# Combinatorics of Exceptional Sequences in Type $\mathbb{A}$

Alexander Garver, Kiyoshi Igusa, Jacob P. Matherne, and Jonah Ostroff
University of Minnesota, Brandeis University, Louisiana State University, University of Washington

## Introduction

Exceptional sequences are certain sequences of quiver representations that are useful in the study of

- i) exceptional vector bundles on projective spaces [GR]
- ii) derived categories [BK], [R]

Exceptional sequences also have connections with

- i) noncrossing partitions [IT]
- ii) cluster algebras [IS], [ST]

We present a combinatorial classification of exceptional sequences of quivers whose underlying graph is a type $\mathbb{A}$ Dynkin diagram.

## Quiver Representations

- A **quiver** $Q$ is a directed graph. We consider quivers of the form $Q = 1 \xrightarrow{+} 2 \xleftarrow{-} 3 \xrightarrow{+} 4$. Such quivers "define" a **sign vector** $\epsilon = (+,+,-,+,+)$ (the two entries on the ends are chosen arbitrarily).
- A **quiver representation** $E$ is an assignment of a $\Bbbk$-vector space to each vertex of $Q$ and linear maps to each arrow.
- A representation $E$ is **indecomposable** if $E = E_1 \oplus E_2$ (the component wise **direct sum**) implies that $E_1 = 0$ or $E_2 = 0$.

### Example

The indecomposable representations of a type $\mathbb{A}$ quiver $Q$ are exactly those of the form $X_{i,j}$ :

$$\begin{array}{ccccccccccccc} 1 & & & & i & & & & j & & & & n \\ 0 & \overset{0}{\longleftrightarrow} & \cdots & \overset{0}{\longleftrightarrow} & 0 & \overset{0}{\longleftrightarrow} & \Bbbk & \overset{1}{\longleftrightarrow} & \cdots & \overset{1}{\longleftrightarrow} & \Bbbk & \overset{0}{\longleftrightarrow} & 0 \overset{0}{\longleftrightarrow} \cdots \overset{0}{\longleftrightarrow} 0 \end{array}$$

where $0 \leq i < j \leq n$.

## Exceptional Sequences

Let $Q$ be an acyclic quiver. An ordered pair of representations $(E_1, E_2)$ of $Q$ is called an **exceptional pair** if

*i)* each $E_i$ is indecomposable,
*ii)* $\text{Ext}^1(E_i, E_i) = 0$ for each $E_i$,
*iii)* $\text{Hom}(E_2, E_1) = 0$, $\text{Ext}^1(E_2, E_1) = 0$.

- A sequence $(E_1, \ldots, E_k)$ ($k \leq n := \#\{\text{vertices of } Q\}$) of representations of $Q$ is an **exceptional sequence** if $(E_i, E_j)$ is an exceptional pair for any $i < j$.
- A set $\{E_1, \ldots, E_k\}$ ($k \leq n$) of representations of $Q$ is an **exceptional collection** if $(E_{\sigma(1)}, \ldots, E_{\sigma(k)})$ is an exceptional sequence for some $\sigma \in \mathfrak{S}_k$.
- An exceptional sequence or collection is **complete** if $k = n$.

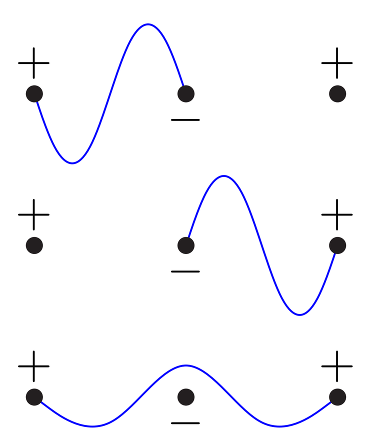
## Strand Diagrams and Labeled Strand Diagrams

Encode collections of representations of a type $\mathbb{A}$ quiver $Q$ using planar diagrams up to isotopy. Regard a representation of $Q$ as a **strand**. Let $Q = 1 \xleftarrow{-} 2$.

$$X_{0,1} = \Bbbk \xleftarrow[-]{0} 0 \quad \overset{\Phi_\epsilon}{\longmapsto}$$

$$X_{1,2} = 0 \xleftarrow[-]{0} \Bbbk \quad \longmapsto$$

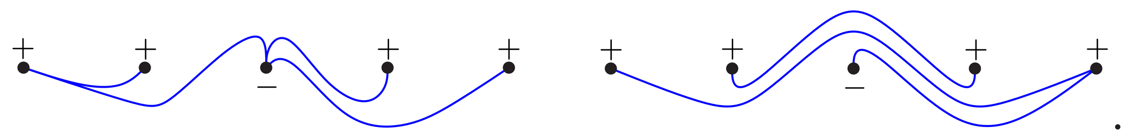
$$X_{0,2} = \Bbbk \xleftarrow[-]{1} \Bbbk \quad \longmapsto$$



A **strand diagram** $d = \{c(i_\ell, j_\ell)\}_{\ell \in [k]}$ is a collection of strands that are pairwise noncrossing and whose underlying graph is a forest. Let $\mathcal{D}_{k,\epsilon}$ denote the set of strand diagrams with $k$ strands.

