Reverse plane partitions via representations of quivers II

Al Garver, UQAM
(joint with Rebecca Patrias and Hugh Thomas)

AMS Special Session on Combinatorics in Algebra and Algebraic Geometry

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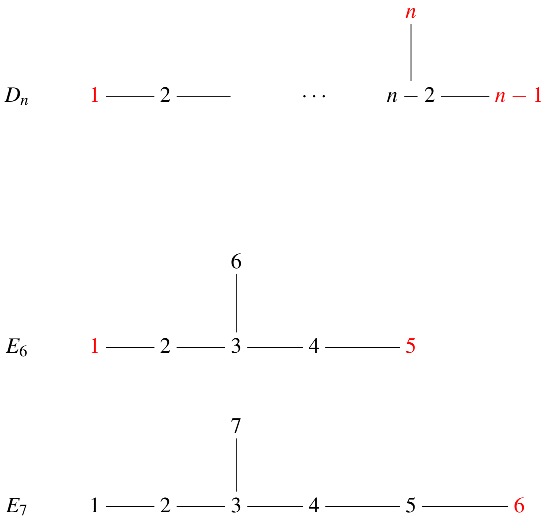
Outline

- minuscule posets
- Auslander–Reiten quivers
- “RSK” in minuscule types
- Periodicity of piecewise-linear promotion

The minuscule posets are defined by choosing a simply-laced Dynkin diagram and a **minuscule vertex** m.



$\mathsf{P}_{A_n,m} = [m] \times [n-m+1]$

$\mathsf{P}_{D_n,1} = J^{n-3}([2] \times [2])$

$\mathsf{P}_{D_n,n-1} = J([2] \times [n-2])$

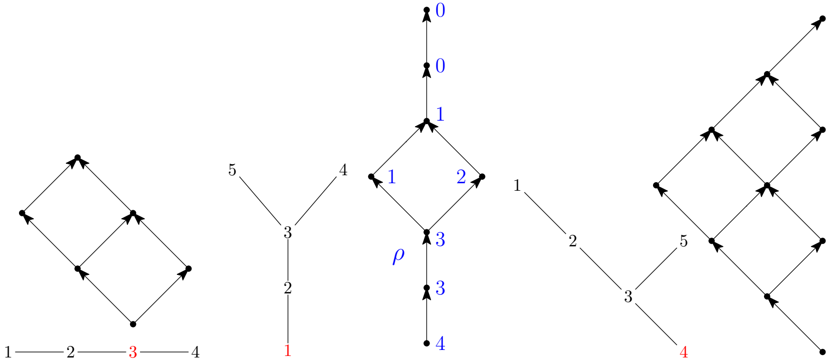
$\mathsf{P}_{D_n,n} = J([2] \times [n-2])$

$\mathsf{P}_{E_6,1} = J^2([2] \times [3])$

$\mathsf{P}_{E_6,5} = J^2([2] \times [3])$

$\mathsf{P}_{E_7,6} = J^3([2] \times [3])$

Goal: Understand the **reverse plane partitions** associated with minuscule posets using the representation theory of quivers that are orientations of the corresponding Dynkin diagrams.



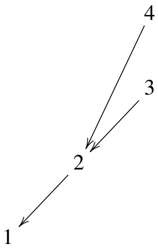
$\mathsf{P}_{A_4,3}$ $\mathsf{P}_{D_5,1}$ $\mathsf{P}_{D_5,4}$

A **reverse plane partition** is an order-reversing map $\rho : P \to \mathbb{Z}_{\geqslant 0}$.

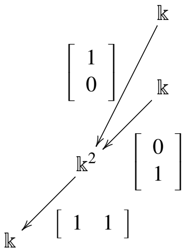
Theorem (Gabriel '72)

The isomorphism classes of indecomposable representations of a Dynkin quiver are in bijection with the positive roots of the associated root system.

