Maximal Green Sequences and Type $\mathbb{A}$ Quivers

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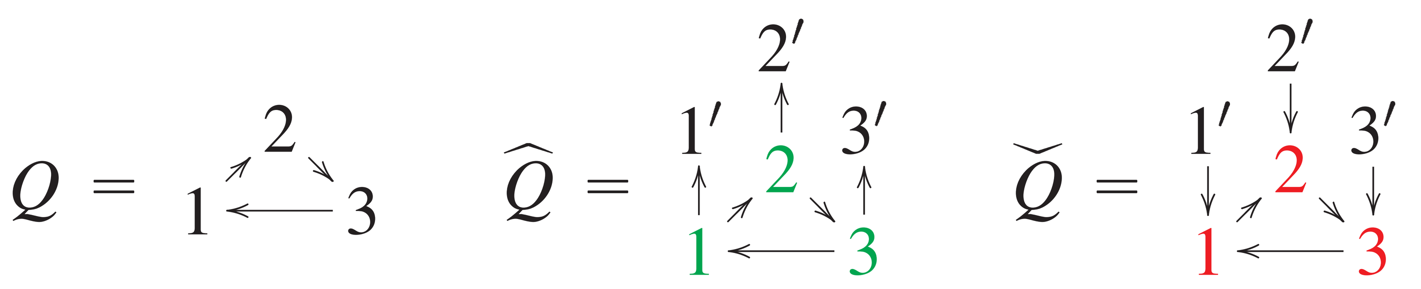
Introduction

Maximal green sequences appear in many areas including
- i) quantum dilogarithm identities in representation theory
- ii) Cambrian lattices in combinatorics
- iii) computations of BPS spectra in physics.

We construct a maximal green sequence for each quiver mutation equivalent to an oriented type $\mathbb{A}$ Dynkin diagram.

Notation

Given a quiver Q, the **framed quiver** (resp. **coframed quiver**) of Q, denoted $\widehat{Q}$ (resp. $\breve{Q}$), is formed by
(i) adding a **frozen vertex** i' for each vertex i in Q
(ii) adding an arrow $i \to i'$ (resp. $i \leftarrow i'$) for each vertex i in Q.
A vertex of i of $\overline{Q} \in \text{Mut}(\widehat{Q})$ is **green** (resp. **red**) if all arrows frozen vertices and i point away from (resp. toward) i.



Theorem (DWZ, "Sign Coherence of c-vectors")

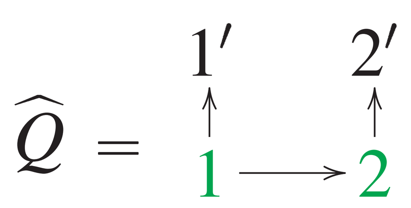
Each vertex i of $\overline{Q} \in \text{Mut}(\widehat{Q})$ is either **green** or **red**, but not both.

Definition

A mutation sequence $\underline{\mu} = \mu_{i_k} \circ \cdots \circ \mu_{i_1}$ is a **maximal green sequence** if
(i) for all $j \in [k]$ vertex i_j is **green** in $\mu_{i_{j-1}} \circ \cdots \circ \mu_{i_1}(\widehat{Q})$
(ii) all vertices in $\underline{\mu}(\widehat{Q})$ are **red**.

Example

The quiver $\widehat{Q}$ has maximal green sequences $\mu_2 \circ \mu_1$ and $\mu_2 \circ \mu_1 \circ \mu_2$.



Theorem (Brüstle-Dupont-Pérotin)

If Q is an *acyclic* orientation of a simply-laced Dynkin diagram or affine Dynkin diagram, then green(Q), the set of maximal green sequences of Q, is non-empty and finite.

Definition

Let Q_1, Q_2 be quivers. Let $\{a_1, \ldots, a_k\}$ be an k-multiset on $(Q_1)_0$ and $\{b_1, \ldots, b_k\}$ be a k-multiset on $(Q_2)_0$. Define the **direct sum** of Q_1 and Q_2, $Q_1 \oplus_{(a_1,\ldots,a_k)}^{(b_1,\ldots,b_k)} Q_2$, to be the quiver with vertices

$$\left(Q_1 \oplus_{(a_1,\ldots,a_k)}^{(b_1,\ldots,b_k)} Q_2\right)_0 := (Q_1)_0 \sqcup (Q_2)_0$$

