Induced representations

① Setup - we have a finite group G and a subgroup $H \leq G$.

Given a representation of H $\rho: H \to GL(V)$, we want to find / construct from it a G-rep in a canonical way.

This canonical G-rep. will be called the induced representation $(\mathrm{Ind}_H^G \rho): G \to GL(?)$.

② Def. $\rho_G: G \to GL(V)$ is said to be induced by $\rho_H: H \to GL(W)$

if: (a) $W \leq V$ an H-invariant subspace of V.

(b) For every H-coset in G $\sigma = gH \in G/H$ there exists a vector subspace $W_\sigma \leq V$ s.t.

$$V = \bigoplus_{\sigma \in G/H} W_\sigma \quad \text{and} \quad W = W_{eH}$$

(c) For every $g \in G$ we have

$$g \cdot (W_{xH}) = W_{gxH}$$

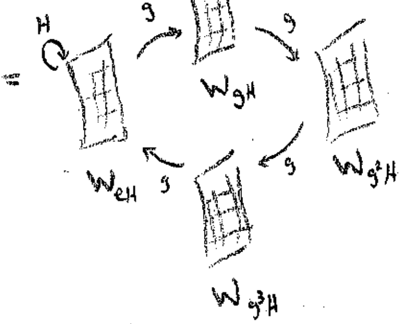
i.e. $\rho(g)$ acts on these subspaces by sending W_σ to $W_{g \cdot \sigma}$.

Note: G acts on G/H by left multiplication $g.(xH) = (gx)H$, and it acts on the subspaces W_{xH} as well.

Requirement (c) asks for the two actions to be equivariant

i.e.
$$\begin{tikzcd} G/H \arrow[r, "g"] \arrow[d] & G/H \arrow[d] \\ \{W_\sigma\}_{\sigma \in G/H} \arrow[r, "\rho(g)"] & \{W_\sigma\}_{\sigma \in G/H} \end{tikzcd}$$
commutes.

In pictures:

$V =$ 

$$V = W \oplus W_{gH} \oplus W_{g^2H} \oplus W_{g^3H}.$$

③ Universal propery

Recall that if an object satisfies some universal property, the property characterizes the object uniquely!

⊛ Universal property of the induced representation:

Let W be an H-rep. and V a G-rep.

We say that $W \xrightarrow{\alpha} V$ is the induction if it satisfies the following universal property:

α is an H-linear map
$$W \xrightarrow{\alpha} V$$

s.t. for every G-rep. Z and a H-linear map $W \xrightarrow{\beta} Z$

$\exists!$ a G-linear map $V \xrightarrow{\bar\beta} Z$

s.t.
$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & V \arrow[d, "\bar\beta"] \\ & Z \end{tikzcd}$$
commutes, i.e. $\bar\beta \circ \alpha = \beta$.

In diagrams -
$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & V \arrow[d, dashed, "\exists!"] \\ & Z \end{tikzcd}$$

Note: This essentially says that V is the universal G-rep. associated to the H-rep. W.

Any other G-rep which W maps into, must factor uniquely through V.

⊛ The universality implies that there are <u>no</u> relations imposed on V other than the fact that H is contained in G.

i.e 1) W must include into V injectively.

2) Every coset of G/H corresponds to a distinct copy of W.

and this is where the definition of Ind_H^G, comes from.

$$\left[\begin{array}{ll} \underset{\text{existence}}{\exists \bar\beta} \implies & \alpha \text{ is injective} + \left(\bigoplus_{\sigma\in G/H} W_\sigma\right) \le V \\ \underset{\text{uniqueness}}{!\bar\beta} \implies & V \le \sum_{g\in G} g.W \quad \text{- V is spanned by the G-images of W} \end{array}\right]$$

⊛ Claim: If $V = \bigoplus_{(\sigma\in G/H)} W_\sigma$ as in the def. then $W = W_{eH} \overset{\alpha}{\hookrightarrow} V$ satisfies the universal property.

Pf. Let $W \xrightarrow{\beta} Z$ be any H-map with Z a G-rep.

1) $!\bar\beta$: Suppose
$$\begin{tikzcd} W \arrow[r, hook, "\alpha"] \arrow[dr, "\beta"'] & V \arrow[d, "\bar\beta"] \\ & Z \end{tikzcd}$$
commutes with $\bar\beta$ G-linear.