Main example: the D_4 quiver



Q



M

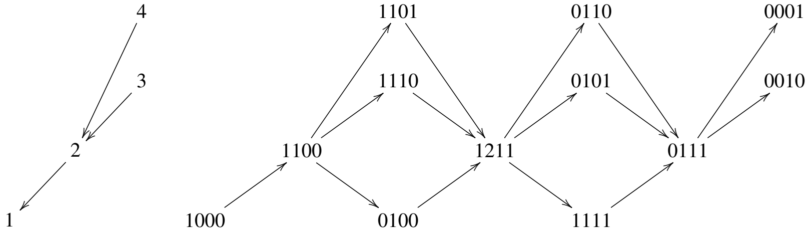
a **representation** of Q

$\mathbf{dim}(M) = 1211$

$\mathbf{dim}(M) \in \Phi^+\left(\overline{Q}\right)$

dimension vector of M

Main example:



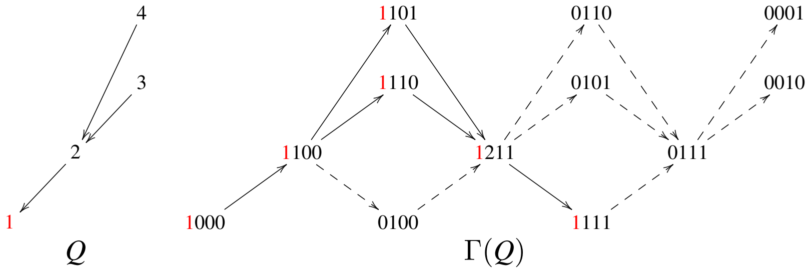
Q $\quad$ $\Gamma(Q)$ **Auslander–Reiten quiver** of Q

$\Gamma(Q)_0 \quad = \quad$ { isomorphisms classes of indecomposable representations of Q }

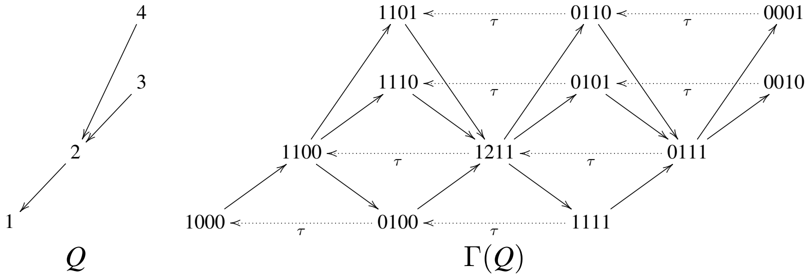
$\Gamma(Q)_1 \quad \leftrightarrow \quad$ { a basis of the space of **irreducible morphisms** between the corresponding representations }

Lemma

Given a Dynkin quiver Q and a minuscule vertex m, the Hasse quiver of the minscule poset $\mathsf{P}_{\overline{Q},m}$ is the full subquiver of $\Gamma(Q)$ on the representations supported at m.



There is a map $\tau : \Gamma(Q)_0 \to \Gamma(Q)_0$ called the **Auslander–Reiten translation**.



The Auslander–Reiten translation partitions the indecomposables into τ**-orbits**.

$$Q_0 \longleftrightarrow \{\tau\text{-orbits}\}$$

“RSK” for minuscule posets

M – representation $\rightsquigarrow$ $\mathrm{NEnd}(M) := \{$nilpotent endomorphisms of $M\}$

$\mathrm{GenJF}(M) = (\lambda^1, \ldots, \lambda^{|Q_0|})$ – generic Jordan form of an element of $\mathrm{NEnd}(M)$

$\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^{|Q_0|})$ – sequence of partitions $\rightsquigarrow$ $\mathrm{rep}_{\boldsymbol{\lambda}}(Q)$ – variety of representations compatible with a nilpotent endomorphism with Jordan blocks given by $\boldsymbol{\lambda}$

Theorem (G.–Patrias–Thomas ‘18)

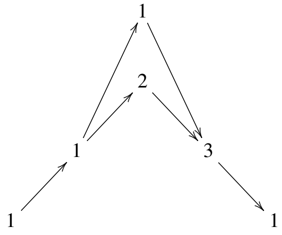
The objects of $\mathcal{C}_{Q,m}$ (i.e., representations of Q whose indecomposable summands are supported at m) are in bijection with $RPP(\mathsf{P}_{\overline{Q},m})$ via

