Reverse plane partitions via representations of quivers

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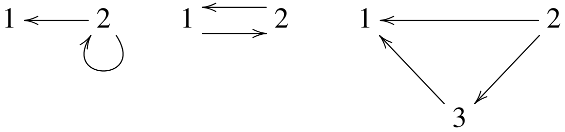
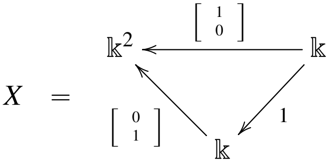
Outline

- representations of quivers (and algebras)
- nilpotent endomorphisms of quiver representations
- minuscule posets
- reverse plane partitions on minuscule posets
- periodicity of promotion

A **quiver** Q is a directed graph.

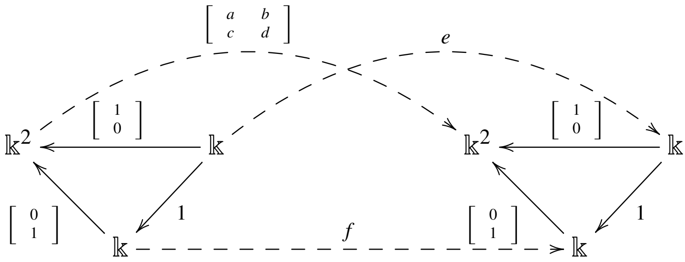


A **representation** X of Q is an assignment of finite-dimensional $\Bbbk$-vector spaces to each vertex and linear maps to each arrow.



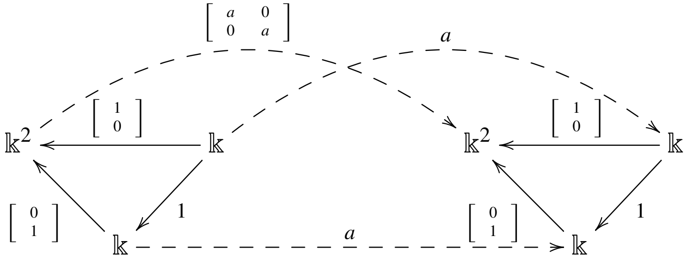
Throughout this talk, $\Bbbk$ will be an algebraically closed field.

A **morphism** of representations is a collection of linear maps so that "every square commutes".



rep(Q) - the category of all representations of Q (over $\mathbb{k}$)

A **morphism** of representations is a collection of linear maps so that "every square commutes".



rep(Q) - the category of all representations of Q (over $\mathbb{k}$)

Why study quiver representations?

Theorem (Gabriel '72)

For any finite-dimensional $\Bbbk$-algebra A,

$$mod(A) \simeq mod(\Bbbk Q/I).$$

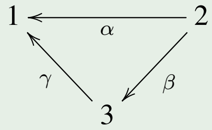
and

$$mod(\Bbbk Q/I) \simeq rep(Q,I).$$

These ideas appeared earlier in [Thrall '47, Grothendieck '57, Gabriel '60].

Example

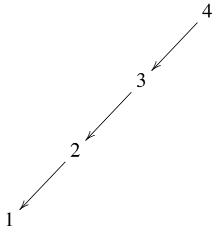
$Q \quad = $



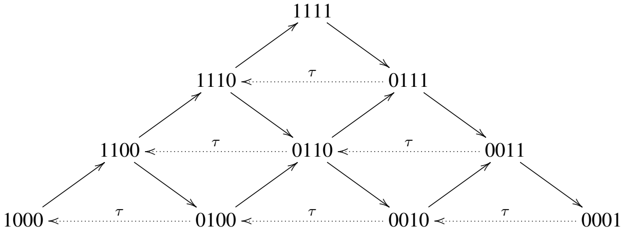
$$\Bbbk Q \cong \Bbbk e_1 \oplus \Bbbk e_2 \oplus \Bbbk e_3 \oplus \Bbbk\alpha \oplus \Bbbk\beta \oplus \Bbbk\gamma \oplus \Bbbk\gamma\beta$$

Here $\mathrm{mod}(\Bbbk Q) \simeq \mathrm{rep}(Q)$.