A **labeled strand diagram** $d(k) = \{c(i_\ell, j_\ell, s_\ell)\}_{\ell \in [k]}$ is a strand diagram whose strands are bijectively labeled with by the integers $s \in [k]$. A labeled strand diagram $d(k)$ has a **good labeling** if for any $i \in [0, n]$ the labels of the strands connected to $i$ increase when one reads through them clockwise.

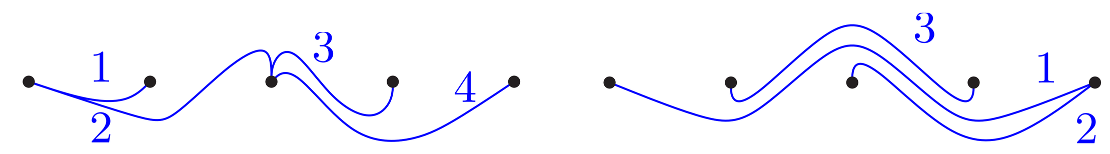
Let $Q = 1 \xrightarrow{+} 2 \xleftarrow{-} 3 \xrightarrow{+} 4$ and $\epsilon = (+,+,-,+,+)$. Here are the strand diagrams $\{c(0,1), c(0,2), c(2,3), c(2,4)\}$ and $\{c(0,4), c(1,3), c(2,4)\}$



.

## Strand Diagrams and Labeled Strand Diagrams (continued)

Here are two labeled strand diagrams where the former has a



good labeling and the latter does not.

## Theorem

Let $Q$ be a type $\mathbb{A}$ quiver with sign vector $\epsilon$. The maps

$$\{X_{i_\ell, j_\ell}\}_{\ell \in [k]} \longmapsto \{c(i_\ell, j_\ell)\}_{\ell \in [k]} \quad \text{and} \quad (X_{i_\ell, j_\ell})_{\ell \in [k]} \longmapsto \{(c(i_\ell, j_\ell), k + 1 - \ell)\}_{\ell \in [k]}$$

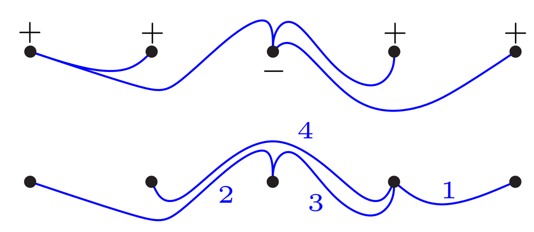
define bijections between exceptional collections (resp. exceptional sequences) and strand diagrams (resp. labeled strand diagrams with a good labeling).

### Example

Let $Q = 1 \xrightarrow{+} 2 \xleftarrow{-} 3 \xrightarrow{+} 4$ and $\epsilon = (+,+,-,+,+)$.

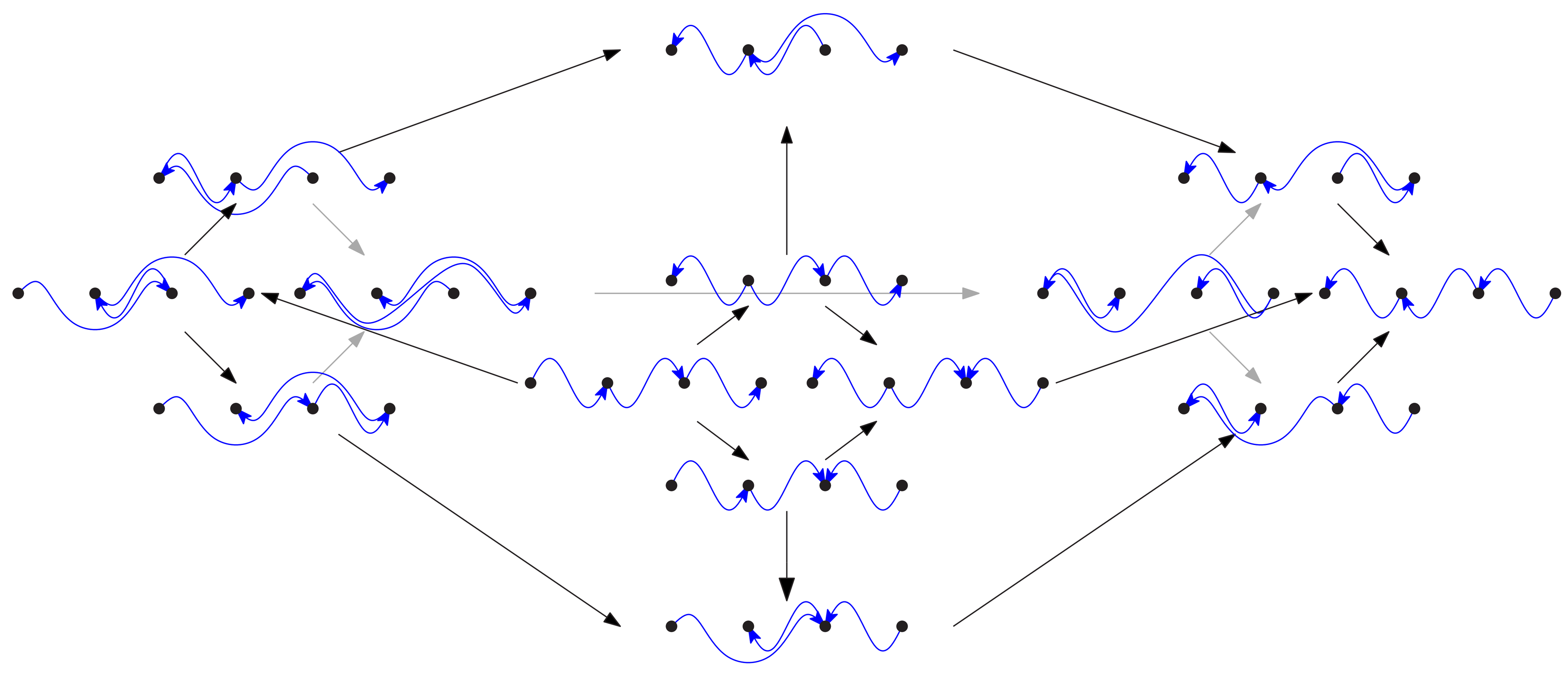
$\{X_{0,1}, X_{0,2}, X_{2,3}, X_{2,4}\} \mapsto$

