Maximal Green Sequences for Type $\mathbb{A}$ Quivers (w/ Gregg Musiker)

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Minnesota Combinatorics Seminar

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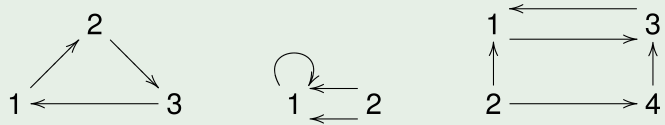
Outline

- Quiver Mutation
- Maximal Green Sequences
- Type $\mathbb{A}$ Quivers
- Direct Sums of Quivers
- Maximal Green Sequences for Type $\mathbb{A}$ Quivers
- Idea of the Proof

- Let $Q = (\underbrace{(Q)_0}_{vertices}, \underbrace{(Q)_1}_{arrows}, s, t)$ be a finite, connected quiver.
- $s\left(i \stackrel{\alpha}{\to} j\right) = i$ and $t\left(i \stackrel{\alpha}{\to} j\right) = j$.

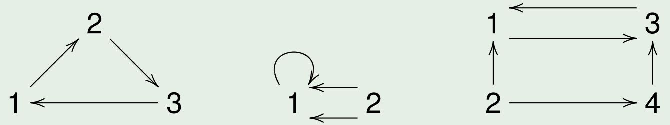
Example



Quiver Mutation

- Let Q be a finite, connected quiver.
- We assume Q has no loops and no 2-cycles.

Example



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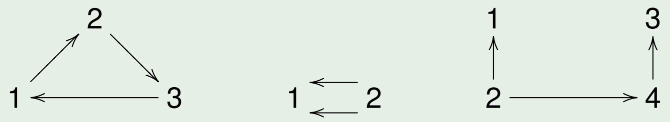

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Quiver Mutation

Definition

Given a quiver Q, we can **mutate** Q at vertex k to obtain a quiver $\mu_k(Q)$. The quiver $\mu_k(Q)$ is obtained from Q by

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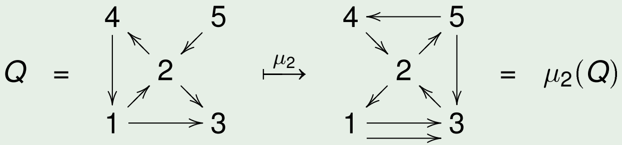


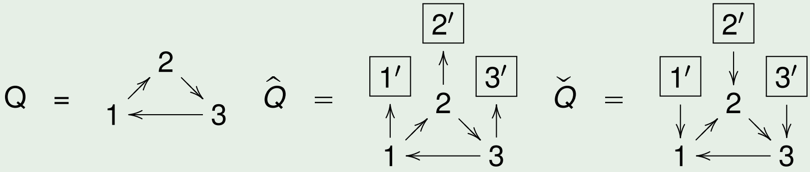
Definition

Given a quiver Q, the **framed quiver** (resp. **coframed quiver**) of Q, denoted $\widehat{Q}$ (resp. $\check{Q}$), is formed by

(i) adding a **frozen vertex** i' for each vertex i in Q

(ii) adding an arrow $i \to i'$ (resp. $i \leftarrow i'$) for each vertex i in Q.

Example



The quiver $\widehat{Q}$ will have vertices $[N] \sqcup [N']$.

Maximal Green Sequences

- Write down the **exchange graph** of $\widehat{Q}$, denoted $EG(\widehat{Q})$
- Write $\mathrm{Mut}(\widehat{Q})$ for the vertices of $EG(\widehat{Q})$.

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Theorem (DWZ, "Sign Coherence of c-vectors")

*Each vertex i of $\overline{Q} \in \text{Mut}(\widehat{Q})$ is either **green** or **red**, but not both.*

Maximal Green Sequences

Definition

A mutation sequence $\underline{\mu} = \mu_{i_k} \circ \cdots \circ \mu_{i_1}$ is a **maximal green sequence** if

(i) for all $j \in [k]$ vertex i_j is **green** in $\mu_{i_{j-1}} \circ \cdots \circ \mu_{i_1}(\widehat{Q})$

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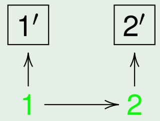
Example

The quiver $\widehat{Q}$ has maximal green sequences $\mu_2 \circ \mu_1$ and $\mu_2 \circ \mu_1 \circ \mu_2$.

$\widehat{Q} =$



Motivation: Oriented Exchange Graphs

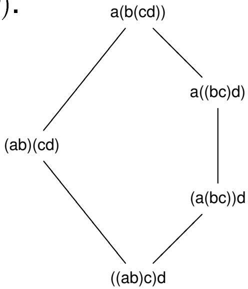
- Form the **oriented exchange graph** $\overrightarrow{EG}(\widehat{Q})$: $\overline{Q}_1 \to \overline{Q}_2$ if $\mu_i \overline{Q}_1 = \overline{Q}_2$ such that i is green in $\overline{Q}_1$.

Theorem (Nagao, 2010)

$\overrightarrow{EG}(\widehat{Q})$ is a poset.

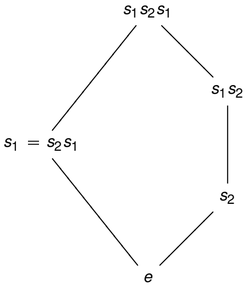
- Maximal green sequences can be thought of as maximal chains in $\overrightarrow{EG}(\widehat{Q})$.

$\overrightarrow{EG}(1 \to 2) \overset{[T.]}{\cong}$



Tamari Lattice

$\overset{[K.]}{\cong}$



Cambrian Lattices

Theorem (Brüstle-Dupont-Pérotin)

Let Q be a quiver. If all of the vertices of $\overline{Q} \in Mut\left(\widehat{Q}\right)$ are red, then $\overline{Q} \cong \check{Q}$.

Known Results

Theorem (Brüstle-Dupont-Pérotin)

Let Q be a quiver. If all of the vertices of $\overline{Q} \in Mut\left(\hat{Q}\right)$ are red, then $\overline{Q} \cong \check{Q}$. Thus there exists $\sigma \in \mathfrak{S}_{(Q)_0} \cong \mathfrak{S}_N$ such that $\overline{Q}\sigma = \check{Q}$. In particular, if $\check{Q}$ is a vertex of $\overrightarrow{EG}(\hat{Q})$, it is the unique sink of $\overrightarrow{EG}(\hat{Q})$.