Any quiver Q has an **Auslander–Reiten quiver** $\Gamma(Q)$ whose vertices are the isomorphism classes of indecomposable representations of Q and whose arrows are irreducible morphisms. [Auslander–Reiten ‘75]



Q



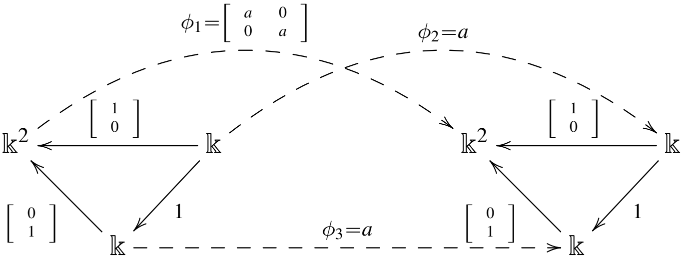
$\Gamma(Q)$ - the Auslander–Reiten quiver of Q

- There is a map τ called the **Auslander–Reiten translation**.
- The Auslander–Reiten translation partitions the indecomposables into τ**-orbits**.
- When Q is Dynkin, $\{\text{vertices of } Q\} \longleftrightarrow \{\tau\text{-orbits}\}$.

Theorem (Gabriel ‘72)

The isomorphism classes of indecomposable representations of a Dynkin quiver are in bijection with the positive roots of the associated root system.

- $A = \Bbbk Q/I$ - a finite dimensional algebra
- $X = ((X_i)_i, (f_a)_a) \in \mathrm{rep}(Q, I)$
- $\phi = (\phi_i)_i$ - a nilpotent endomorphism of X
- $\mathrm{NEnd}(X)$ - all nilpotent endomorphisms of X



Lemma

The space NEnd(X) is an irreducible algebraic variety.

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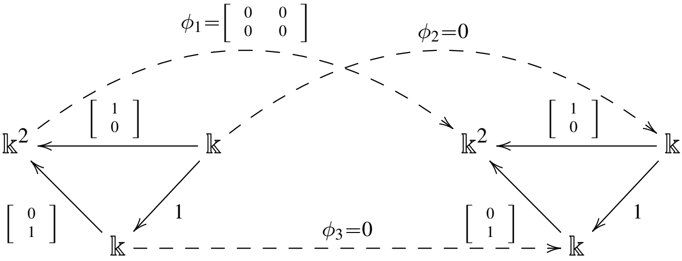


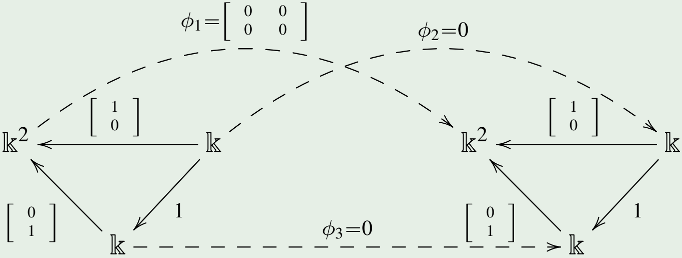
Lemma

The space NEnd(X) is an irreducible algebraic variety.

For each i, $\phi_i \rightsquigarrow \lambda^i = (\lambda^i_1 \geqslant \cdots \geqslant \lambda^i_r)$ where partition λ^i records the sizes of the Jordan blocks of ϕ_i.

$$\mathrm{JF}(\phi) := (\lambda^1, \ldots, \lambda^n) \quad \text{the } \textbf{Jordan form data} \text{ of } \phi$$

Example



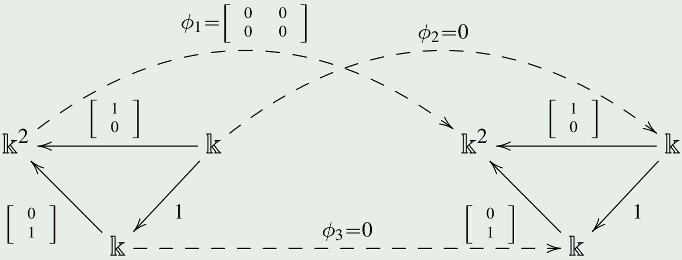
$$\mathrm{JF}(\phi) = ((1,1),(1),(1))$$

For $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_r)$ and $\lambda' = (\lambda'_1 \geqslant \cdots \geqslant \lambda'_{r'})$, one has $\lambda \leqslant \lambda'$ in **dominance order** if $\lambda_1 + \cdots + \lambda_\ell \leqslant \lambda'_1 + \cdots + \lambda'_\ell$ for each $\ell \geqslant 1$.

