Reverse plane partitions via representations of quivers

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A reverse plane partition on a poset $\mathcal{P}$ is an order-reversing map $\rho : \mathcal{P} \rightarrow \{0, 1, \ldots, N\}$ for some $N$. 

[[3] × [2]]
Toggling on reverse plane partitions

Given $x \in P$, one can **toggle** a reverse plane partition $\rho$ as follows to produce a new reverse plane partition $t_x \rho$.

$$t_x \rho(y) = \begin{cases} \max_{y < y_1} \rho(y_1) + \min_{y_2 < y} \rho(y_2) - \rho(y) & : \text{if } y = x \\ \rho(y) & : \text{if } y \neq x, \end{cases}$$
Periodicity for promotion \( \text{pro} = t_1 t_2 t_4 t_3 \)
Theorem (Grinberg–Roby ‘15, Musiker–Roby ‘18)

Promotion on reverse plane partitions of $P = [a] \times [b]$ has order $a + b$.

- Proved using a birational version of the promotion operator and then “tropicalizing”.
- We have a new proof of periodicity by showing that reverse plane partitions are equivalent to certain representations of quivers.

\[
\begin{align*}
\mathbf{4} & \quad \mathbf{k}^5 \\
\downarrow & \downarrow \text{f}_{4,3} \\
\mathbf{3} & \quad \mathbf{k}^8 \\
\uparrow & \uparrow \text{f}_{2,3} \\
\mathbf{2} & \quad \mathbf{k}^5 \\
\uparrow & \uparrow \text{f}_{1,2} \\
\mathbf{1} & \quad \mathbf{k}^3
\end{align*}
\]

$Q$ quiver and $V$ quiver representation with dimension vector $\text{dim}(V) = 3585$
Theorem (G.–Patrias–Thomas ‘18)

Let $Q$ be a type A Dynkin quiver, $m$ a vertex of $Q$, and $P_{Q,m}$ the associated (minuscule) poset. There is a bijection between representations of $Q$ all of whose indecomposable summands are supported at $m$ and reverse plane partitions of $P_{Q,m}$.

To obtain $\rho_V$ from $V$, one calculates the Jordan blocks of a generic nilpotent endomorphism of $V$.

In the example, the Jordan blocks are $((3), (4,1), (5,3), (5))$. 
Auslander–Reiten quivers

\[ Q \xrightarrow{\sim} \Gamma(Q) \]

quiver \quad \text{Auslander–Reiten quiver}

\{\text{vertices of } \Gamma(Q)\} \leftrightarrow \{\text{indecomposable representations of } Q\}
\{\text{arrows of } \Gamma(Q)\} \leftrightarrow \{\text{irreducible morphisms up to rescaling}\}

\[ \begin{array}{ccc}
111 & \tau & 011 \\
\downarrow & \downarrow & \downarrow \\
110 & \tau & 010 \\
\downarrow & \downarrow & \downarrow \\
100 & \tau & 001
\end{array} \]

\[ \Gamma(Q) \text{ where } Q = 1 \leftarrow 2 \leftarrow 3 \]

The functor \( \tau \) is known as the \textbf{Auslander–Reiten translation}. 
The category of representations of $Q$ sits inside the root category $\mathcal{R}_Q$.

$\Gamma(\mathcal{R}_Q)$ where $Q = 1 \leftarrow 2 \leftarrow 3$

**Lemma**

*Let $Q$ be a type $A$ Dynkin quiver. For any $X \in \mathcal{R}_Q$,*

$$\tau^{|\text{vertices of } Q|+1}(X) = X.$$
Promotion is compatible with $\tau$

**Lemma**

Let $Q$ be a type A Dynkin quiver. For any $X \in R_Q$,

$$\tau|\{\text{vertices of } Q\}|+1(X) = X.$$

For $Q = 1 \leftarrow 2 \leftarrow 3$, the possible associated posets are $[3]$ (when $m = 1, 3$) and $[2] \times [2]$ (when $m = 2$).

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**Lemma**

Let $Q$ be a type A Dynkin quiver. For any $X \in \mathcal{R}_Q$,

$$\tau|\{\text{vertices of } Q\}| \cdot X = X.$$

For $Q = 1 \leftarrow 2 \leftarrow 3$, the possible associated posets are $[3]$ (when $m = 1, 3$) and $[2] \times [2]$ (when $m = 2$).
Theorem (G.–Patrias–Thomas ‘18)

Let $Q$ be a type A Dynkin quiver and $m$ a vertex of $Q$. Let $V$ be a representation of $Q$ where $\rho_V$ is its corresponding reverse plane partition. Then $\rho_{\tau V} = \text{pro}\rho_V$ so $\text{pro}^{a+b}\rho_V = \rho_{\tau^{a+b}V} = \rho_V$.

We obtain the periodicity result.