They are coplanar!

(12.5) Equations of Lines and Planes

Q: What determines a line L in 3-dimensional space?

* two points on L

* a point \( P_0 \) on L and a vector \( \mathbf{v} \) parallel to L

Let \( P \) be any other point on L, and let \( \mathbf{a} \) be the vector going from \( P_0 \) to \( P \).

\[ \mathbf{a} = t \mathbf{v} \] for some scalar \( t \).

Since \( P \) was an arbitrary point on L, we have that L is given by

\[ \mathbf{r} = \mathbf{r}_0 + t \mathbf{v} \].

+ Idea: L is traced out by the vectors \( \mathbf{r} \).

Let \( \mathbf{r} = \langle x, y, z \rangle \), \( \mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle \), \( \mathbf{v} = \langle a, b, c \rangle \).

The parametric equations of line L containing \((x_0, y_0, z_0)\) that is parallel to \( \mathbf{v} = \langle a, b, c \rangle \) are

\[ x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct. \]
Solve these equations for $t$.

\[ t = \frac{x-x_0}{a}, \quad t = \frac{y-y_0}{b}, \quad t = \frac{z-z_0}{c} \]

The symmetric equations of $L$ are

\[ \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \]

Ex Find the three types of equations for the line containing $(0, \frac{1}{2}, 1)$ and $(3, 1, 3)$.

\[ \vec{r}_0 = \langle 2, 1, -3 \rangle \quad \vec{v} = \langle 2, \frac{1}{2}, 4 \rangle \]

\[ \vec{r} = \langle 2+2t, 1+\frac{1}{2}t, -3-4t \rangle \]

\[ x = 2+2t, \quad y = 1+\frac{1}{2}t, \quad z = -3-4t \]

\[ \frac{x-2}{2} = 2y-2 = \frac{z+3}{-4} \]

Where does this line intersect the $xz$-plane?

Find a point $(x, 0, z)$ that's on the line.

\[ 1 + \frac{1}{2}t = 0 \]

\[ t = -2 \]

\[ (-2, 0, 5) \] is such a point.
Planes

What information determines a plane?

- 3 non collinear points in the plane
- 2 non parallel vectors and a point \( P_0 \) in the plane
- a vector \( \mathbf{n} \) orthogonal to the plane and a point \( P_0 \) in the plane

Let \( \mathbf{n} \) be such a vector, \( P_0 = (x_0, y_0, z_0) \), and \( P = (x, y, z) \) any point in the plane.

\( x - x_0 \) lies in the plane

\[ \mathbf{n} \cdot (x - x_0) = 0 \]

Vector equation of the plane

\[ \mathbf{r} = \langle x, y, z \rangle \]

\[ \mathbf{r_0} = \langle x_0, y_0, z_0 \rangle \]

\[ \mathbf{n} = \langle a, b, c \rangle \]

\[ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \]

Scalar equation of the plane containing \( P = (x_0, y_0, z_0) \) with normal vector \( \mathbf{n} \).

Linear equation of the plane

\[ ax + by + cz + d = 0 \]

\[ d = -ax_0 - by_0 - cz_0 \]
Ex: Find an equation of the plane containing \( P_0 = (2,0,1) \) and that is orthogonal to the line \( x = 3t, y = 2-t, z = 3+4t \).

**Answer:** \( \mathbf{v} = \langle 3, -1, 4 \rangle = \mathbf{n} \)

\[
3(x-2) + (-1)(y) + 4(z-1) = 0
\]

\[
3x - y + 4z = 10
\]

Ex: Find an equation of the plane containing \( P = (3,0,-1), Q = (-3,2,3) \), and \( R = (7,1,-4) \).

\[
\mathbf{PQ} = \langle -5, -2, 4 \rangle
\]

\[
\mathbf{PR} = \langle 4, 1, -3 \rangle
\]

\[
\mathbf{n} = \mathbf{PQ} \times \mathbf{PR} = \langle 6-4, -(15-6), -5+8 \rangle
\]

\[
= \langle 2, 1, 3 \rangle
\]

\[
2(x-3) + y + 3(z+1) = 0
\]

\[
2x + y + 3z - 3 = 0
\]