Known Results

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Let Q be a quiver. If all of the vertices of $\overline{Q} \in Mut\left(\hat{Q}\right)$ *are red, then* $\overline{Q} \cong \check{Q}$*. Thus there exists* $\sigma \in \mathfrak{S}_{(Q)_0} \cong \mathfrak{S}_N$ *such that* $\overline{Q}\sigma = \check{Q}$*. In particular, if* $\check{Q}$ *is a vertex of* $\overrightarrow{EG}(\hat{Q})$*, it is the unique sink of* $\overrightarrow{EG}(\hat{Q})$*.*

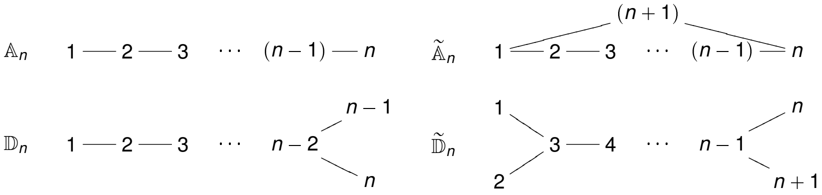
The known proofs of this use algebraic techniques, but the statements are combinatorial.

Problem

Find a combinatorial proof.

Theorem (Brüstle-Dupont-Pérotin)

If Q is an acyclic orientation of a simply-laced Dynkin diagram or affine Dynkin diagram, then green(Q) is non-empty and finite.



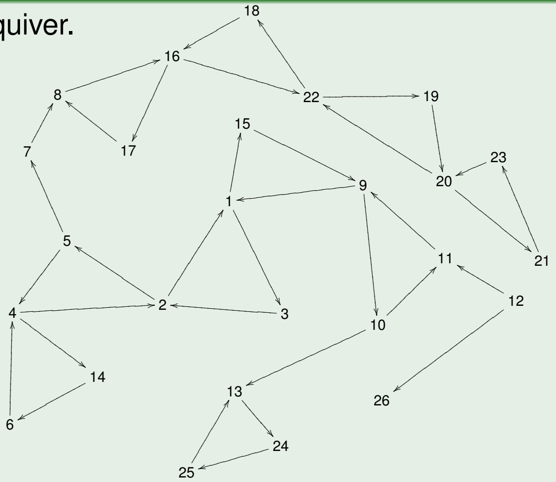
$\mathbb{E}_6$ $\mathbb{E}_7$ $\mathbb{E}_8$ $\tilde{\mathbb{E}}_6$ $\tilde{\mathbb{E}}_7$ $\tilde{\mathbb{E}}_8$

What about maximal green sequences for non-acyclic quivers?

- We will call a quiver Q a **type $\mathbb{A}$ quiver** if it is mutation equivalent to an orientation of a type $\mathbb{A}$ Dynkin diagram.

Example

A type $\mathbb{A}$ quiver.



Type $\mathbb{A}$ quivers

Lemma (Buan-Vatne)

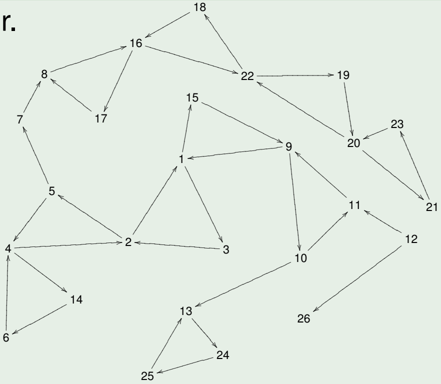
A quiver Q is of type $\mathbb{A}$ if and only if Q satisfies the following:

- i) *All non-trivial cycles in the underlying graph of Q are oriented and of length 3.*
- ii) *Any vertex has at most four neighbors.*
- iii) *If a vertex has four neighbors, then two of its adjacent arrows belong to one 3-cycle, and the other two belong to another 3-cycle.*
- iv) *If a vertex has exactly three neighbors, then two of its adjacent arrows belong to a 3-cycle, and the third arrow does not belong to any 3-cycle.*

- A type $\mathbb{A}$ quiver Q is obtained by gluing 3-cycles  and $\mathbb{A}_2$ quivers $1 \longrightarrow 2$ together so that no additional cycles are created, and each vertex of Q is part of at most 1 attachment.

Example

A type $\mathbb{A}$ quiver.



Direct Sums of Quivers

Let Q_1, Q_2 be quivers.

Definition

Let $\{a_1, \ldots, a_k\}$ be an k-multiset on $(Q_1)_0$ and $\{b_1, \ldots, b_k\}$ be a k-multiset on $(Q_2)_0$. Define the **direct sum** of Q_1 and Q_2, $Q_1 \oplus_{(a_1,\ldots,a_k)}^{(b_1,\ldots,b_k)} Q_2$, to be the quiver with vertices

$$\left(Q_1 \oplus_{(a_1,\ldots,a_k)}^{(b_1,\ldots,b_k)} Q_2\right)_0 := (Q_1)_0 \sqcup (Q_2)_0$$

and arrows

$$\left(Q_1 \oplus_{(a_1,\ldots,a_k)}^{(b_1,\ldots,b_k)} Q_2\right)_1 := (Q_1)_1 \sqcup (Q_2)_1 \sqcup \left\{a_i \xrightarrow{\alpha_i} b_i : i \in [k]\right\}.$$

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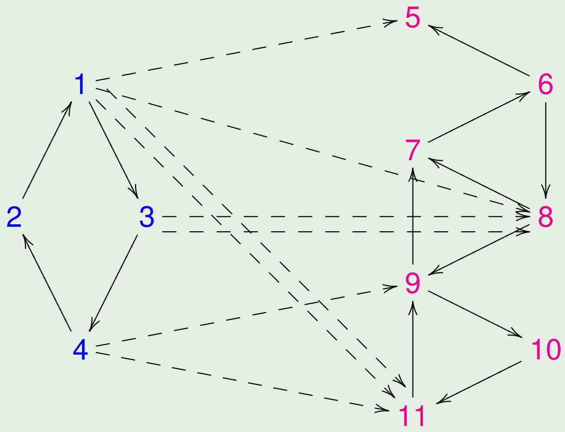
Definition

We say a quiver Q is **irreducible** if $Q = Q_1 \oplus_{(a_1,\ldots,a_k)}^{(b_1,\ldots,b_k)} Q_2$ implies that $Q_1 = \varnothing$ or $Q_2 = \varnothing$.

