The ring $\mathbb{C}[G]$ and the Fourier transform.

① The space $\mathbb{C}[G]$ has a natural ring structure given by

$$e_g \cdot e_h = e_{gh},$$

i.e. $\left(\sum_g a_g e_g\right)\left(\sum_h b_h e_h\right) = \sum_{g,h} (a_g b_h) e_{gh}$.

⊛ When thinking about $\mathbb{C}[G]$ as the space of functions $\{f : G \to \mathbb{C}\}$ (really when considering the dual space $\mathbb{C}[G]^*$, with $f(g) = a_g$ being the coefficients of vectors in $\mathbb{C}[G]$)

this multiplication operation becomes the convolution of functions:

$(f * f') : G \to \mathbb{C}$ given by

$$(f * f')(g) = \sum_{h \in G} f(gh^{-1}) f'(h).$$

② The space of polynomials $\mathbb{C}[x]$:

Note that the ring $\mathbb{C}[G]$ and the convolution are quite familiar in this context:

⊛ Consider $\mathbb{C}[\mathbb{N}] = \text{span}\langle e_n : n \in \mathbb{N} \rangle$.

An alternative notation is

$$e_n \rightsquigarrow x^n.$$

The natural multiplication in this ring is given by

$$x^n \cdot x^m = x^{n+m}$$

$\left[\begin{array}{c}\text{The operation in the semi-group} \\ \mathbb{N} = \langle 0,1,2,\ldots ; + \rangle\end{array}\right]$

This is the same as when we define $e_g \cdot e_h = e_{g \cdot h}$.

$\Longrightarrow$ The ring $\mathbb{C}[\mathbb{N}]$ is the standard polynomial ring $\mathbb{C}[x]$.

⊛ Consider the coefficiens of a polynomial

$$P(x) = \sum_{n=0}^{N} a_n x^n \rightsquigarrow \left[\begin{array}{c} f : \mathbb{N} \to \mathbb{C} \\ f(n) = a_n \end{array}\right].$$

Multiplication in the ring $\mathbb{C}[x]$ corresponds to the convolution of the coefficients:

$$P(x) \cdot Q(x) = \left(\sum a_n x^n\right)\left(\sum_m b_m x^m\right)$$

$$= \sum_{n,m} a_n b_m x^{n+m} = \sum_k \Big(\underbrace{\sum_n a_n b_{k-n}}_{a*b\,(k)} \Big) x^k$$

i.e. if we denote

$P_f(x) = \sum_{n=0}^{N} f(n) x^n$ then we have a description of multiplication in terms of the convolution and vice-versa:

$$P_f \cdot P_{f'} = P_{f * f'}.$$

<u>Rem:</u> Another way of expressing this is by saying the the vector space iso.

$$\left[\mathbb{C}[\mathbb{N}] ; \; x^n \cdot x^m = x^{n+m}\right] \simeq \Big[\underbrace{\{f : \mathbb{N} \to \mathbb{C}\}}_{\text{who vanish almost everywhere}} ; f * g\Big]$$

is an isomorphism of <u>rings</u>,

and the same will be true for all finite groups G – $[\mathbb{C}[G], \cdot] \simeq [\{f : G \to \mathbb{C}\}, *]$.

⊛ Every formal polynomial defines a function $P(x) : \mathbb{C} \to \mathbb{C}$ by substituting x by any complex number –

$$P(r) = \sum_{n=0}^{N} a_n r^n.$$

This is a map $\mathbb{C}[x] \to \underbrace{\{f : \mathbb{C} \to \mathbb{C}\}}_{\text{complex valued functions on } \mathbb{C}}$.

<u>Note:</u> This map is in fact a ring homomorphism (even a hom. of $\mathbb{C}$-algebras),

when the operations on $\{f : \mathbb{C} \to \mathbb{C}\}$ are taken pointwise:

$$(\lambda f)(r) = \lambda \cdot f(r)$$

$$(f + g)(r) = f(r) + g(r)$$

$$(f \cdot g)(r) = f(r) \cdot g(r).$$

In particular, $P \cdot Q\,(r) = P(r) \cdot Q(r) \quad \forall P, Q \in \mathbb{C}[x]$

i.e. $P_{f*g}(r) = P_f \cdot P_g\,(r) = P_f(r) \cdot P_g(r)$.

so this is our first example of the Fourier transform turning a convolution $f*g$ into a product $\rho_f \cdot \rho_g : \mathbb{C} \to \mathbb{C}$.

③ What is the Fourier transform (FT)?

In previous problem sessions we described how given a G-rep $\rho: G \to GL(V_\rho)$ we can turn any function $f: G \to \mathbb{C}$ into a linear self map of V_ρ by taking $\rho(f) := \sum_{g \in G} \underset{\mathbb{C}}{f(g)} \cdot \underset{GL(V_\rho)}{\rho(g)} \in End(V_\rho)$.