Theorem (G.–Patrias–Thomas, '18)

There is a unique maximum value of JF(·) on NEnd(X) with respect to componentwise dominance order, denoted by GenJF(X). It is attained on a dense open subset of NEnd(X).

Example



$$\mathrm{GenJF}(X) = ((1,1),(1),(1))$$

Goal: Study the invariant GenJF(X)

Question

For which subcategories $\mathcal{C}$ of rep(Q, I) is it the case that any object $X \in \mathcal{C}$ may be recovered from GenJF(X)? We say such a subcategory is ***Jordan recoverable****.*

Example

Usually GenJF(X) is not enough information to recover X. Let $Q = 1 \leftarrow 2$.

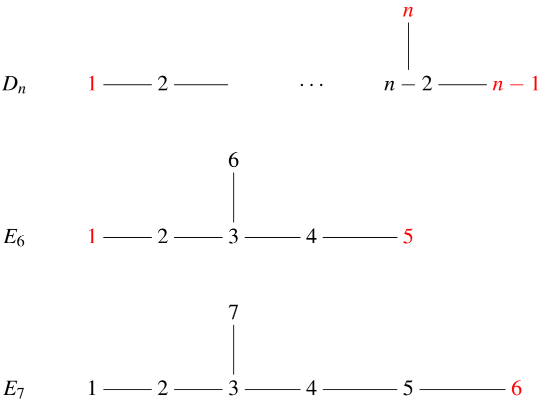
- $X = \mathbb{k} \xleftarrow{1} \mathbb{k}$ has GenJF$(X) = ((1), (1))$
- $X' = \mathbb{k} \xleftarrow{0} \mathbb{k}$ has GenJF$(X') = ((1), (1))$

Theorem (G.–Patrias–Thomas '18)

Let Q be a Dynkin quiver and m a ***minuscule vertex*** *of Q. The category $\mathcal{C}_{Q,m}$ of representations of Q all of whose indecomposable summands are supported at m is Jordan recoverable.*

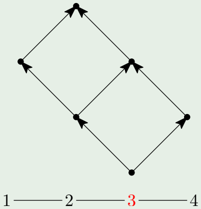
Moreover, we classify the objects in $\mathcal{C}_{Q,m}$ in terms of the combinatorics of the **minuscule poset** associated with Q and m.

The minuscule posets are defined by choosing a simply-laced Dynkin diagram and a **minuscule vertex** m.

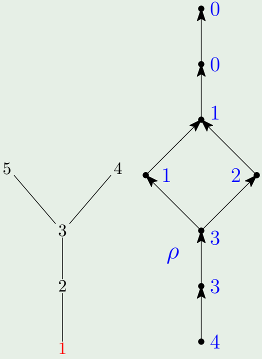


Minuscule posets

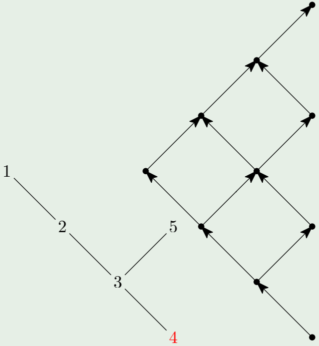
Example (Three families of minuscule posets)



$\mathsf{P}_{A_4,3}$



$\mathsf{P}_{D_5,1}$

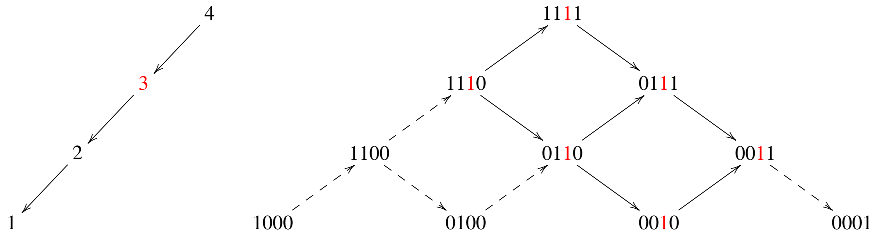


$\mathsf{P}_{D_5,4}$

A **reverse plane partition** is an order-reversing map $\rho : \mathsf{P} \to \mathbb{Z}_{\geqslant 0}$.
The objects of $\mathcal{C}_{Q,m}$ will be parameterized by **reverse plane partitions** defined on the minuscule poset associated with Q and m.