Fix $\sigma = gH \in G/H$ and $\upsilon \in W_\sigma$.

By assumption $g^{-1}(W_\sigma) = W_{g^{-1}gH} = W_{eH} = W$.

$$\implies g^{-1}.\upsilon \in W.$$

Since the diagram commutes $\beta = \bar\beta\circ\alpha$ and α is just the inclusion $W \hookrightarrow V$ we have $\alpha(g^{-1}\upsilon) = g^{-1}\upsilon$

$$\longrightarrow \beta(g^{-1}\upsilon) = \bar\beta\circ\alpha(g^{-1}\upsilon) = \bar\beta(g^{-1}\upsilon)$$

But $\bar\beta$ is G-linear $\longrightarrow \bar\beta(g^{-1}\upsilon) = g^{-1}.\bar\beta(\upsilon)$.

We found, $\bar\beta(\upsilon) = g.\beta(g^{-1}\upsilon)$ ⊛

for all $\upsilon \in W_{gH}$.

This shows that $\bar\beta$ is uniquely defined by β on $\bigoplus_{\sigma\in G/H} W_\sigma = V$.

2) $\exists\bar\beta$: Define $\bar\beta$ by the formula ⊛:

$$\bar\beta(\upsilon) = g.\beta(g^{-1}.\upsilon) \quad \forall \upsilon \in W_{gH}$$

and extend linearly to $V = \bigoplus_{\sigma\in G/H} W_\sigma$.

We need to show that $\beta = \bar\beta\circ\alpha$ and $\bar\beta$ is G-linear.

Indeed, $\forall \upsilon \in W = W_{eH}$ we have

$$\bar\beta\circ\alpha(\upsilon) = \bar\beta(\upsilon) \underset{\text{by def.}}{=} e.\beta(e^{-1}.\upsilon) = \beta(\upsilon) \quad \checkmark$$

and $\forall \upsilon \in W_{gH}$ and $x \in G$ we have

$x.\upsilon \in x(W_{gH}) = W_{xgH}$ by assumption on V

$$\begin{aligned} \implies \bar\beta(x.\upsilon) &= (xg).\beta((xg)^{-1}.(x\upsilon)) \\ &= x.(g.\beta(g^{-1}(x^{-1}x\upsilon))) \\ &= x.(g\beta(g^{-1}\upsilon)) = x.\bar\beta(\upsilon) \end{aligned}$$

This shows that $\bar\beta\circ x = x\circ\bar\beta$ on a generating set of $V \Rightarrow$ it's true on every element of V! $\checkmark$

☒

(4) <u>Explicit construction of</u> $\mathrm{Ind}_H^G W = V$.

(or alternatively, $\mathrm{Ind}_H^G W$ exists!)

Given an H-rep. W, we will construct $\mathrm{Ind}_H^G W$.

First, pick representatives $e = g_1, g_2, \dots, g_n \in G$ for all the cosets in G/H, i.e. $g_iH \neq g_jH$ $\forall i\neq j$ and $\forall g\in G$ $\exists i$ s.t. $gH = g_iH$.

Define a vector space

$$V = W \oplus \underbrace{W \oplus \dots}_{n \text{ times}} \oplus W = W^n$$

where the subspaces are labeled by the cosets $W_{eH}, W_{g_2H}, \dots, W_{g_nH}$.

Define a G-action on V in the following way:

$\forall g \in G$ and $v \in W_{g_i H}$,

first find the unique j s.t.

$g(g_i H) = gg_i H = g_j H$

i.e. $\exists h \in H$ s.t. $gg_i = g_j h$.

Define $g.v$ by setting

$g.v := hv$ in the $W_{g_j H}$ copy,

i.e. make v move from $W_{g_i H} \to W_{g_j H}$ and act on it by $h: W \to W$.

Ex. This is a G-action on V and it coincides with the H-action on $W = W_{eH}$.

In particular,

$$gv = (g_j h g_i^{-1})v = (g_j h)(\underbrace{g_i^{-1} v})$$

in the W_{eH} copy, on which H-acts as perscribed by W.

$$= g_j[\underbrace{h(g_i^{-1}v)}]$$

h acts on W, and g_j moves the result to the $W_{g_j H}$ copy!

The resulting G-rep V satisfies the requirements of $\mathrm{Ind}_H^G W$ and we are done!