Direct Sums of Quivers

Example



$$Q_1 \oplus_{(1,1,1,1,3,3,4,4)}^{(5,8,11,11,8,8,9,11)} Q_2$$

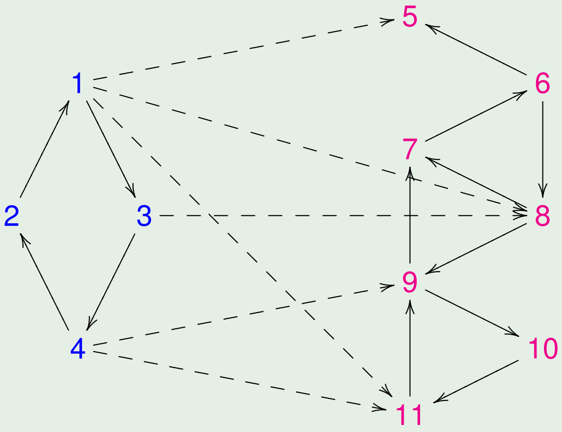
Direct Sums of Quivers

Definition

We say that a direct sum of quivers is ***t*-colored** if $t = \#\{\text{distinct elements of } \{a_1, \ldots, a_k\}\}$ and there does not exist i and j such that $\left|\#\{a_i \xrightarrow{\alpha} b_j\}\right| \geqslant 2$.

Example

A 3-colored direct sum.



$$Q_1 \oplus^{(5,8,11,8,9,11)}_{(1,1,1,3,4,4)} Q_2$$

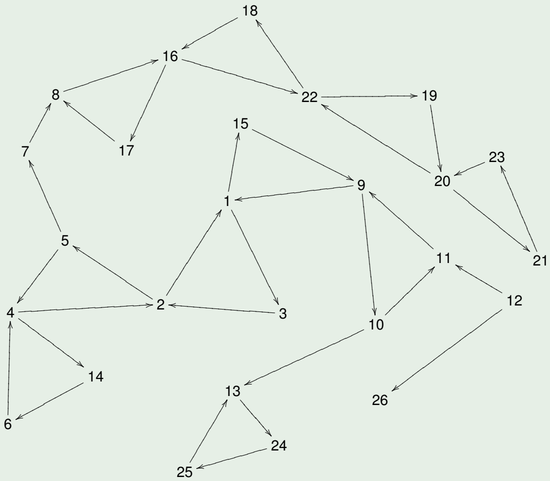
Direct Sums of Quivers

Proposition (G.-Musiker)

Let $Q = Q_1 \oplus_{(a_1,\ldots,a_k)}^{(b_1,\ldots,b_k)} Q_2$ *be a* t*-colored direct sum of quivers. If* $\underline{\mu}_1 \in green(Q_1)$ *and* $\underline{\mu}_2 \in green(Q_2)$, *then* $\underline{\mu}_2 \circ \underline{\mu}_1 \in green(Q)$.

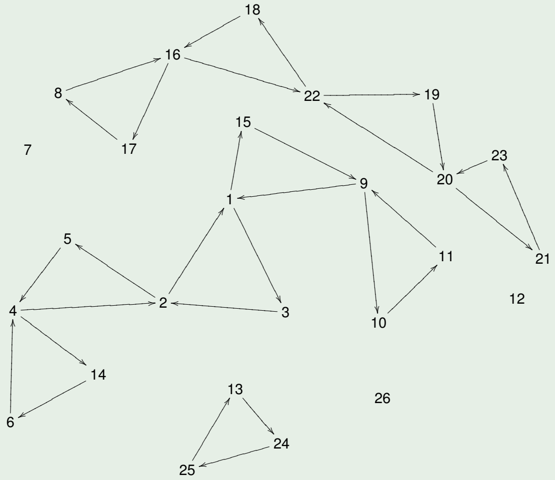
Example

A type $\mathbb{A}$ quiver.



Example

A type $\mathbb{A}$ quiver split into its irreducible components.



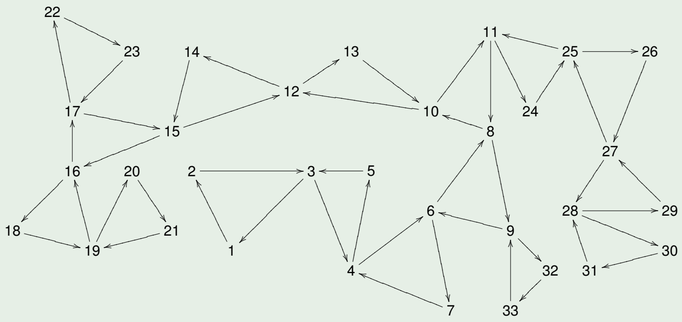
Lemma

The irreducible type $\mathbb{A}$ quivers are "collections of 3-cycles glued together" and the quiver of type $\mathbb{A}_1$.

- The quiver of type $\mathbb{A}_1$ has exactly 1 maximal green sequence.
- We show how to write down a maximal green sequence for any "collection of 3-cycles glued together", which we will call **irreducible type $\mathbb{A}$ quivers**

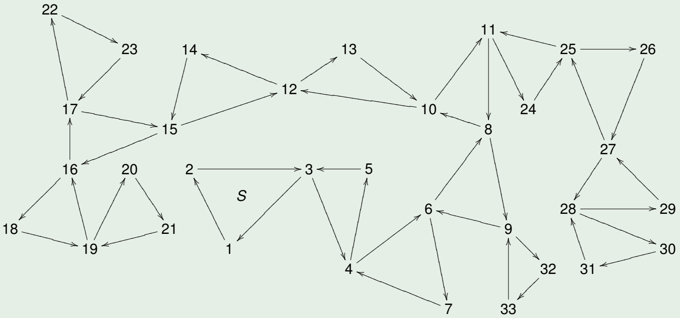
- Let Q an irreducible type $\mathbb{A}$ quiver.

Example

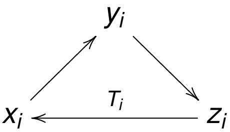


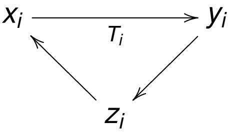
- Choose a **leaf** 3-cycle S.