Lemma

Given a Dynkin quiver Q and a minuscule vertex m, the Hasse quiver of the minscule poset $P_{Q,m}$ is isomorphic to the full subquiver of $\Gamma(Q)$ on the representations supported at m.



Q $\quad$ $\Gamma(Q)$ - the Auslander–Reiten quiver of Q

Definition

A **minuscule vertex** m of a Dynkin quiver Q is one where every indecomposable representation X of Q has $\dim(X_m) \in \{0, 1\}$.

Theorem (Proctor '84)

For any minuscule poset P, the generating function for reverse plane partitions on P is

$$\sum_{\rho:P\to\mathbb{Z}_{\geqslant 0}\in RPP(P)} q^{|\rho|} = \prod_{x\in P} \frac{1}{1-q^{rk(x)}}$$

where $|\rho| := \sum_{x\in P}\rho(x)$ and $rk : P \to \mathbb{Z}_{\geqslant 1}$ is the rank function on P.

- Analogous identities for order filters of certain minuscule posets (Stanley '71, Hillman–Grassl '76, Gansner '81, Pak '01, Sulzgruber 2017)

Theorem (MacMahon '1916)

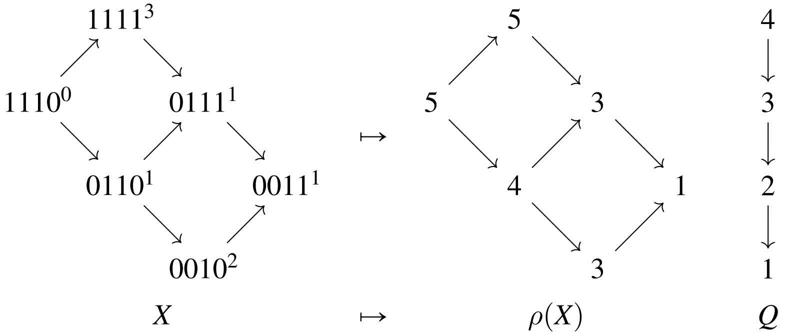
If P is a type A minuscule poset, the generating function for reverse plane partitions on P whose largest entry is at most t is

$$\sum q^{|\rho|} = \prod_{i=1}^{r}\prod_{j=1}^{s}\prod_{k=1}^{t} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}.$$

Theorem (G.–Patrias–Thomas ‘18)

The objects of $\mathcal{C}_{Q,m}$ are in bijection with $RPP(\mathsf{P}_{Q,m})$ via

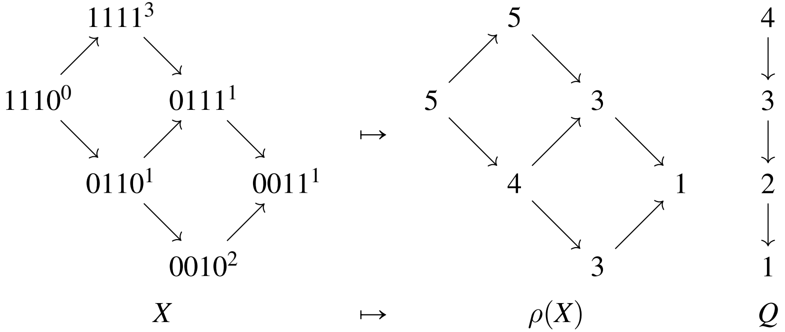
$$X \mapsto \rho \text{ – reverse plane partition from filling the } \tau\text{-orbits of } \mathsf{P}_{Q,m} \text{ with the Jordan block sizes in GenJF}(X)$$



$$\begin{aligned} \mathbf{dim}(X) &= 3585 \\ \text{GenJF}(X) &= ((3),(4,1),(5,3),(5)) \end{aligned}$$

Theorem (G.–Patrias–Thomas, ‘18)

The objects of $\mathcal{C}_{Q,m}$ are in bijection with $RPP(P_{Q,m})$.

