15.7 Cylindrical Coordinates

A coordinate system for \( \mathbb{R}^3 \)
similar to polar coordinates

A point in cylindrical coordinates
is a triple \((r, \theta, z)\)

To go from cylindrical to rectangular coordinates, use equations

\[
x = r \cos \theta, \quad y = r \sin \theta, \quad z = z
\]

To go from rectangular to cylindrical coordinates, use the equations

\[
R^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}, \quad z = z
\]

Example 1: Describe the following surfaces

\[a) \ r = 2\]

\[b) \ \theta = \frac{\pi}{6}\]

\[c) \ r^2 + z^2 = 4\]

\[d) \ r = \sin(2\theta)\]

Answers:

\[a) \text{Vertical cylinder of radius 2, centered at } z\text{-axis}\]
Vertical plane forming angle of $\frac{\pi}{6}$ with xz-plane

Integration Consider continuous

$E = \iiint_E f(x,y,z) \, dx \, dy \, dz$ in $D$, $u_1(x,y) = z = u_2(x,y)$

$D$ is region in the xy-plane that is describable using polar coordinates

$D = \{(r,\theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$

$\iiint_D f(x,y,z) \, dV = \iiint_D \left( \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z) \, dz \right) \, dA$

$= \int_0^{2\pi} \left( \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta, z) \, dz \right) \, r \, dr \, d\theta$

For each value of $z$, we get the same flower curve with 4 leaves.
Example 2: Evaluate \( \iiint_V \) where \( E \) is below \( z = x + 2 \) and above the xy-plane and between \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 4 \).

\[
\iiint_E r^2 \sin \theta dz \, dr \, d\theta = 0
\]

Easy to see:

\[
\iiint_E r^2 \sin \theta dz \, dr \, d\theta = \iiint_{E_1} r^2 \sin \theta dz \, dr \, d\theta + \iiint_{E_2} r^2 \sin \theta dz \, dr \, d\theta = 0
\]

these two regions have same volume

Example 3: Express \( \iiint_{E_3} xyz \, dz \, dx \, dy \) in cylindrical coordinates.

The inequalities \(-1 \leq y \leq 1, 0 \leq x \leq \sqrt{1 - y^2}\) describe a polar region \( D \) in the xy-plane.

\[
D = \{ (r, \theta) \mid 0 \leq r \leq 1, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \}
\]

Also, \( x^2 + y^2 \leq z \leq \sqrt{x^2 + y^2} \) becomes \( r^2 \leq z \leq r \) in cylindrical coordinates.
Example 4 Change the order of integration of \( \iiint r^3 \cos \theta \sin \theta \, dz \, dr \, d\theta \) so that the differential is \( dz \, dr \, d\theta \).

The equation \( z/\sqrt{3} = r \) is an infinite double cone with vertex at \((0,0,0)\). Therefore, the region we are integrating over is the solid cone in image 15-7-ex4.pdf.

Examine the region in the \( yz \)-plane.

The integral becomes
\[
\iiint \frac{r^3}{3} \, dz \, dr \, d\theta. \quad \left( = \frac{3\pi}{10} \right)
\]

Example 5 Find the volume of the solid enclosed by \( z = \sqrt{x^2 + y^2} \) and \( x^2 + y^2 + z^2 = 2 \).

The surfaces intersect in a circle with center \((0,0,1)\) in rectangular coordinates and radius 1. Also it is parallel to the \( xy \)-plane.

\[\Rightarrow \text{Volume of } E = \iiint_E dV \]
\[= \iiint r \, dz \, dr \, d\theta \]
\[= \frac{4\pi}{3} \left( \sqrt{2} - 1 \right).\]
Spherical Coordinates

ρ ≥ 0
\[ \text{distance from origin to the point} \]

φ = angle between positive z-axis and the line from origin to the point

0 ≤ φ ≤ π

θ same as for cylindrical coordinates

Spherical \rightarrow Cylindrical

z = ρ \cos φ, r = ρ \sin φ

Spherical \rightarrow Rectangular

x = ρ \sin φ \cos θ, y = ρ \sin φ \sin θ, z = ρ \cos φ

Rectangular \rightarrow Spherical \[ p^2 = x^2 + y^2 + z^2 \]

Example: Describe the following surfaces

\( \rho = c \) \( \Rightarrow \) c is constant

\( \theta = \frac{\pi}{3} \)

\( \phi = c \) \( \Rightarrow \) c is constant where \( 0 < c < \frac{\pi}{2} \)

\( \rho - 3 \rho + 2 = 0 \)

\( \rho \cos \phi = 1 \)
(a) sphere of radius $c$

(b) half plane

(c) cone

(d) $p^2 - 3p + 2 = 0$

$(p-2)(p-1) = 0$

two concentric spheres both centered at origin

Example Convert

$$E = \int_{0}^{3} \int_{0}^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2 + y^2 + z^2 \, dz \, dx \, dy$$

Find bounds for $\phi$.

Where do $z = \sqrt{x^2+y^2}$ and $x^2 + y^2 + z^2 = 18$ intersect?

horizontal plane containing $(0,0,1)$ in rectangular coordinates

Example Convert

$$E = \int_{0}^{\sqrt{3}} \int_{0}^{\pi/4} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} r \, dr \, d\theta \, d\phi$$

$0 \leq \rho \leq 3\sqrt{2}$

$0 \leq \theta \leq \frac{\pi}{2}$

$\rho \cos \phi = 3 \Rightarrow 3\sqrt{2} \cos \phi = \frac{3\sqrt{2}}{2}$

$\phi = \frac{\pi}{4}$

$z = 3$

$x^2 + y^2 = 9$

$z^2 = 18$
Integral becomes

\[
\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} (p^2 \sin^2 \phi \cos^2 \theta + p^2 \sin \phi \sin^2 \theta + p^2 \cos^2 \phi) \, p^2 \sin \phi \, dp \, d\phi \, d\theta
\]

\[
= p^2 \sin^2 \phi
\]

\[
= \frac{972\pi \sqrt{2}}{2} \left(1 - \frac{\sqrt{2}}{2}\right)
\]

\[
= \frac{972\pi \sqrt{2}}{2} \left(1 - \frac{\sqrt{2}}{2}\right)
\]

Example: Find the volume of the solid enclosed by \( Z = \sqrt{x^2 + y^2} \), \( x^2 + y^2 + z^2 = 2 \)

Volume of \( E \) is \( \iiint_E dV \)

\[
= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} p^2 \sin \phi \, dp \, d\phi \, d\theta
\]

\[
= \int_0^{\frac{\pi}{2}} \left( \frac{2\pi}{3} \right) \left( \frac{\sqrt{2}}{2} \right) \left( 1 - \frac{\sqrt{2}}{2} \right)
\]

\[
= \frac{4\pi}{3} \left( \sqrt{2} - 1 \right)
\]