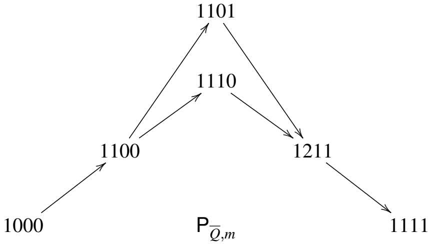
$M \mapsto \rho$ – *reverse plane partition from filling the τ-orbits of $\mathsf{P}_{\overline{Q},m}$ with the partitions in $GenJF(M)$*

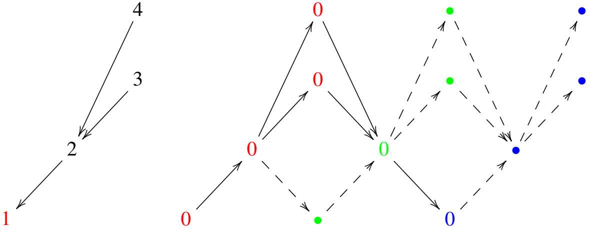
M – *isomorphism class of representations in dense open subset of $rep_{\boldsymbol{\lambda}(\rho)}(Q)$* $\leftarrow$ ρ.

To prove the theorem, we construct a sequence of piecewise-linear maps that turns a representation into a reverse plane partition.

Let $M \in \mathcal{C}_{Q,m}$ (i.e., M is a representation of Q whose indecomposable summands correspond to elements of $\mathsf{P}_{\overline{Q},m}$).

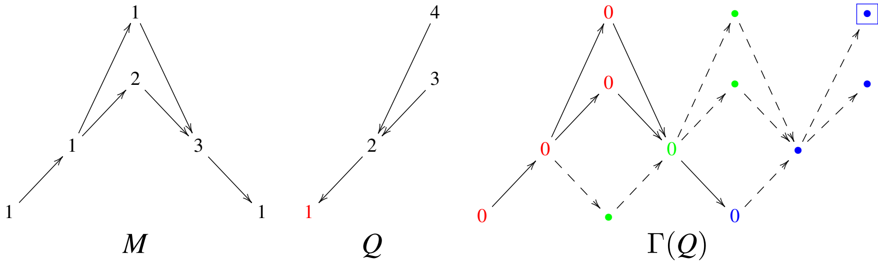
$M =$ 





Starting with ρ_0, the zero reverse plane partition, and the "rightmost" vertex X of $\Gamma(Q)$.

- if X is a summand of M, replace $\rho_0(X)$ with $\rho_1(X) = \max_{X \lessdot Y} \rho_0(Y) + \text{mult}(X)$,
- for each X' in the τ-orbit of X with $X < X'$, replace $\rho_0(X')$ with $\rho_1(X') = \max_{X' \lessdot Y} \rho_0(Y) + \min_{Y \lessdot X'} \rho_0(Y) - \rho_0(X')$.



Continue this process with the next "rightmost" vertex X of $\Gamma(Q)$.

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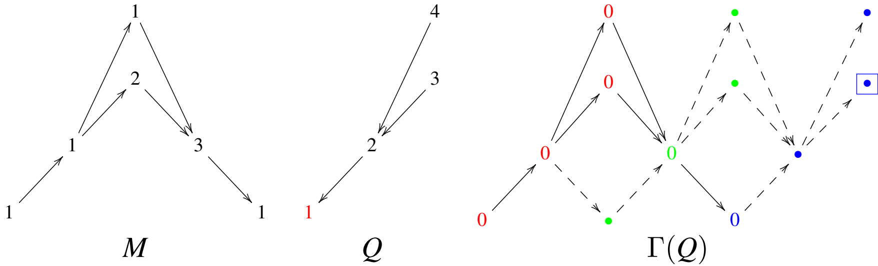