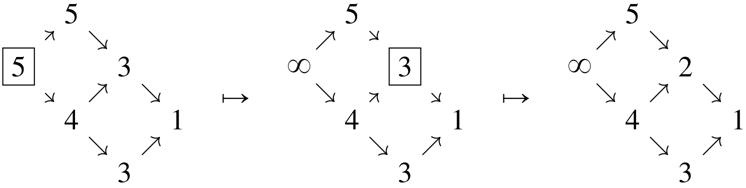


$$\mathrm{GenJF}(X) = ((3), (4,1), (5,3), (5))$$

Corollary

$$\sum_{\rho \in RPP(P)} q^{|\rho|} = \sum_{X \in \mathcal{C}_{Q,m}} q^{dim(X)} = \prod_{X^i \in ind(\mathcal{C}_{Q,m})} \frac{1}{1 - q^{dim(X^i)}} = \prod_{x \in P} \frac{1}{1 - q^{rk(x)}}$$

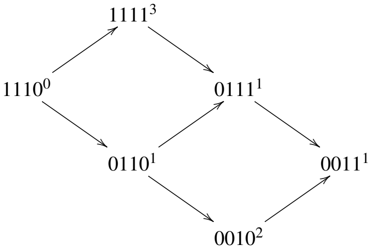
Idea of the proof



Given $\mathsf{x} \in \mathsf{P}$, one can **toggle** a reverse plane partition ρ as follows to produce a new reverse plane partition $t_{\mathsf{x}}\rho$. [Berenstein–Kirillov '95]

$$t_{\mathsf{x}}\rho(\mathsf{y}) = \begin{cases} \max\limits_{\mathsf{y} < \mathsf{y}_1} \rho(\mathsf{y}_1) + \min\limits_{\mathsf{y}_2 < \mathsf{y}} \rho(\mathsf{y}_2) - \rho(\mathsf{y}) & : \text{ if } \mathsf{y} = \mathsf{x} \\ \rho(\mathsf{y}) & : \text{ if } \mathsf{y} \neq \mathsf{x}. \end{cases}$$

Build up the representation and corresponding reverse plane partition inductively, using toggles.



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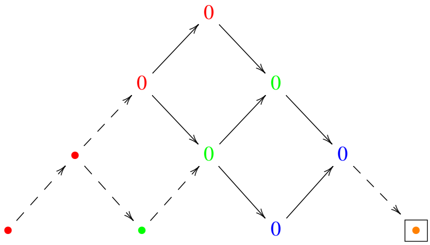
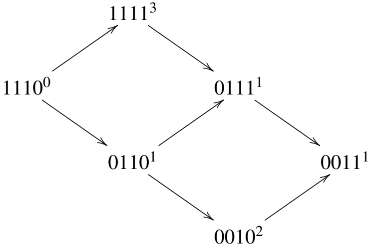


$\Gamma(Q)$

Idea of the proof



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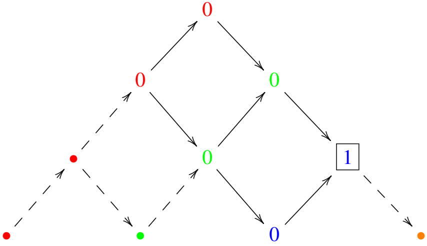

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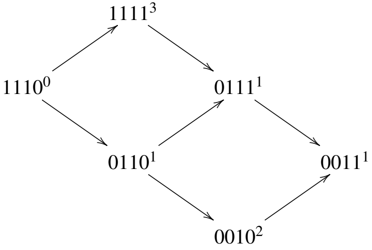

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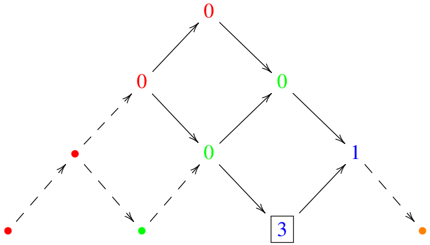

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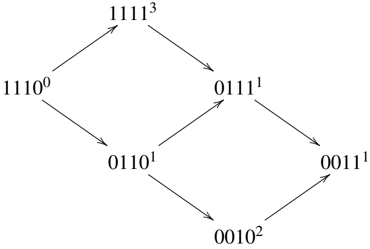