Note: $\rho(f)$ might not be invertible, and might not be G-linear. It is just some linear map $V_\rho \to V_\rho$.

$\Longrightarrow$ f defines a way of associating to every G-rep $\rho \rightsquigarrow$ some linear transformation $\rho(f) \in End(V_\rho)$.

We call this association the Fourier transform of f denoted by $\hat{f}$.

$\hat{f}$ take any G-rep ρ to some endomorphism $\hat{f}(\rho) \in End(V_\rho)$.

Note: In terms of $\mathbb{C}[G]$, every G-rep ρ defines a map

$$\rho\left(\sum_{g \in G} a_g e_g\right) = \sum_{g \in G} a_g \rho(g)$$

and this is clearly a homomorphism of rings $[\mathbb{C}[G], \cdot] \to [End(V_\rho), \circ]$.

⊛ Instead of considering $\hat{f}$ as acting on every G-rep. ρ, we may want to consider only it's action on irreps. Since every G-rep. is a direct sum of irreps, this will be sufficient to describe the action of $\hat{f}$ on any G-rep.

Let $\rho_1, \dots, \rho_k$ be all the distinct G-irreps: we consider

$\hat{f}(\rho_i) \in End(V_{\rho_i})$ $\forall i$ simultaniously

Define $FT: \mathbb{C}[G] \to End(V_{\rho_1}) \oplus \dots \oplus End(V_{\rho_k})$ by $f \mapsto (\hat{f}(\rho_1), \hat{f}(\rho_2), \dots, \hat{f}(\rho_k))$.

Ex. This is a ring homomorphism, and in fact it is an isomorphism (!!)

i.e. every combination of linear maps $T_i: V_{\rho_i} \to V_{\rho_i}$ can be realized by a unique function $f: \mathbb{C} \to G$.

Example: take $T_{i_0} = id_{V_{i_0}}$

$T_j = 0 \quad \forall j \neq i_0$

the projection that sends every G-rep V to the sum of irreducible copies $V_{i_0}^n \leq V$ iso to V_{i_0}.

We've found this function explicitly in the past: $f(g) = \frac{\dim V_{i_0}}{|G|} \cdot \overline{\chi_{i_0}(g)}$.

④ Inversion formula:

We know now that FT is an iso. so for every $(T_1, \dots, T_k)$ we can find a unique function $f: G \to \mathbb{C}$ that will generate these transformations.

But can we write down f explicitly in terms of the T_i's? In other words, what is $(FT)^{-1}$?

⊛ Recall that the irreps of G contain all of the information that appears in any G-rep., but there is one rep.

that provides direct information into the group G itself, namely the regular rep. $\mathbb{C}[G] = \bigoplus_i V_i^{\dim V_i}$

(Recall that we used this idea before, when saying that $\rho(g) = id \ \forall \text{irrep } \rho$
$\iff g = id$,
and similarly $\rho(g)\rho(h) = \rho(h)\rho(g) \ \forall \text{irrep } \rho$
$\iff gh = hg$ in G.)

⊛ In $\mathbb{C}[G]$, the trace of any element $\rho(g)$ is $\begin{cases} |G| & g = e \\ 0 & g \neq e \end{cases}$, so we can tell elements of g appart using Tr.

⊛ Let $f: G \to \mathbb{C}$ be any function.
$\hat{f}(\rho_{reg}): \mathbb{C}[G] \to \mathbb{C}[G]$ is the linear map $T = \sum_{g \in G} f(g)\rho(g)$, so

$$Tr(T \circ \rho(h)^{-1}) = Tr\Big(\sum_{g \in G} f(g)\rho(gh^{-1})\Big)$$
$$= \sum_{g \in G} f(g)\, Tr(gh^{-1}) = |G|\, f(h)$$

i.e. $f(h) = \frac{1}{|G|} Tr(\hat{f}(\rho_{reg}) \circ \rho(h)^{-1})$ ⊛

reconstructs f from the endomorphism $\hat{f}(\rho_{reg})$.

On the other hand, $\mathbb{C}[G] = \bigoplus_i V_i^{\dim V_i}$

$\implies \hat{f}(\rho_{reg})$ has the block-diagonal form:

$$\begin{bmatrix} \boxed{\hat{f}(\rho_1)} & & & & \\ & \ddots & & & \\ & & \boxed{\hat{f}(\rho_2)} & & \\ & & & \ddots & \\ & & & & \ddots \end{bmatrix} \quad \begin{array}{l} \}\ \dim V_1 \text{ many times} \\ \\ \}\ \dim V_2 \text{ many times} \end{array}$$

and so, it's trace is:

$$\dim V_{\rho_1} \cdot Tr(\hat{f}(\rho_1)) + \dots + \dim V_{\rho_k} Tr(\hat{f}(\rho_k))$$
$$= \sum_{\rho \text{ irrep}} \dim \rho \cdot Tr(\hat{f}(\rho)).$$

Replace $\hat{f}(\rho_i)$ by $\hat{f}(\rho_i) \cdot \rho(h)^{-1}$ as in ⊛ and compute:

$$Tr(\hat{f}(\rho_{reg})\rho_{reg}(h)^{-1}) = \sum_{\rho \text{ irrep}} \dim \rho \cdot Tr(\hat{f}(\rho) \cdot \rho(h)^{-1})$$

and by ⊛, the LHS is exactly $|G| f(h)$.