Rem: Another way to get $V = W \oplus \dots \oplus W = W^n$ is by looking at the space of functions

$\{f : G/H \to W\}$ - this is like saying $\mathbb{C}[G]$ = the space of functions on G.

H acts on $f: G/H \to W$ by acting on it's values $(h.f)(\sigma) = h.(f(\sigma))$, and G acts on f by acting on G/H before applying, and at the same time acting by H:

$\forall g \in G$, find the unique g_i and $h \in H$ s.t. $g = g_i h$.

Define,

$$(g.f)(\sigma) = h.(f(g_i^{-1}\sigma)) = h.f(g^{-1}\sigma)$$

(g_i and g act identically on G/H)

Note: ⊛ $W_\sigma = \{f \text{ s.t. } f(\tau) = 0 \ \forall \tau \neq \sigma\}$

i.e. function that take values $\neq 0$ only on σ.

⊛ gW_σ takes values only on the coset $g\sigma \implies gW_\sigma = W_{g\sigma}$

$[(gf)(g\sigma) = h f(g^{-1}g\sigma) = h f(\sigma)]$.

⊛ The space $W = W_{eH}$ is H-invariant, and on it the two actions of H coincide.

Cor. $\{f : G/H \to W\}$ with this G-action is the induced rep.

(5) Examples:

① $\mathrm{Ind}_H^G(\mathbb{C}_{triv}) = \mathbb{C}[G/H]$.

Pf. The H-invariant subspace W is the 1-dim space $\mathbb{C} \cdot e_H$.

⊛ W is H invariant, since $\forall h \in H$

$h.e_H = e_{hH} = e_H \in W$

⊛ The H-action restricted to W is trivial.

$\implies W = \mathbb{C}_{triv}$ as an H-rep.

⊛ $\mathbb{C}[G/H] = \bigoplus_{\sigma \in G/H} W_\sigma = \bigoplus_{\sigma \in G/H} \mathbb{C} e_\sigma$

and the G-action on $\{W_\sigma\}_{\sigma \in G/H}$ is precisely the action on G/H: for $e_\sigma \in W_\sigma$,

$g e_\sigma = e_{g\sigma} \in W_{g\sigma}$.

Thus $\mathbb{C}[G/H] = \bigoplus_{\sigma \in G/H} W_\sigma$ in the right way, with $W = W_{eH}$ the trivial H-rep

$\Longrightarrow \mathbb{C}[G/H] = \mathrm{Ind}_H^G(\mathbb{C}_{triv})$.

② In particular,

$\mathrm{Ind}_{\{e\}}^G(\mathbb{C}_{triv}) = \mathbb{C}[G/\{e\}] = \mathbb{C}[G]$

the standard rep. is induced from the trivial rep. of $\{e\} \leq G$.

③ $\mathrm{Ind}_H^G(\mathbb{C}[H]) = \mathbb{C}[G]$.

Pf. Let $W = \mathbb{C}\langle e_h : h \in H \rangle$

$= \mathbb{C}e_1 \oplus \mathbb{C}e_{h_1} \oplus \dots \oplus \mathbb{C}e_{h_n}$

the space spanned be the elements of H.

⊛ W is H-invariant - $\forall e_{h_j} \in W$
$h_i \cdot (e_{h_j}) = e_{h_i h_j} \in W$

⊛ The H-action restricted to W is precisely the regular rep. $\mathbb{C}[H]$.

⊛ Let $\sigma = gH \in G/H$ be a coset and define $W_\sigma = \mathbb{C}\langle e_{gh} : h \in H \rangle$

$\mathbb{C}\langle e_x : x \in \sigma = gH \rangle$.

Then $g' e_{gh} = e_{g'gh} \in W_{g'gH}$

$\Longrightarrow g'(W_{gH}) = W_{g'gH}$

and $\mathbb{C}[G] = \bigoplus_{\sigma \in G/H} \mathbb{C}\langle e_x : x \in \sigma \rangle$

$= \bigoplus_{\sigma \in G/H} W_\sigma$

as required from the induced representation.

④. This gives another proof for

$\mathrm{Ind}_{\{e\}}^G(\mathbb{C}_{triv}) = \mathbb{C}[G]$

since $\mathbb{C}_{triv} = \mathbb{C}[\{e\}]$ the regular rep. of the group with 1 elements.