Example 3: A thin wire is bent into the shape of a semicircle $x^2 + y^2 = 4$, $x > 0$. The density of the wire is $\rho(x,y) = k$ for some constant $k$. Find the mass and center of mass of the wire.

$$m = \int_C k \, ds = \int_{-\pi/2}^{\pi/2} 2k \, dt = 2k \pi$$

$C$ is given by $r(t) = \langle 2 \cos t, 2 \sin t \rangle$ $\frac{-\pi}{2} \leq t \leq \frac{\pi}{2}$

$$r'(t) = \langle -2 \sin t, 2 \cos t \rangle \quad |r'(t)| = \sqrt{4 \sin^2 t + 4 \cos^2 t} = \sqrt{4} = 2$$

$$\int_{-\pi/2}^{\pi/2} 2k \cdot 2 \cos t \, dt = 4k \int_{-\pi/2}^{\pi/2} \cos t \, dt = 4k \left[ \sin t \right]_{-\pi/2}^{\pi/2} = 8k$$

$$\int_{-\pi/2}^{\pi/2} 2k \cdot \sin t \, dt = 0 \quad \Rightarrow \text{Center of mass} \quad \bar{y} = \left( \frac{4}{\pi}, 0 \right)$$

---

16.2 Line Integrals (continued) Line integrals w.r.t. $x, y$. $C$ given by $r(t) = \langle x(t), y(t) \rangle$, $a \leq t \leq b$

$$\int_C f(x,y) \, dx = \int_a^b f(x(t), y(t)) x'(t) \, dt$$

$$\int_C f(x,y) \, dy = \int_a^b f(x(t), y(t)) y'(t) \, dt$$

We often write

$$\int_C f(x,y) \, dx + \int_C g(x,y) \, dy = \int_C f(x,y) \, dx + g(x,y) \, dy$$

Fact: If $C$ is a piecewise-smooth curve (i.e., a union of smooth curves $C_1, \ldots, C_n$), then

(a) $\int_C f(x,y) \, ds = \int_{C_1} f(x,y) \, ds + \ldots + \int_{C_n} f(x,y) \, ds$
Evaluate
Example 1: \[ \int (y+z)\,dx + (x+z)\,dy + (x+y)\,dz \]
where \( C \) consists of the line segment from \((0,0,0)\) to \((1,0,1)\) and the line segment from \((1,0,1)\) to \((0,1,2)\).

The above integral becomes

\[
\begin{align*}
&\int_c (y+z)\,dx + \int_c (x+z)\,dy + \int_c (x+y)\,dz \\
&\quad + \int_{C_2} (y+z)\,dx + \int_{C_2} (x+z)\,dy + \int_{C_2} (x+y)\,dz.
\end{align*}
\]

The line segments are parameterized as follows:

\[ C_1: \vec{r}_1(t) = \langle t, 0, 1 \rangle \quad 0 \leq t \leq 1 \]
\[ C_2: \vec{r}_2(t) = \langle 1-t, t, 1+t \rangle \quad 0 \leq t \leq 1 \]

\[ \vec{r}'_1(t) = \langle 1, 0, 0 \rangle \]
\[ \vec{r}'_2(t) = \langle -1, 1, 1 \rangle \]

Final answer: \( \frac{1}{2} + \frac{1}{2} + (-2) + 1 = 2 \)

Example 2: A 160-lb man carries a 25-lb can of paint up a helical staircase that encircles a silo with radius of 20 ft. If the silo is 90 ft high and the man makes exactly three revolutions climbing to the top, how much work is done by the man against gravity?

Vector Field \( \vec{F} = \langle 0, 0, 185 \rangle \)

\[ \vec{r}(t) = \langle 20\cos(t), 20\sin(t), 15 \rangle \]
\[ 0 \leq t \leq 6\pi \]

\[ \vec{r}'(t) = \langle -20\sin(t), 20\cos(t), \frac{15}{\pi} \rangle \]
\[ 0 \leq t \leq 6\pi \]
\[
\text{total work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{6\pi} (0, 0, 185) \cdot (-20\sin(t), 20\cos(t), \frac{15}{\pi}) \, dt \\
= \int_0^{6\pi} (185) \, dt = (185)(\frac{15}{\pi})(6\pi) = 16650 \pi - 16
\]

16.3 Fundamental Theorem of Line Integrals

Thm Let \( C \) be a smooth curve parameterized by \( \mathbf{r}(t) \) with \( a = t = b \), and let \( f \) be a differentiable scalar function. Then \( \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \).

\[\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r} \]

Corollary: If \( C \) is a closed curve, then \( \int_C \nabla f \cdot d\mathbf{r} = 0 \).

Example: \( \mathbf{F} = \langle \ln(yz), \frac{x}{y}, \frac{x}{z} \rangle \) and let \( C \) be a curve from \((3,1,2)\) to \((4,2,1)\).

Evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \).

\[f(x,y,z) = x\ln(yz) + y(yz)\]

Find a potential function \( f \).

\( f(x,y,z) = x\ln(yz) \) works.

Now, by the Fundamental Theorem of Line Integrals,

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(4,2,1) - f(3,1,2) = 4\ln(z) - 3\ln(z) = \ln(z).
\]

Example: Show \( \mathbf{F} = \langle -y, x \rangle \) is not conservative.

Show there is no scalar function \( f \) such that \( \mathbf{F} = \nabla f \).

Suppose there was a scalar function \( f \) such that \( \mathbf{F} = \nabla f \).

Last time, we showed that \( \int_C \mathbf{F} \cdot d\mathbf{r} = 8\pi \) where \( C \) is the circle of radius 2 centered at \((0,0)\) and oriented counterclockwise.
Since $C$ is a closed curve, this contradicts the Corollary. Therefore, no such function $\mathbf{F}$ exists. So $\mathbf{F}$ is not conservative.

**Example 3:** Which of the vector fields are conservative?

(a) $\mathbf{F}(x, y) = (xy + y^2)\mathbf{i} + (x^2 + 2xy)\mathbf{j}$

(b) $\mathbf{F}(x, y) = y^2 e^{xy} \mathbf{i} + (1 + xy) e^{xy} \mathbf{j}$

(c) See Figure

(a) $\nabla \times \mathbf{F}(x, y) = \frac{1}{2} x^2 y + xy^2 + g(y)$ for some $g(y)$

\[ \mathbf{F}(x, y) = xy^2 + xy^2 + h(x) \quad \text{for some} \quad h(x) \]

\[ \implies \mathbf{F} \neq \nabla \times \mathbf{F} \quad \text{for any} \quad \mathbf{F}(x, y) \]

(b) $\mathbf{F}(x, y) = ye^{xy} + g(y)$

$\mathbf{F}(x, y) = ye^{xy}$ works