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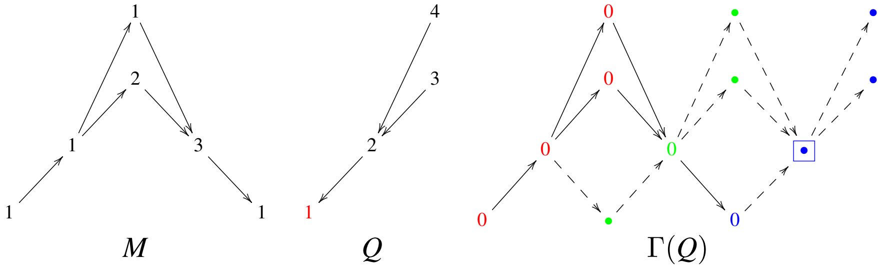

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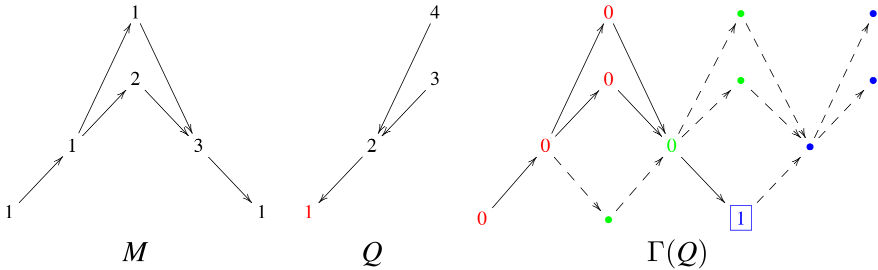

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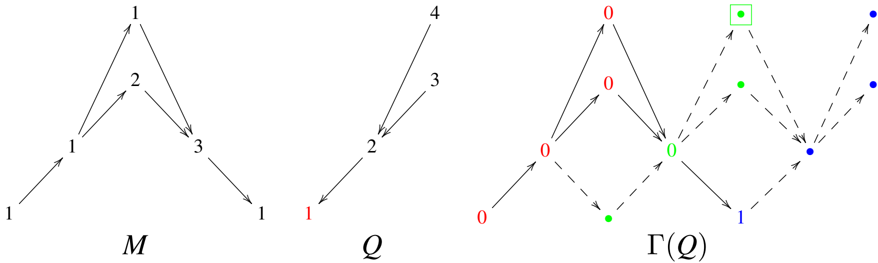

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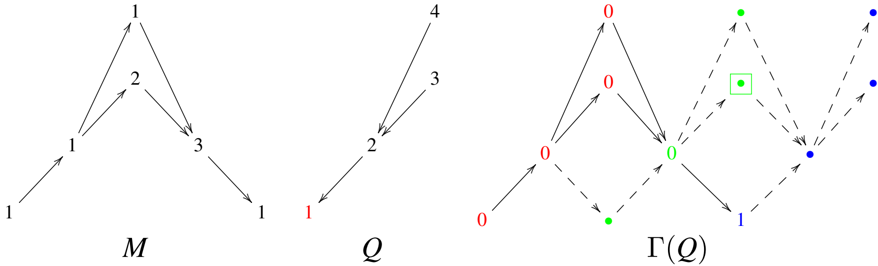

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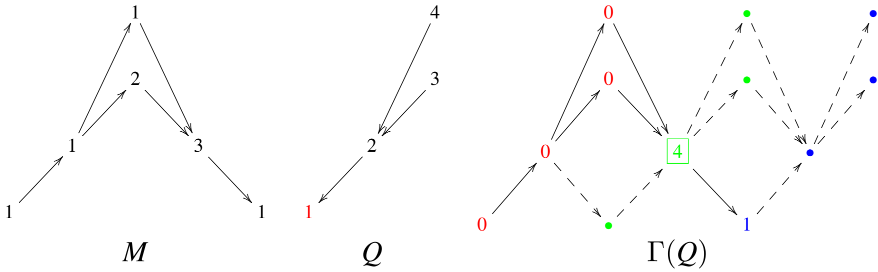

