorem 5, first, observe that $\widetilde{Q}$ satisfies Definition 2. One thus applies Theorem 2 to reduce the problem of calculating l to calculating the minimal length of a maximal green sequence of Q, which is obtained by removing all vertices of $\widetilde{Q}$ belonging to $Q_0^i \backslash \widetilde{Q}_0$ for some i. If $\widetilde{Q}$ is of Type I, II, or III, the resulting family of quivers Q consists of exactly six quivers. Thus, it is a finite calculation to verify the theorem.

On the other hand, if $\widetilde{Q}$ is of Type IV or of cluster type $\widetilde{\mathbb{A}}_{n-1}$, the same family of quivers Q is infinite. However, one can parameterize this family of quivers by triangulations of the once-punctured disk, in the former case, and triangulations of the unpunctured annulus, in the latter case, with certain conditions on the triangles. We then use the corresponding triangulation to construct a minimal length maximal green sequence of each quiver Q. We will sketch this approach when Q is a Type IV quiver.

Let $(\mathbf{S}, \mathbf{M})$ be the once-punctured disk with n marked points on the boundary and unique puncture p. A triangulation $\mathbf{T}$ defining Q is determined by the following properties (see the top left triangulation in Figure 10):

- if γ is not connected to p, there is a triangle whose only internal arc is γ, and
- if γ is connected to p, it is tagged plain at p.

The triangulation $\mathbf{T}$ is the unique triangulation satisfying $Q_{\mathbf{T}} = Q$, up to the action of the mapping class group of the surface and up to simultaneously changing the tagging of all ends of arcs connected to p.

The maximal green sequence we construct has five components, which we denote by $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, $\mathbf{i}_4$, and $\mathbf{i}_5$. We construct the sequence in the context of Figure 10 wherein

$\gamma_{a_s}, \gamma_{b_t} \in \mathbf{T}$ are the arcs corresponding to vertices $a_s, b_t \in Q_{\mathbf{T}}$, respectively. We do not justify it here, but the length of $\mathbf{i}$ is $n + k - 2 + \#\{a_i : \deg(a_i) = 4\}$[7].

The first components, $\mathbf{i}_1 = (a_7, b_6, a_4, b_3, b_2, a_2, b_1)$ and $\mathbf{i}_2 = (a_3, a_5, a_6, a_1)$, perform flips at each of the arcs in the initial triangulation $\mathbf{T}$ exactly once so that the total length of these two sequences is n. The vertices in $\mathbf{i}_j$ with $j = 1, 2$ are ordered so that if two vertices in $\mathbf{i}_j$ are connected by an arrow $\alpha \in (Q_{\mathbf{T}})_1$, then $s(\alpha)$ is mutated before $t(\alpha)$ in $\mathbf{i}_j$.

The sequence $\mathbf{i}_3$ consists of all green vertices of $\mu_{\mathbf{i}_2}\mu_{\mathbf{i}_1}(\widehat{Q}_{\mathbf{T}})$ where flipping the corresponding arcs produces ones that can be obtained from arcs of $\mathbf{T}$ not connected to p by moving their endpoints clockwise along the boundary to the next two marked points. As shown in Figure 10, we have $\mathbf{i}_3 = (i_\alpha)$. Moreover, the new arc α' is obtained by moving the endpoints of γ_{b_2} clockwise along the boundary to the next two marked points.

The sequence $\mathbf{i}_4$ consists of all vertices of $\mu_{\mathbf{i}_3}\mu_{\mathbf{i}_2}\mu_{\mathbf{i}_1}(\widehat{Q}_{\mathbf{T}})$ whose corresponding tagged arcs are connected to p and is tagged plain at p. Similar to $\mathbf{i}_2$, we also require that if two vertices in $\mathbf{i}_4$ are connected by an arrow $\alpha \in \mu_{\mathbf{i}_3}\mu_{\mathbf{i}_2}\mu_{\mathbf{i}_1}(\widehat{Q}_{\mathbf{T}})$, then $s(\alpha)$ is mutated before $t(\alpha)$ in $\mathbf{i}_4$. As shown in Figure 10, we have $\mathbf{i}_4 = (i_\delta)$. It turns out that the length of $\mathbf{i}_3$ plus the length of $\mathbf{i}_4$ is $\#\{a_i : \deg(a_i) = 4\}$.

Lastly, the sequence $\mathbf{i}_5$ is defined inductively. First, mutate at all green vertices of $\mu_{\mathbf{i}_4}\mu_{\mathbf{i}_3}\mu_{\mathbf{i}_2}\mu_{\mathbf{i}_1}(\widehat{Q}_{\mathbf{T}})$ whose corresponding tagged arcs appear in a triangle whose other two sides are tagged arcs notched at p. In the context of Figure 10, one mutates at the vertices corresponding to $\sigma_1^{(1)}$ and $\sigma_2^{(1)}$. Next, repeat this process, each time with the new ice quiver, until there are no remaining green vertices. In the example in Figure 10, this process must be repeated twice: in the second (respectively, third) iteration one mutates at the vertices corresponding to $\sigma_1^{(2)}$ and $\sigma_2^{(2)}$ (respectively, $\sigma_1^{(3)}$). The length of $\mathbf{i}_5$ is $k - 2$.