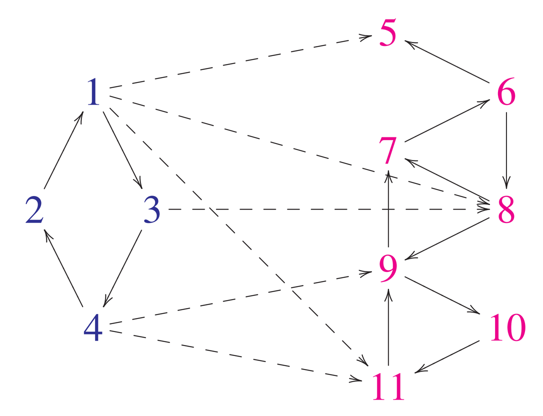
and arrows

$$\left(Q_1 \oplus_{(a_1,\ldots,a_k)}^{(b_1,\ldots,b_k)} Q_2\right)_1 := (Q_1)_1 \sqcup (Q_2)_1 \sqcup \left\{a_i \xrightarrow{\alpha_i} b_i : i \in [k]\right\}.$$

We say a quiver Q is **irreducible** if $Q = Q_1 \oplus_{(a_1,\ldots,a_k)}^{(b_1,\ldots,b_k)} Q_2$ implies that $Q_1 = \varnothing$ or $Q_2 = \varnothing$.

Example

$$Q_1 \oplus_{(1,1,1,3,4,4)}^{(5,8,11,8,9,11)} Q_2 =$$



Proposition (G. - Musiker)

Let $Q = Q_1 \oplus_{(a_1,\ldots,a_k)}^{(b_1,\ldots,b_k)} Q_2$ where $\#\{a_i \xrightarrow{\alpha} b_j\} \leq 1$ for all $i, j \in [k]$. If $\underline{\mu}_1 \in \text{green}(Q_1)$ and $\underline{\mu}_2 \in \text{green}(Q_2)$, then $\underline{\mu}_2 \circ \underline{\mu}_1 \in \text{green}(Q)$.

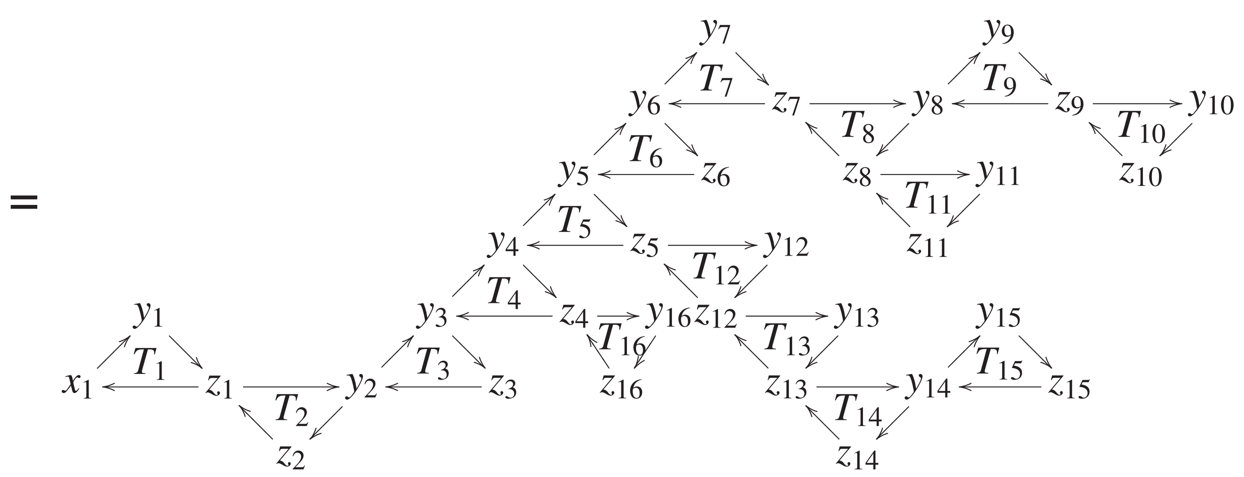
Lemma

Let $N \geq 3$. The irreducible type $\mathbb{A}_N$ quivers are exactly those quivers Q that can be obtained from a collection of 3-cycles $\{T_i\}_{i \in [k]}$ by identifying their vertices in such a way that the following are satisfied.
(i) any $i, j \in (Q)_0$ satisfy $\|\#\{i \xrightarrow{\alpha} j\}\| \leq 1$
(ii) each $i \in (Q)_0$ is identified with at most one other vertex in $(Q)_0$
(iii) each cycle in the underlying graph of Q is induced by a T_i.

Example

We can label any irreducible type $\mathbb{A}$ quiver Q as follows.