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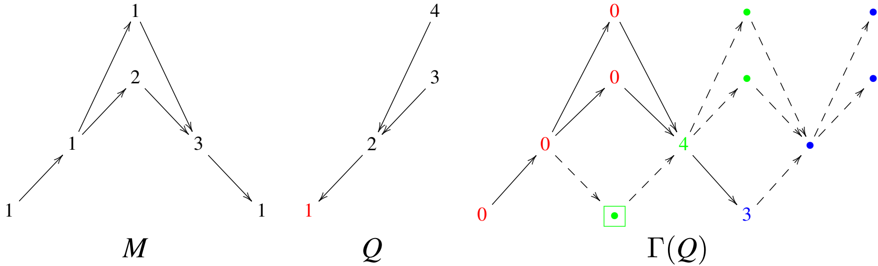

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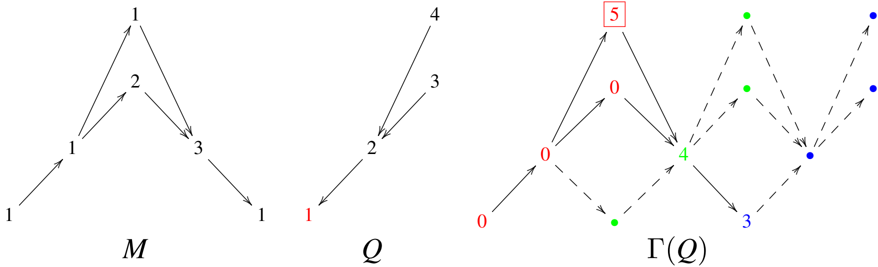

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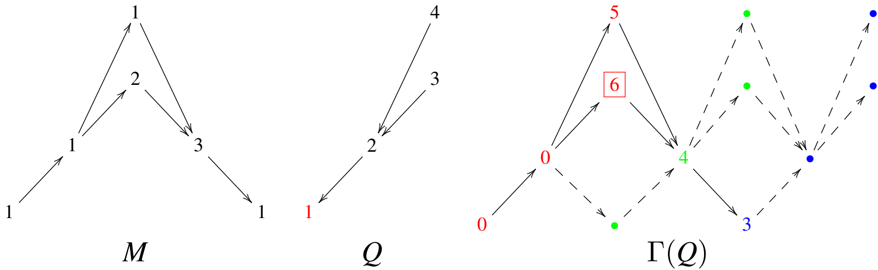

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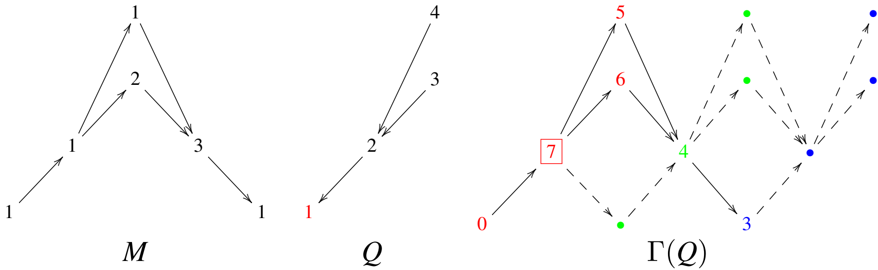

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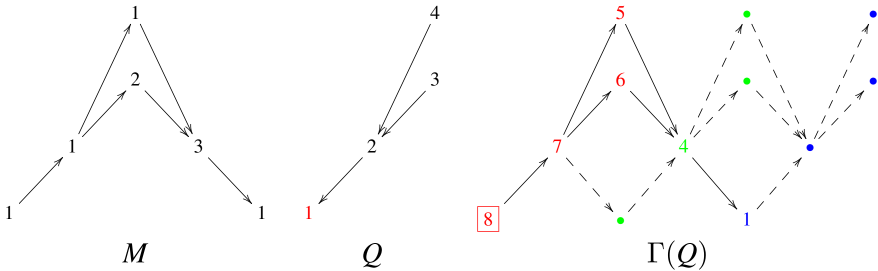

The resulting map $\rho : \mathsf{P}_{\overline{Q},m} \to \mathbb{Z}_{\geqslant 0}$ is a reverse plane partition.

Theorem (G.–Patrias–Thomas)

The map $\mathcal{C}_{Q,m} \to RPP(\mathsf{P}_{\overline{Q},m})$ is a bijection.

Thanks! (And piecewise-linear promotion pro = $t_4 t_3 t_2 t_1$)