Example



- Form the **embedded**, irreducible type $\mathbb{A}$ quiver, $\mathcal{Q} = (Q, S, \rho)$, w.r.t S.
- In $\mathcal{Q}$, each 3-cycle is **upward-pointing** or **downward-pointing**



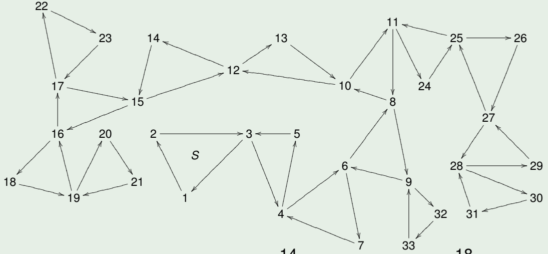


- $\rho(S) := T_1$ must be upward-pointing
- T_2, if it exists, must be downward-pointing

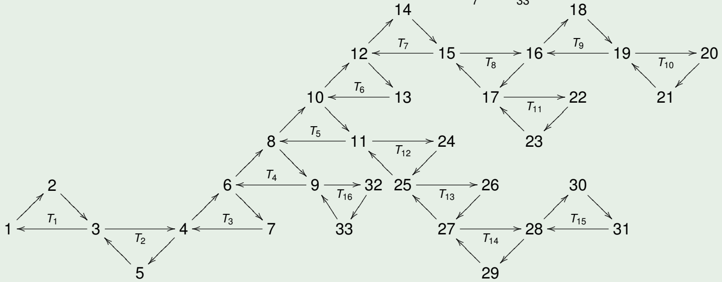
- Form the **embedded**, irreducible type $\mathbb{A}$ quiver, $\mathcal{Q} = (Q, S, \rho)$, w.r.t S.

Example

(Q, S) =



$\mathcal{Q}$ =



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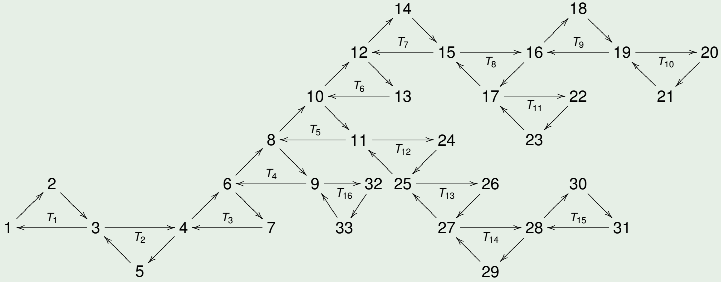

Maximal Green Sequences for Type $\mathbb{A}$ Quivers

Example

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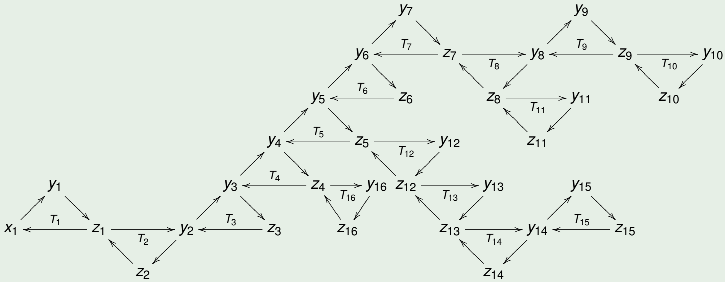
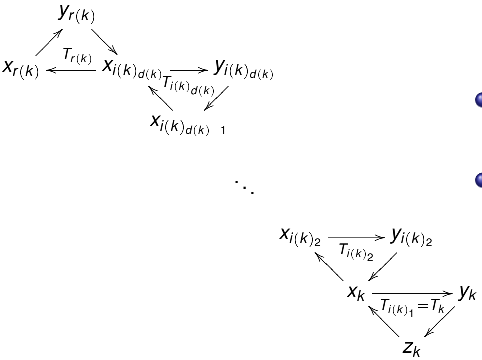


We will define a mutation sequence $\underline{\mu} = \underline{\mu}_n \circ \underline{\mu}_{n-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0$ of $\mathcal{Q}$ where $n = \#(\text{3-cycles in } \mathcal{Q})$.

Suppose T_k is a 3-cycle of $\mathcal{Q}$. We define the **special** upward-pointing 3-cycle before T_k as follows.



- If T_k is upward-pointing, $T_{r(k)} := T_k$.
- If T_k is downward-pointing, $T_{r(k)}$ is defined by the picture on the left.

- We call $(T_{i(k)_1}, T_{i(k)_2}, \ldots, T_{i(k)_{d(k)}})$ the **path** between T_k and $T_{r(k)}$.

Suppose T_k is a 3-cycle of $\mathcal{Q}$. If T_k is downward-pointing, let $(T_{i_1}, \ldots, T_{i_d})$ denote its path where $T_k = T_{i_1}$.

$$v(k) := \begin{cases} x_k & : \quad T_k \text{ is upward-pointing} \\ x_{r(k)} & : \quad T_k \text{ is downward-pointing and} \\ & \quad\ \deg(y_{r(k)}) = \deg(y_{i_j}) = 2\ \forall j \in [2, d] \\ z_{s_t} & : \quad T_k \text{ is downward-pointing and} \\ & \quad\ \deg(y_{r(k)}) = 4 \text{ or } \deg(y_{i_s}) = 4 \\ & \quad\ \text{for some } s \in [2, d]. \end{cases}$$

What is z_{s_t}? Let $s := \max\{j \in \{r(k), i_2, \ldots, i_d\} : \deg(y_j) = 4\}$.

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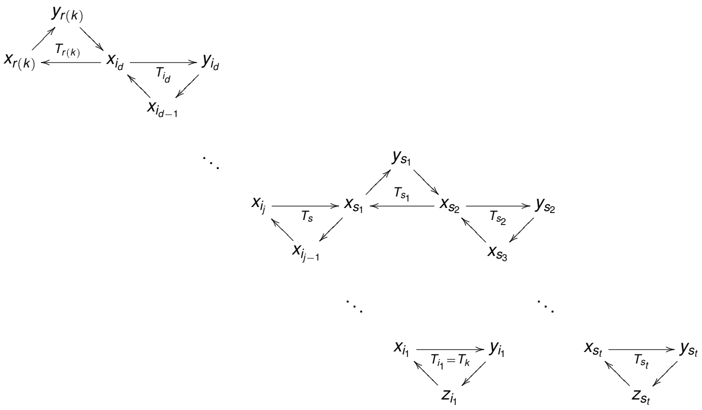
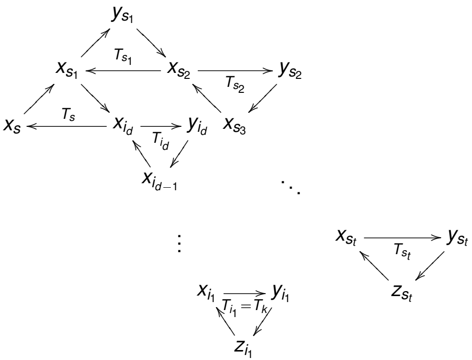
Then $\mathcal{Q}$ locally looks like



or

$\mathcal{Q}$ locally looks like



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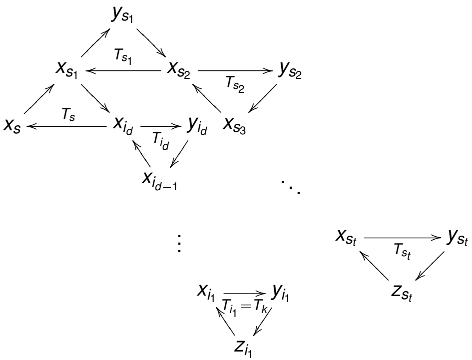

We call $(T_{s_1}, T_{s_2}, \ldots, T_{s_t})$ the **twig** above T_k.