$\implies$ we get an inversion formula:

$$f(h) = \frac{1}{|G|} \sum_{\rho \text{ irrep}} \dim \rho \cdot Tr(\hat{f}(\rho) \cdot \rho(h)^{-1}).$$

⊛ Since we know that any combination $(T_1, \dots, T_k) \in End(V_{\rho_1}) \oplus \dots \oplus End(V_{\rho_k})$ is equal to $\hat{f}$ for some $f: G \to \mathbb{C}$, we find the inversion formula:

$$f(h) = \frac{1}{|G|} \sum_{\rho \text{ irrep}} \dim \rho \ Tr(T_\rho \cdot \rho(h)^{-1})$$

<u>Example</u>: Apply this to the projection,
$T_{i_0} = id_{i_0}$, $T_j = 0 \ \forall j \neq i_0$.

$$f(h) = \frac{1}{|G|}\Big[\dim \rho_{i_0} Tr(id \circ \rho_{i_0}(h)^{-1}) + \sum_{j \neq i_0} \dim \rho_j\, Tr(0 \circ \rho_j(h)^{-1})\Big]$$

$$= \frac{\dim \rho_{i_0}}{|G|} Tr(\rho_{i_0}(h)^{-1}) = \frac{\dim \rho_{i_0}}{|G|} \overline{\chi_{i_0}(h)}$$

and this is precisely the formula we already know!

⑤ Finite cyclic groups.

<u>Example</u>: Let $G = \mathbb{Z}/d\mathbb{Z}$.

A function $f: G \to \mathbb{C}$ is equivalent to a function $\tilde{f}: \mathbb{Z} \to \mathbb{C}$ periodic with period d.

⊛ The irreps. of G are parametrized by $\mathbb{Z}/d\mathbb{Z}$ again:
For $k = 0, 1, \dots, d-1$ we define
$\rho_k: \mathbb{Z}/d\mathbb{Z} \to \mathbb{C}^\times = GL(\mathbb{C})$ by

$\rho_k(n) = e^{2\pi i \frac{k}{d} n} = \left(e^{2\pi i \frac{k}{d}}\right)^n$ - indeed a hom. from $\mathbb{Z}/d\mathbb{Z}$ to $\mathbb{C}^\times$.

* The FT of a (periodic) function f is given by:

$$\hat{f}(k) := \hat{f}(\rho_k) = \rho_k(f) = \sum_{n=0}^{d-1} f(n)\rho_k(n)$$
$$= \sum_{n=0}^{d-1} f(n) e^{2\pi i \frac{nk}{d}}.$$

* The inversion formula gives a way of going back:

$$f(n) = \frac{1}{d}\sum_{k=0}^{d-1} \dim(\rho_k)\cdot \mathrm{Tr}\big(\hat{f}(\rho_k)\underbrace{\rho_k(n)^{-1}}_{\text{just a } 1\times 1 \text{ matrix}}\big)$$
$$= \frac{1}{d}\sum_{k=0}^{d-1} 1\cdot \hat{f}(\rho_k)\rho_k(n)^{-1}$$
$$= \frac{1}{d}\sum_{k=0}^{d-1} \hat{f}(k)\left(e^{2\pi i \frac{kn}{d}}\right)^{-1}$$
$$= \frac{1}{d}\sum_{k=0}^{d-1} \hat{f}(k) e^{-2\pi i \frac{kn}{d}} \quad (**)$$

* This is just the discrete FT (DFT) and it is extremely useful in computer science!

Essentially, any periodic function on $\mathbb{Z}$ can be written uniquely as a sum of "simple coherent waves"

$n \longmapsto e^{-2\pi i \frac{k}{d} n}$ - wave of frequency k/d (period length d/k)

and $\frac{1}{d}\hat{f}(k)$ is the (complex) amplitude of the wave of frequency k/d, as is apparent by writing down

(**) $f = \sum_k \frac{1}{d}\hat{f}(k)\cdot[\text{wave of freq } k/d]$.

[Now we see that DFT is a simple case of a very general situation, and we also see how this generalizes to non-cyclic, and even non-abelian groups!]