Ex: For \( f(x,y) = \ln(1+xy) \), find unit vectors starting at point \( P = (2,3) \) where

(a) \( \mathbf{u} \) points in the direction of greatest increase
(b) \( \mathbf{u} \) points in the direction of greatest decrease
(c) \( \mathbf{u} \) points in the direction of no change.

(a) \( \nabla f = \left< \frac{y}{1+xy}, \frac{x}{1+xy} \right> \)

\( |\nabla f(2,3)| = \sqrt{\frac{1}{49} + \frac{4}{49}} = \frac{\sqrt{15}}{7} \)

\( \nabla f(2,3) = \left< \frac{3}{7}, \frac{2}{7} \right> \) \( \Rightarrow \mathbf{u} = \frac{1}{\sqrt{\frac{13}{7}}} \left< \frac{3}{7}, \frac{2}{7} \right> = \left< \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right> \)

(b) \( \mathbf{u} = \left< \frac{-2}{\sqrt{13}}, \frac{-2}{\sqrt{13}} \right> \)

(c) A vector in the direction of no change is orthogonal to \( \left< \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right> \).

The unit vectors orthogonal to a unit vector \( \langle a, b \rangle \) are \( \langle -b, a \rangle \) and \( \langle b, -a \rangle \).

Thus, \( \mathbf{u} = \left< \frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right> \).

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Idea: \( \nabla f(x,y) \) is perpendicular to level curve \( f(x,y) = f(x_0,y_0) \)

Directions of no change at \( (x_0, y_0) \) are parallel to the tangent line to the level curve \( f(x,y) = f(x_0,y_0) \) at \( (x_0, y_0) \).
Tangent planes to level surfaces

Let $S$ be the surface $F(x,y,z) = k$, (a level surface of a function $F(x,y,z)$)

Let $\mathbf{P} = (x_0, y_0, z_0)$ be a point on $S$.

Let $C: x = x(t), \ y = y(t), \ z = z(t)$ be a curve in $S$ passing through $\mathbf{P}$. A point on $C$ satisfies $F(x(t), y(t), z(t)) = k$ for some $k$.

**Chain Rule**

\[
\frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} = 0
\]

\[\nabla F \cdot \mathbf{r}'(t) = 0\]

where $\mathbf{r}(t) = (x(t), y(t), z(t))$

Let $t_0$ be such that $x_0 = x(t_0), \ y_0 = y(t_0), \ z_0 = z(t_0)$.

Then

\[\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0\]

\[\nabla F(x_0, y_0, z_0)\] is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ of any curve $C$ in $S$ that passes through $\mathbf{P}$.

It therefore makes sense to define the tangent plane to $F(x,y,z) = k$ at $(x_0, y_0, z_0)$ as:

\[F_x(x_0, y_0, z_0)(x-x_0) + F_y(x_0, y_0, z_0)(y-y_0) + F_z(x_0, y_0, z_0)(z-z_0) = 0\]
We also define the normal line to that plane as

\[ x = x_0 + F_x(x_0, y_0, z_0) t, \quad y = y_0 + F_y(x_0, y_0, z_0) t, \]
\[ z = z_0 + F_z(x_0, y_0, z_0) t \]

Ex: Find the (a) tangent plane and (b) normal line to

\[ xy^2z^3 = 8 \quad \text{at} \quad (2, 2, 1) \]

(a) \[ F_x = yz^3 \quad F_x(2, 2, 1) = 4 \]
\[ F_y = 2xyz^3 \quad F_y(2, 2, 1) = 8 \]
\[ F_z = 3xyz^2 \quad F_z(2, 2, 1) = 24 \]

tangent plane

\[ 4(x - 2) + 8(y - 2) + 24(z - 1) = 0 \]

\[ \implies x + 2y + 6z = 12 \]

(b) normal line

\[ x = 2 + 4t \]
\[ y = 2 + 8t \]
\[ z = 1 + 24t. \]
True or False? (p. 921)

1. True, set \( s = t^3 \). Then \( r(s) = \langle s, 2s, 3s \rangle \).

2. True,

\[
\frac{\dot{z}}{16} = y, \quad x = 0
\]

3. False; it is a line, but not one through the origin.

4. True

5. False, \( \frac{d}{dt} (u(t) \times v(t)) = u'(t) \times v(t) + u(t) \times v'(t) \)

6. False, \( r(t) = \langle 1, 1, t \rangle \) \( \Rightarrow \) \( |r(t)| = \sqrt{2 + t^2} \)

\[
\frac{d}{dt} |r(t)| = \frac{t}{\sqrt{2 + t^2}}
\]

\( \dot{r}(t) = \langle 0, 0, 1 \rangle \), \( |\dot{r}(t)| = 1 \) for all \( t \).

When \( t = 0 \), \( \frac{d}{dt} |r(0)| = 0 \).