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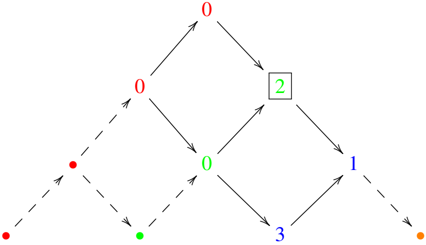

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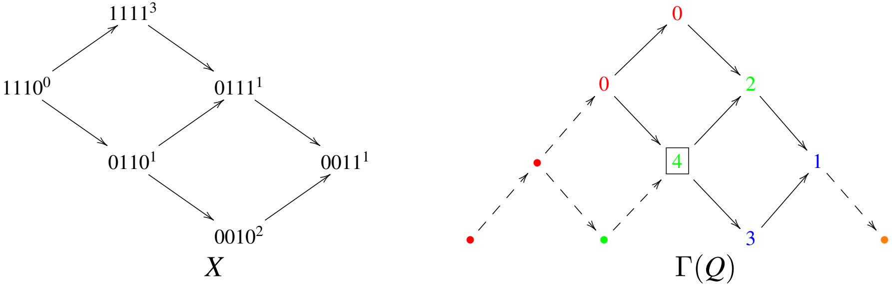

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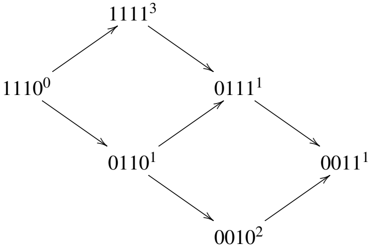

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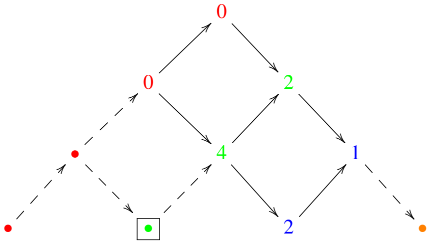

$\Gamma(Q)$

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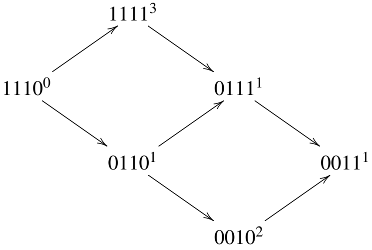

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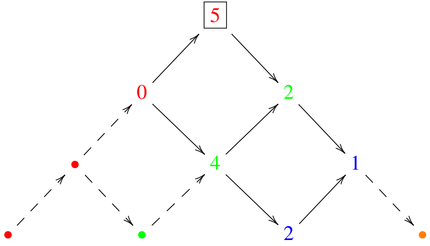
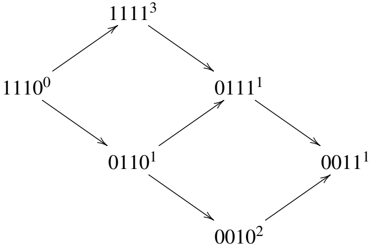


$\Gamma(Q)$

Idea of the proof



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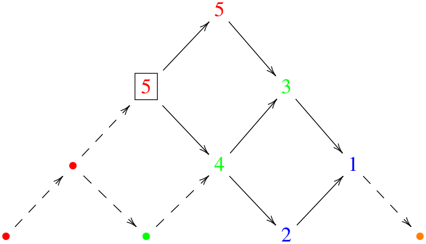

$\Gamma(Q)$

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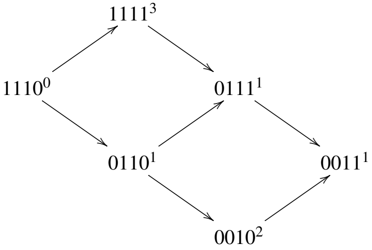

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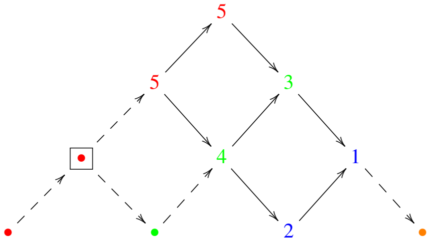