Definition

Let $\mathcal{Q}$ be an embedded, irreducible type $\mathbb{A}$ quiver. Define the **associated mutation sequence** of $\mathcal{Q}$ by $\underline{\mu} = \underline{\mu}_n \circ \cdots \underline{\mu}_1 \circ \underline{\mu}_0$ ($n := \#$(3-cycles in $\mathcal{Q}$)) where $\underline{\mu}_0 := \mu_{x_1}$ and

$$\underline{\mu}_k := \underline{\mu}_{A(k)} \circ \underline{\mu}_{B(k)} \circ \underline{\mu}_{C(k)} \circ \underline{\mu}_{D(k)}$$

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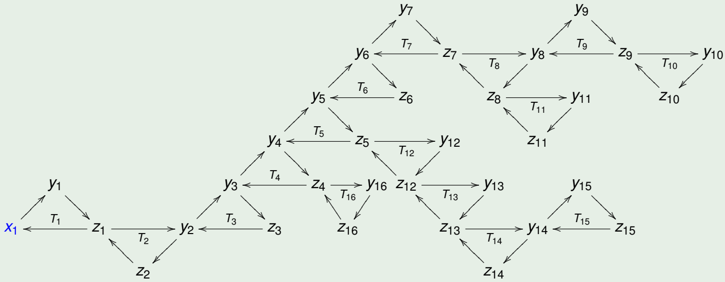
$$\underline{\mu}_k := \underline{\mu}_{A(k)} \circ \underline{\mu}_{B(k)} \circ \underline{\mu}_{C(k)} \circ \underline{\mu}_{D(k)}$$

where

$$\begin{aligned}
\underline{\mu}_{D(k)} &:= \mu_{z_k} \circ \mu_{y_k} \\
\underline{\mu}_{C(k)} &:= \begin{cases} \mu_{x_{i(k)_{d(k)}}} \circ \cdots \circ \mu_{x_{i(k)_1}} & : \ T_k \text{ is downward-pointing} \\ \varnothing & : \ T_k \text{ is upward-pointing} \end{cases} \\
\underline{\mu}_{B(k)} &:= \begin{cases} \mu_{v(r(k)-1)} & : \ \text{if } r(k) \neq 1 \\ \varnothing & : \ \text{if } r(k) = 1 \end{cases} \\
\underline{\mu}_{A(k)} &:= \mu_{v(k)}.
\end{aligned}$$

Example

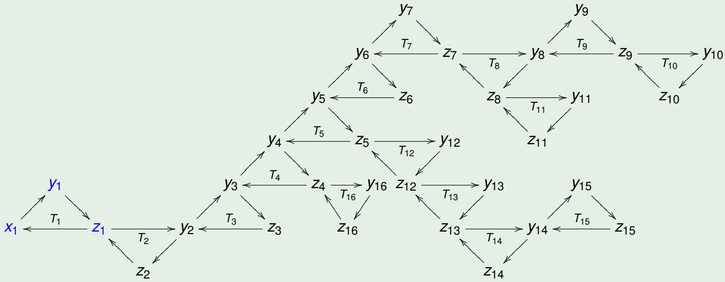
$\mathcal{Q} \quad =$



- $\underline{\mu}_0 = \mu_{x_1}$
- $\underline{\mu}_1 = \underbrace{\mu_{x_1}}_{\underline{\mu}_{A(1)}} \circ \underbrace{\mu_{z_1} \circ \mu_{y_1}}_{\underline{\mu}_{D(1)}} \quad (v(1) = x_1)$
- $\underline{\mu}_2 = \underbrace{\mu_{x_1}}_{\underline{\mu}_{A(2)}} \circ \underbrace{\mu_{x_2}}_{\underline{\mu}_{C(2)}} \circ \underbrace{\mu_{z_2} \circ \mu_{y_2}}_{\underline{\mu}_{D(2)}} \quad (v(2) = x_{r(2)} = x_1)$

Example

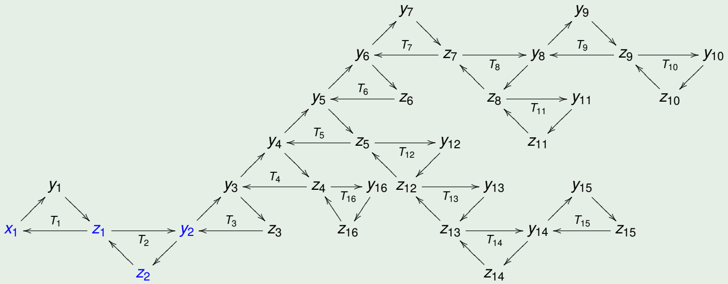
$\mathcal{Q}$ =



- $\underline{\mu}_0 = \mu_{x_1}$
- $\underline{\mu}_1 = \underbrace{\mu_{x_1}}_{\underline{\mu}_{A(1)}} \circ \underbrace{\mu_{z_1} \circ \mu_{y_1}}_{\underline{\mu}_{D(1)}} \quad (v(1) = x_1)$
- $\underline{\mu}_2 = \underbrace{\mu_{x_1}}_{\underline{\mu}_{A(2)}} \circ \underbrace{\mu_{x_2}}_{\underline{\mu}_{C(2)}} \circ \underbrace{\mu_{z_2} \circ \mu_{y_2}}_{\underline{\mu}_{D(2)}} \quad (v(2) = x_{r(2)} = x_1)$