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[7] To show that this number is the minimal length of a maximal green sequence requires its own intricate argument. Our technique is to use the combinatorics of tagged triangulations to show that at least $n + k - 2 + \#\{a_i : \deg(a_i) = 4\}$ tagged arcs must be flipped in any maximal green sequence.

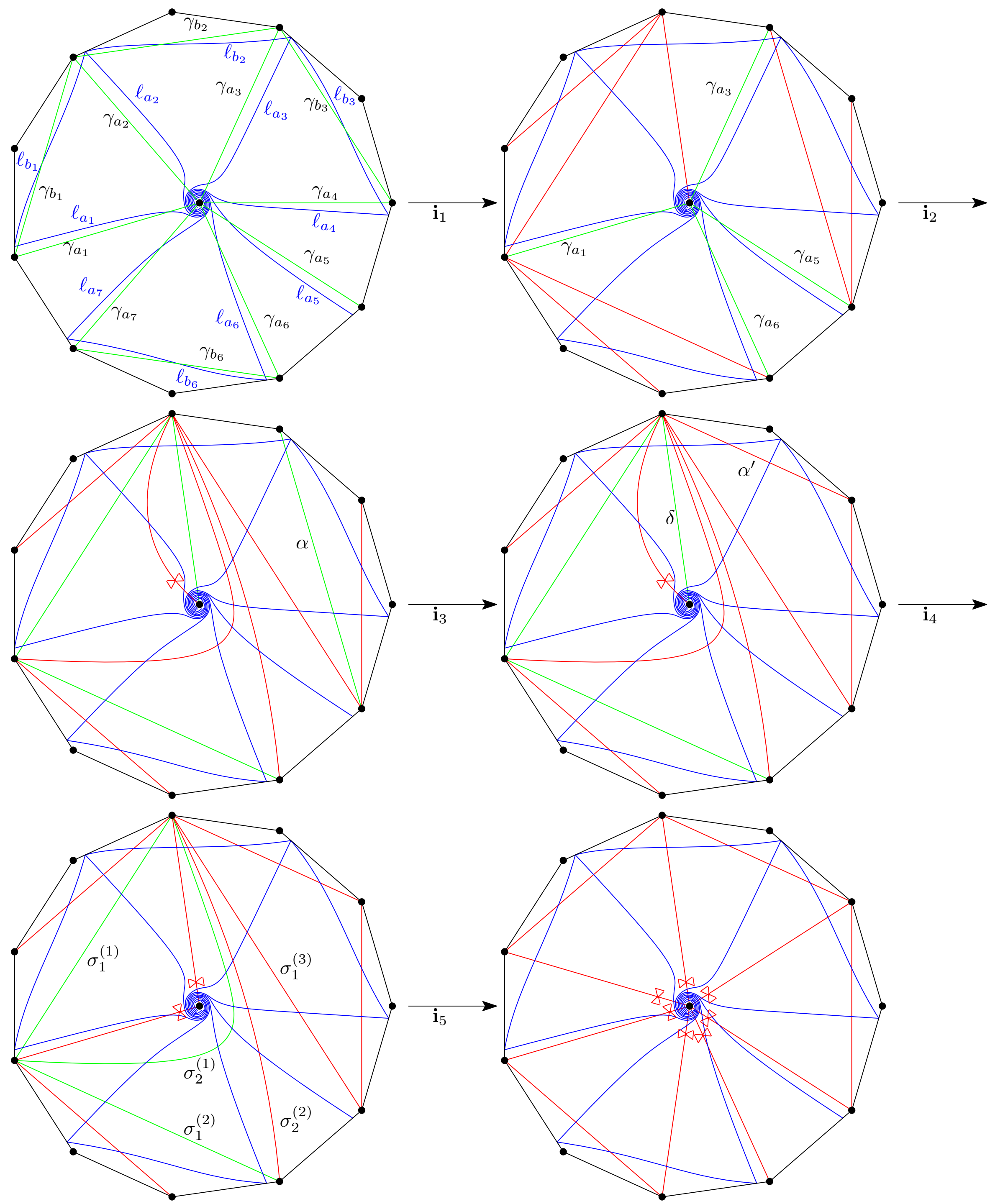

Figure 10: The maximal green sequence $\mathbf{i} = \mathbf{i}_1 \circ \mathbf{i}_2 \circ \mathbf{i}_3 \circ \mathbf{i}_4 \circ \mathbf{i}_5$ for a Type IV $\mathbb{D}_{11}$ quiver.

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