$\Gamma(Q)$

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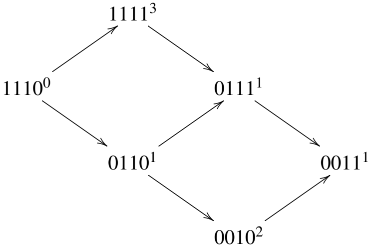

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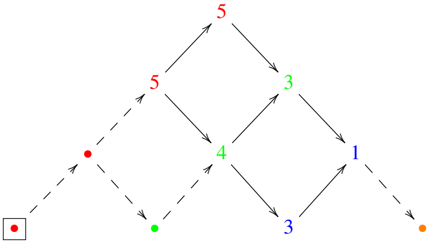
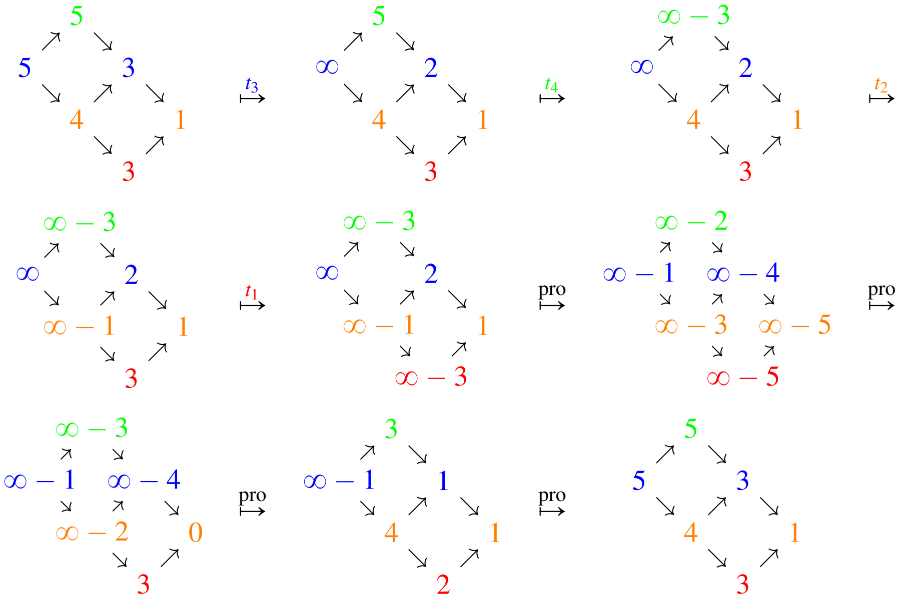


$\Gamma(Q)$

Periodicity for promotion pro $= t_1 t_2 t_4 t_3$



Let t_i be the operation of toggling every entry of $\rho \in RPP(\mathsf{P}_{Q,m})$ in τ-orbit i. Let pro $= t_n \cdots t_1$ where arrows of Q go from larger vertices point to smaller vertices.

Theorem (G.–Patrias–Thomas '18)

We have $\mathrm{pro}^h = \mathrm{id}$ *where* h *is the Coxeter number of the root system.*

- Brouwer and Schrijver '74 proved this in type A for rpps with $p(x) \in \{0, 1\}$.
- Fon-der-Flaass '92 calculated the orbit sizes in this same setting.
- Further properties and generalizations worked out by Panyushev '09, Armstrong–Stump–Thomas '11, Striker–Williams '11, Rush–Shi '12.
- Birational promotion (actually, birational "rowmotion") was introduced Einstein–Propp '13; our promotion is the "tropicalization".
- Our theorem can be deduced from work of Grinberg–Roby '15 and Musiker–Roby '18; ours is the first uniform proof.

Additional questions

- Understand GenJF(X) for other families of algebras
- Other periodicity (or integrability) results?
- What is the connection to Lie theory?
- What about MacMahon's theorem?

Theorem (MacMahon '1916)

If P is a type A minuscule poset, the generating function for reverse plane partitions on P whose largest entry is at most t is

$$\sum q^{|\rho|} = \prod_{i=1}^{r} \prod_{j=1}^{s} \prod_{k=1}^{t} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}.$$

Thanks!