Example

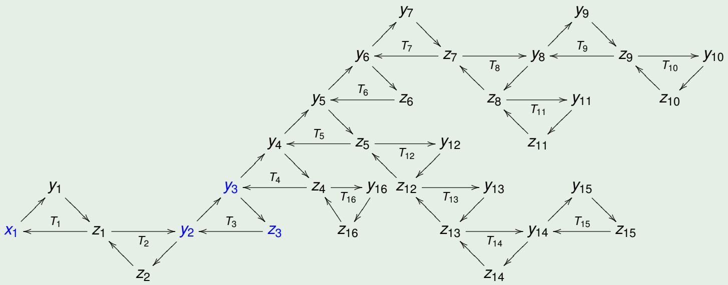
$\mathcal{Q}$ =



- $\underline{\mu}_0 = \mu_{x_1}$
- $\underline{\mu}_1 = \underbrace{\mu_{x_1}}_{\underline{\mu}_{A(1)}} \circ \underbrace{\mu_{z_1} \circ \mu_{y_1}}_{\underline{\mu}_{D(1)}} \quad (v(1) = x_1)$
- $\underline{\mu}_2 = \underbrace{\mu_{x_1}}_{\underline{\mu}_{A(2)}} \circ \underbrace{\mu_{x_2}}_{\underline{\mu}_{C(2)}} \circ \underbrace{\mu_{z_2} \circ \mu_{y_2}}_{\underline{\mu}_{D(2)}} \quad (v(2) = x_{r(2)} = x_1)$

Example

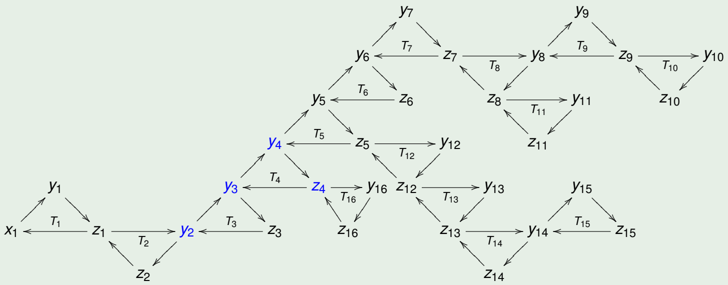
$\mathcal{Q}$ =



- $\underline{\mu}_3 = \underbrace{\mu_{x_3}}_{\underline{\mu}_{A(3)}} \circ \underbrace{\mu_{x_1}}_{\underline{\mu}_{B(3)}} \circ \underbrace{\mu_{z_3} \circ \mu_{y_3}}_{\underline{\mu}_{D(3)}}$
- $\underline{\mu}_4 = \underbrace{\mu_{x_4}}_{\underline{\mu}_{A(4)}} \circ \underbrace{\mu_{x_3}}_{\underline{\mu}_{B(4)}} \circ \underbrace{\mu_{z_4} \circ \mu_{y_4}}_{\underline{\mu}_{D(4)}}$

Example

$\mathcal{Q} =$



- $\underline{\mu}_3 = \underbrace{\mu_{x_3}}_{\underline{\mu}_{A(3)}} \circ \underbrace{\mu_{x_1}}_{\underline{\mu}_{B(3)}} \circ \underbrace{\mu_{z_3} \circ \mu_{y_3}}_{\underline{\mu}_{D(3)}}$
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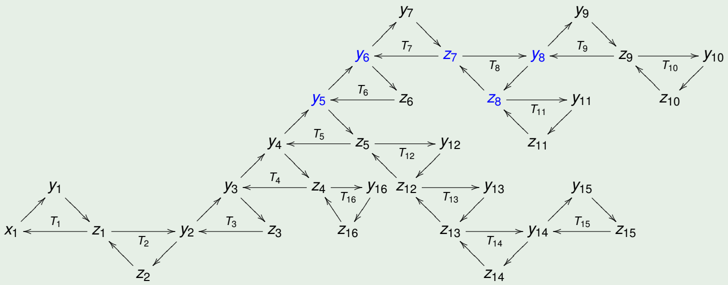
Maximal Green Sequences for Type $\mathbb{A}$ Quivers

Example

$\mathcal{Q}$ =



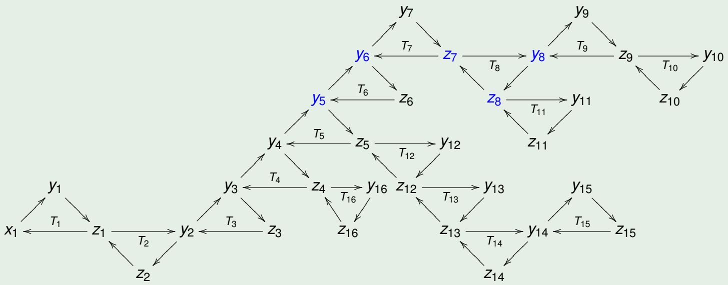
- $\underline{\mu}_8 = \underbrace{\mu_{x_7}}_{\underline{\mu}_{A(8)}} \circ \underbrace{\mu_{x_6}}_{\underline{\mu}_{B(8)}} \circ \underbrace{\mu_{x_8}}_{\underline{\mu}_{C(8)}} \circ \underbrace{\mu_{z_8} \circ \mu_{y_8}}_{\underline{\mu}_{D(8)}}$

$v(r(8)-1) = v(7-1) = v(6) = x_6$

$v(8) = x_{r(8)} = x_7$

Example

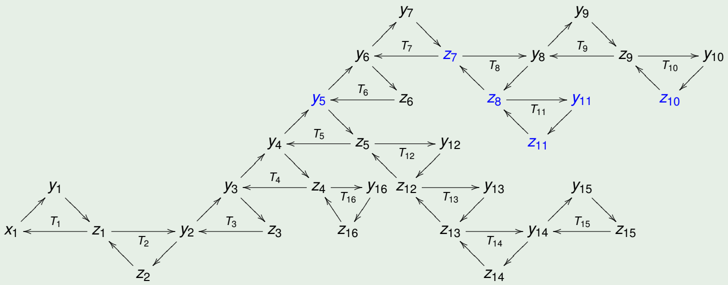
$\mathcal{Q}$ =



- $\underline{\mu}_8 = \underbrace{\mu_{x_7}}_{\underline{\mu}_{A(8)}} \circ \underbrace{\mu_{x_6}}_{\underline{\mu}_{B(8)}} \circ \underbrace{\mu_{x_8}}_{\underline{\mu}_{C(8)}} \circ \underbrace{\mu_{z_8} \circ \mu_{y_8}}_{\underline{\mu}_{D(8)}}$
- $\underline{\mu}_{11} = \underbrace{\mu_{z_{10}}}_{\underline{\mu}_{A(11)}} \circ \underbrace{\mu_{x_6}}_{\underline{\mu}_{B(11)}} \circ \underbrace{\mu_{x_8} \circ \mu_{x_{11}}}_{\underline{\mu}_{C(11)}} \circ \underbrace{\mu_{z_{11}} \circ \mu_{y_{11}}}_{\underline{\mu}_{D(11)}}$