$(X_{1,3}, X_{2,3}, X_{0,2}, X_{3,4}) \mapsto$



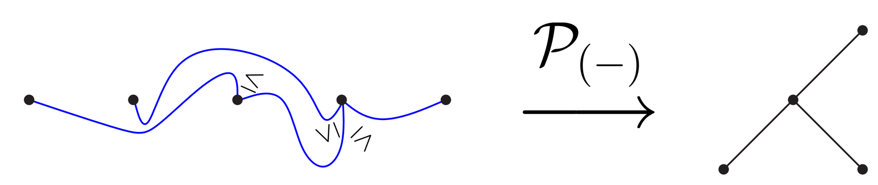
## Cluster Algebras

Oriented diagrams $\vec{d} \in \vec{\mathcal{D}}_{n,\epsilon}$ index the **clusters** in $\mathcal{A}(Q)$. Let $Q = 1 \xrightarrow{+} 2 \xleftarrow{-} 3$ and $\epsilon = (-,+,-,+)$.



## Linear Extensions

Each $d \in \mathcal{D}_{n,\epsilon}$ defines a poset.



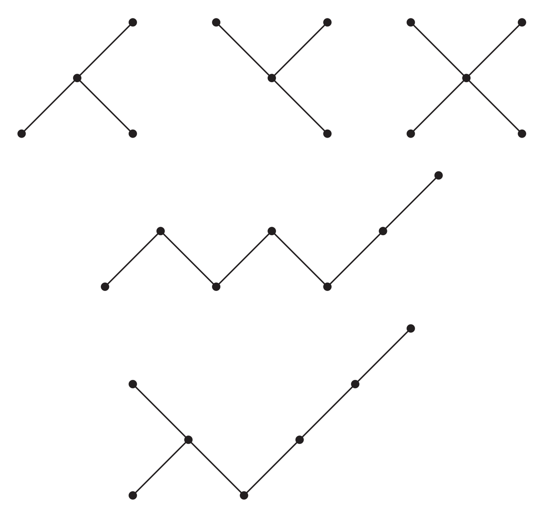
Let $\overline{\xi}$ be a CEC whose diagram is $d \in \mathcal{D}_{n,\epsilon}$. Then the natural map $\{\text{CESs determined by } \overline{\xi}\} \longrightarrow \mathscr{L}(\mathcal{P}_d)$ is a bijection.

$$(X_{i_\ell, j_\ell})_{\ell \in [n]} \mapsto \{(c(i_\ell, j_\ell), n + 1 - \ell)\}_{\ell \in [n]} \mapsto (f : c(i_\ell, j_\ell) \longmapsto n + 1 - \ell)$$

## Posets Arising From Strand Diagrams

Let $\epsilon = (-,-,\ldots,-)$ or $(+,+,\ldots,+)$. A poset $\mathcal{P} \in \mathscr{P}(\mathcal{D}_{n,\epsilon}) := \{\mathcal{P}_d : d \in \mathcal{D}_{n,\epsilon}\}$ if and only if

1. each $x \in \mathcal{P}$ has at most two covers and covers at most two elements
2. the underlying graph of the Hasse diagram of $\mathcal{P}$ has no cycles
3. the Hasse diagram of $\mathcal{P}$ is connected.



## Noncrossing Partitions

Exceptional sequences of $Q = 1 \leftarrow \cdots \leftarrow n$ are in bijection with **saturated** chains in $NC(\mathfrak{S}_{n+1})$, the lattice of noncrossing partitions, that contain its bottom element $\{\{i\}\}_{i \in [n+1]}$.