$Q =$



Definition

Let $\mathcal{Q}$ be an embedded, irreducible type $\mathbb{A}$ quiver. Define the **associated mutation sequence** of $\mathcal{Q}$ by $\underline{\mu} = \underline{\mu}_n \circ \cdots \underline{\mu}_1 \circ \underline{\mu}_0$ ($n := \#(3\text{-cycles in } \mathcal{Q})$) where $\underline{\mu}_0 := \mu_{x_1}$ and

$$\underline{\mu}_k := \underline{\mu}_{A(k)} \circ \underline{\mu}_{B(k)} \circ \underline{\mu}_{C(k)} \circ \underline{\mu}_{D(k)}$$

where

$$\underline{\mu}_{D(k)} := \mu_{z_k} \circ \mu_{y_k}$$
$$\underline{\mu}_{C(k)} := \begin{cases} \mu_{x_{i(k)_{d(k)}}} \circ \cdots \circ \mu_{x_{i(k)_1}} & : T_k \text{ is downward-pointing} \\ \varnothing & : T_k \text{ is upward-pointing} \end{cases}$$
$$\underline{\mu}_{B(k)} := \begin{cases} \mu_{v(r(k)-1)} & : \text{if } r(k) \neq 1 \\ \varnothing & : \text{if } r(k) = 1 \end{cases}$$
$$\underline{\mu}_{A(k)} := \mu_{v(k)}.$$

Theorem (G. - Musiker)

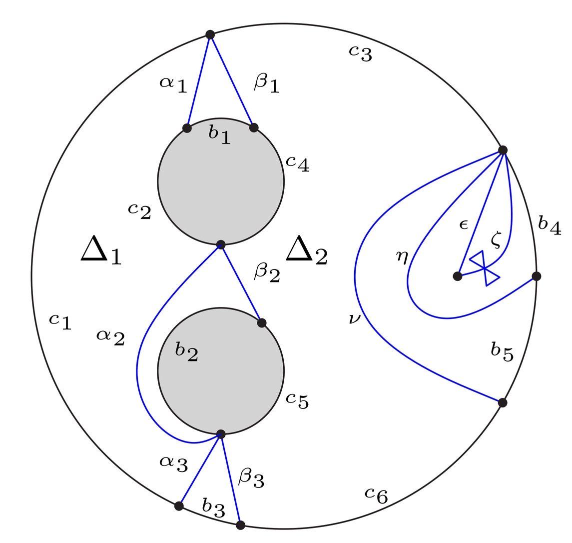
Let $\mathcal{Q}$ be an embedded, irreducible type $\mathbb{A}$ quiver. Then $\underline{\mu} \in \text{green}(\mathcal{Q})$.

Quivers Arising from Triangulated Surfaces

The triangulation $\Delta_1 \sqcup \Delta_2 \sqcup \{\eta, \epsilon, \zeta\}$ of the following surface S determines the quiver

$$Q = Q_1 \oplus_{(v_{\alpha_1}, v_{\alpha_2}, v_{\alpha_3})}^{(v_{\beta_1}, v_{\beta_2}, v_{\beta_3})} Q_2 \oplus_{(v_\nu)}^{(v_\eta)} R$$

where $(R)_0 = \{v_\eta, v_\zeta, v_\epsilon\}$. Since Q_1 and Q_2 are type $\mathbb{A}$ quivers and R is acyclic, our theorem gives a maximal green sequence of Q.



Combinatorial DT-Invariants

Let Q be an irreducible type $\mathbb{A}_N$ quiver, let

$$\widehat{\mathbb{A}}_Q := \mathbb{Q}(q^{1/2})\langle\langle y^\alpha : \alpha \in \mathbb{N}^N, y^\alpha y^\beta = q^{\lambda(\alpha,\beta)/2} y^{\alpha+\beta}\rangle\rangle$$

denote the **formal quantum affine space** with $y_i := y^{e_i}$ and

$$\lambda(e_i, e_j) := \#(\text{arrows } i \to j \text{ in Q}) - \#(\text{arrows } j \to i \text{ in Q}).$$

Denote the **quantum dilogarithm series** by

$$\mathbb{E}(y_i) := 1 + \frac{q^{1/2}}{q-1} y_i + \cdots + \frac{q^{n^2/2}}{(q^n - 1)(q^n - q)\cdots(q^n - q^{n-1})} y_i^n + \cdots \in \widehat{\mathbb{A}}_Q.$$

Using Keller's construction we can compute $\mathbb{E}_Q$, the **combinatorial DT-invariant** of $\mathcal{Q}$. Since $\underline{\mu} \in \text{green}(Q)$, we have

$$\mathbb{E}_Q = \mathbb{E}(\underline{\mu}) := \mathbb{E}(\underline{\mu}_0)\mathbb{E}(\underline{\mu}_1)\cdots\mathbb{E}(\underline{\mu}_n)$$

where each $\mathbb{E}(\underline{\mu}_i)$ is a product of quantum dilogarithm series.

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