Example

$\mathcal{Q}$ =



- $\underline{\mu}_8 = \underbrace{\mu_{x_7}}_{\underline{\mu}_{A(8)}} \circ \underbrace{\mu_{x_6}}_{\underline{\mu}_{B(8)}} \circ \underbrace{\mu_{x_8}}_{\underline{\mu}_{C(8)}} \circ \underbrace{\mu_{z_8} \circ \mu_{y_8}}_{\underline{\mu}_{D(8)}}$
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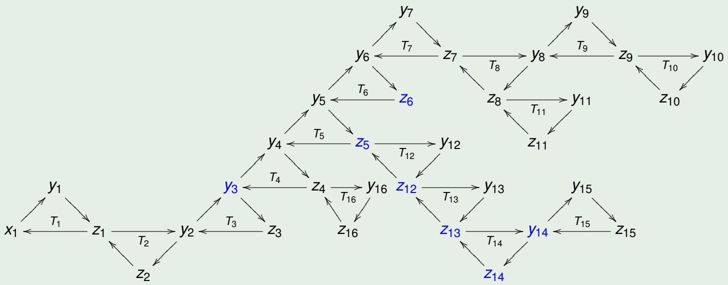
Example

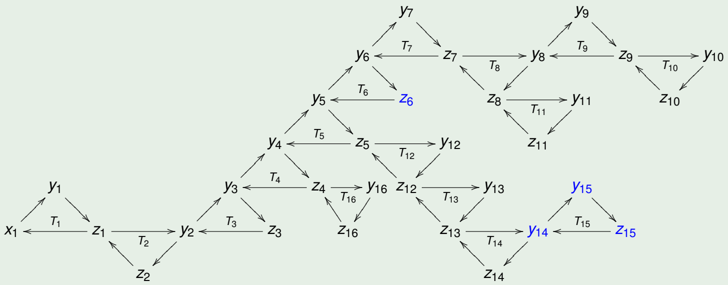
$\mathcal{Q}$ =



- $\underline{\mu}_{14} = \underbrace{\mu_{z_6}}_{\underline{\mu}_{A(14)}} \circ \underbrace{\mu_{x_4}}_{\underline{\mu}_{B(14)}} \circ \underbrace{\mu_{x_{12}} \circ \mu_{x_{13}} \circ \mu_{x_{14}}}_{\underline{\mu}_{C(14)}} \circ \underbrace{\mu_{z_{14}} \circ \mu_{y_{14}}}_{\underline{\mu}_{D(14)}}$
- $\underline{\mu}_{15} = \underbrace{\mu_{x_{15}}}_{\underline{\mu}_{A(15)}} \circ \underbrace{\mu_{z_6}}_{\underline{\mu}_{B(15)}} \circ \underbrace{\mu_{z_{15}} \circ \mu_{y_{15}}}_{\underline{\mu}_{D(15)}}$

Example

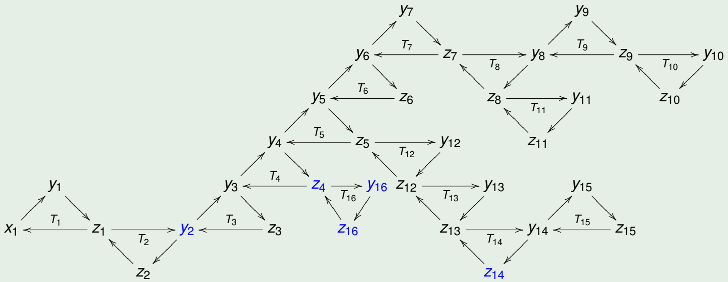
$\mathcal{Q} =$



- $\underline{\mu}_{14} = \underbrace{\mu_{z_6}}_{\underline{\mu}_{A(14)}} \circ \underbrace{\mu_{x_4}}_{\underline{\mu}_{B(14)}} \circ \underbrace{\mu_{x_{12}} \circ \mu_{x_{13}} \circ \mu_{x_{14}}}_{\underline{\mu}_{C(14)}} \circ \underbrace{\mu_{z_{14}} \circ \mu_{y_{14}}}_{\underline{\mu}_{D(14)}}$
- $\underline{\mu}_{15} = \underbrace{\mu_{x_{15}}}_{\underline{\mu}_{A(15)}} \circ \underbrace{\mu_{z_6}}_{\underline{\mu}_{B(15)}} \circ \underbrace{\mu_{z_{15}} \circ \mu_{y_{15}}}_{\underline{\mu}_{D(15)}}$

Example

$\mathcal{Q}$ =



- $\underline{\mu}_{16} = \underbrace{\mu_{z_{14}}}_{\underline{\mu}_{A(16)}} \circ \underbrace{\mu_{x_3}}_{\underline{\mu}_{B(16)}} \circ \underbrace{\mu_{x_{16}}}_{\underline{\mu}_{C(16)}} \circ \underbrace{\mu_{z_{16}} \circ \mu_{y_{16}}}_{\underline{\mu}_{D(16)}}$

Theorem (G. - Musiker)

Let Q be an embedded, irreducible type $\mathbb{A}$ quiver with respect to a leaf 3-cycle S. The associated mutation sequence $\underline{\mu}$ is a maximal green sequence of Q.

Idea of the Proof

We describe $\left(\underline{\mu}_{k-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0\right)\left(\widehat{\mathcal{Q}}\right)$ for each $k \in [n+1]$.

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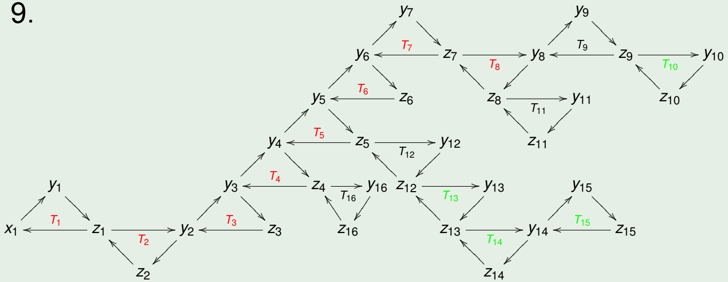
- For each k, split $\left(\underline{\mu}_{k-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0\right)\left(\hat{\mathcal{Q}}\right)$ in 3 sets of 3-cycles

$$\mathfrak{Q} = \mathfrak{Q}_{k-1} \sqcup (\mathfrak{T}_{k-1} \cup \{T_k\}) \sqcup \mathfrak{R}_{k-1}.$$

Example

Let $k = 9$.

$\mathcal{Q} =$



Then $\mathfrak{Q} = \underbrace{\{T_1, \ldots, T_8\}}_{\mathfrak{Q}_{k-1}} \sqcup \underbrace{\{T_9, T_{11}, T_{12}, T_{16}\}}_{(\mathfrak{T}_{k-1} \cup \{T_k\})} \sqcup \underbrace{\{T_{10}, T_{13}, T_{14}, T_{15}\}}_{\mathfrak{R}_{k-1}}$

Let $N := \#(\text{vertices of } Q)$.

$$\mathrm{Mut}\left(\widehat{Q}\right) \hookrightarrow \text{Skew-symm. matrices in } \mathbb{Z}^{N\times 2N}$$

$$\overline{Q} \to B_{\overline{Q}} = [b_{ij}] \in \mathbb{Z}^{N\times 2N}$$

$$b_{ij} = \#\left\{i \to j \in \left(\overline{Q}\right)_1\right\} - \#\left\{j \to i \in \left(\overline{Q}\right)_1\right\}$$

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Example

$$\hat{Q} \to \left[B_Q \middle| I_N\right] \in \mathbb{Z}^{N\times 2N}$$
$$\check{Q} \to \left[B_Q \middle| -I_N\right] \in \mathbb{Z}^{N\times 2N}$$

Idea of the Proof

Let $\mathcal{Q}_{k-1}$ and $\mathcal{R}_{k-1}$ be the full subquivers of Q on $\mathfrak{Q}_{k-1}$ and $\mathfrak{R}_{k-1}$, respectively.

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Lemma

For each $k \in [n+1]$, $\left(\underline{\mu}_{k-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0\right)\left(B_{\hat{\mathcal{Q}}}\right) = \mathcal{M}_{k-1}$ *where*

$$\mathcal{M}_{k-1} = \left[\begin{array}{ccc|ccc} B_{\mathcal{Q}_{k-1}}\sigma_{k-1} & -A^t & 0 & -M(\sigma_{k-1}) & 0 & 0 \\ A & B & -C^t & M\left(\widetilde{\mathbf{c}}(v)\right) & I & 0 \\ 0 & C & B_{\mathcal{R}_{k-1}} & 0 & 0 & I \end{array}\right].$$

Idea of the Proof

Let $\mathcal{Q}_{k-1}$ and $\mathcal{R}_{k-1}$ be the full subquivers of Q on $\mathfrak{Q}_{k-1}$ and $\mathfrak{R}_{k-1}$, respectively.

Lemma

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- A gives connections between red vertices and vertices in partially mutated 3-cycles.
- B gives connections between vertices in partially mutated 3-cycles.
- C gives connections between vertices in partially mutated 3-cycles and green vertices.

Idea of the Proof

Lemma

For each $k \in [n+1]$, $\left(\underline{\mu}_{k-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0\right)\left(B_{\hat{\mathcal{Q}}}\right) = \mathcal{M}_{k-1}$ *where*

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- $\sigma_{k-1} = \tau_{k-1} \cdots \tau_1$ where $\tau_j = (j_2, \ldots, j_d)$ where $\underline{\mu}_j = \mu_{j_d} \circ \cdots \circ \mu_{j_2} \circ \mu_{j_1}$ for $j \in [k-1]$.
- $-M(\sigma_{k-1})_{i,j} = \begin{cases} -1 & : \text{ if } i \cdot \sigma_{k-1} = j \\ 0 & : \text{ otherwise.} \end{cases}$
- $\left(B_{\mathcal{Q}_{k-1}}\sigma_{k-1}\right)_{i,j} = \left(B_{\mathcal{Q}_{k-1}}\right)_{i\cdot\sigma_{k-1},\, j\cdot\sigma_{k-1}}$

Idea of the Proof

Lemma

For each $k \in [n+1]$, $\left(\underline{\mu}_{k-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0\right)\left(B_{\widehat{\mathcal{Q}}}\right) = \mathcal{M}_{k-1}$
where

$$\mathcal{M}_{k-1} = \left[\begin{array}{ccc|ccc} B_{\mathcal{Q}_{k-1}}\sigma_{k-1} & -A^t & 0 & -M(\sigma_{k-1}) & 0 & 0 \\ A & B & -C^t & M\left(\widetilde{\mathbf{c}}(v)\right) & I & 0 \\ 0 & C & B_{\mathcal{R}_{k-1}} & 0 & 0 & I \end{array}\right].$$

- $M\left(\widetilde{\mathbf{c}}(v)\right)$ gives connections between vertices in partially mutated 3-cycles and frozen vertices corresponding to red vertices.
- If $v = y_i$, $\widetilde{\mathbf{c}}(v) = \mathbf{0}$. If $v = z_i$,

$$\widetilde{\mathbf{c}}(v) = \mathbf{e}_{x'_{r(i)}} + \chi_{\{r(i)>1\}}\mathbf{e}_{z'_{r(i)-1}} + \left(\sum_{u \in \text{supp}\left(\underline{\mu}_{C(i)}\right)} \mathbf{e}_{u'}\right).$$

Lemma

For each $k \in [n+1]$, $\left(\underline{\mu}_{k-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0\right)\left(B_{\hat{\mathcal{Q}}}\right) = \mathcal{M}_{k-1}$ *where*

$$\mathcal{M}_{k-1} = \left[\begin{array}{ccc|ccc} B_{\mathcal{Q}_{k-1}}\sigma_{k-1} & -A^t & 0 & -M(\sigma_{k-1}) & 0 & 0 \\ A & B & -C^t & M\left(\tilde{\mathbf{c}}(v)\right) & I & 0 \\ 0 & C & B_{\mathcal{R}_{k-1}} & 0 & 0 & I \end{array}\right].$$

By definition, $\mathcal{M}_n = \left[B_{\mathcal{Q}_n}\sigma_n \middle| -M(\sigma_n)\right]$.

Corollary

We have $\underline{\mu}\hat{\mathcal{Q}} \cong \check{Q}$.

The End

Thanks!