History

⊛ Group characters first appeared long before representations!

In analytic number theory, series were used. The most famous of which is the Riemann zeta function

$$\sum_{n=1}^{\infty} 1/n^s = \prod_{p \text{ prime}} (1 - 1/p^s)^{-1}$$

but also functions of the form

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

for some multiplicative function

$$\chi : \mathbb{N} \to \mathbb{C}^{\times}$$

$$\chi(nm) = \chi(n)\,\chi(m) .$$

(The zeta function corresponding to the case $\chi \equiv 1$).

Their importance fueled the study of <u>characters</u>:

homomorphisms $\chi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^{\times}$

and more generally, homomorphisms

$$\chi : G \to S^1 \subseteq \mathbb{C}^{\times} .$$

⊛ Dedekind discovered that for an abelian group G, one can write the multiplication table of G as

	g_1	g_2	---	g_n
h_1	$g_1h_1^{-1}$	$g_2h_1^{-1}$	---	
h_2	$\vdots$			
$\vdots$				
h_n				

and taking the determinant

$$P(x_{g_1}, \dots, x_{g_n}) = \det\left(x_{g_i h_j^{-1}}\right)$$ - poly of deg. $n = |G|$

one finds that P <u>splits</u> as a product of linear factors

$$P = \prod_{\chi_i} \left(\sum_{g \in G} \chi(g) \cdot x_g \right) = \prod_{\chi} \left(\chi(g_1) x_{g_1} + \chi(g_2) x_{g_2} \right.$$

where χ_i are all the irreducible characters of G (meaning that they can't be realized as a character of a quotient of G).

But when taking G non-abelian, Dedekind couldn't generalize his formula.

He sent this puzzle to Frobenius to solve.

⊛ Frobenius realized that there is an orthonormal basis of class functions $\chi_1, \dots, \chi_k$ s.t. $d_i = \chi_i(1)$ and P splits as a product

$P = \prod_{\chi_i} F_i^{d_i}$ where the coeff. of $x_g x_e^{d_i - 1}$ in F_i is $\chi_i(g)$.

⊛ Still very mysterious, he found an explicit constuction:

If $\rho_i : G \to GL_{d_i}(\mathbb{C})$ is a homomorphisms, and there is no basis in which $\rho(g) = \left(\begin{array}{c|c} A_g & 0 \\ \hline * & B_g \end{array}\right)$ $\forall g \in G$

[i.e. no invariant subspace = irrep.]

then $\chi_i(g) = \mathrm{Tr}(\rho_i(g))$.

- But still he cared only about the functions χ_i and not about ρ_i.

⊛ Later, Dedekind and Schur descided to study the ρ_i's in their own right and made them the focus of the theory.

The center $Z(G)$ in the character table

⊛ In your HW you defined
$$\ker(\chi) = \{g \in G : \chi(g) = \dim V\}$$
and worked out that
$$\ker(\chi) = \ker(\rho) \trianglelefteq G \text{ a normal subgroup.}$$
$\Longrightarrow$ The Character table detects normal subgroups of G.

⊛ Claim: $Z(G) = \bigcap_{\chi \text{ irrep}} \{g \in G : |\chi(g)| = \dim V\}$.

Pf. We showed in a previous problem session (2) that
$$|\chi(g)| = \dim V \iff \rho(g) = \lambda I$$
for $\lambda \in \mathbb{C}$ some root of 1.

Thus $|\chi(g)| = \dim V \Rightarrow \rho(g)$ commutes with any other matrix.

In particular $\rho(g)\rho(h) = \rho(h)\rho(g)$ $\forall h \in G$.

But if this is the case on every irrep. it will be true on any G-rep.

In particular on $\mathbb{C}[G]$:
$$e_{gh} = \rho(gh)e_1 = \rho(g)\rho(h)e_1 = \rho(h)\rho(g)e_1 = e_{hg}$$
$$\Longrightarrow gh = hg \quad \forall h \in G,$$
i.e. $g \in Z(G)$.

Conversely, if $g \in Z(G)$ then the map $\rho(g) : V \to V$ is G-linear

[$\rho(g)\rho(h) = \rho(gh) = \rho(hg) = \rho(h)\rho(g)$
i.e. $T \circ \rho(h) = \dots = \rho(h) \circ T$
where $T = \rho(g)$]

and thus by schur's lemma $\rho(g) = \lambda I$ on every irrep. ▨

Algebraic integers

① In a general setting:

$S \leq R$ two integral domains.

We proved in class that if $\alpha \in R$ satisfies
$$\alpha^n + s_1\alpha^{n-1} + \dots + s_n = 0$$
then $S[\alpha]$ is finitely generated over S, e.g. by $1, \alpha, \dots, \alpha^{n-1}$.

Claim: The converse is also true.

Pf. Suppose $r_1, \dots, r_n \in S[\alpha]$ are a generating set. Then $\forall i$,
$$\alpha \cdot r_i \in S[\alpha] \Rightarrow \exists s_{ji} \in S \text{ s.t.}$$
$$\alpha r_i = \sum_{j=1}^{n} s_{ji} r_j$$
or, in matrix form $A = (s_{ji})$
$$\begin{pmatrix} \alpha r_1 \\ \vdots \\ \alpha r_n \end{pmatrix} = A \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \Rightarrow (\alpha I - A)\begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = 0$$
Let $\mathrm{adj}(\alpha I - A)$ be the adjoint matrix (this can be defined over any commutative ring, and has the same properties).

Multiplying by it: $\mathrm{adj}(X) \cdot X = \det(X) \cdot I$
$$0 = \det(\alpha I - A)\begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$$
But since R is a domain,
$$\det(\alpha I - A) = 0.$$
But $\det(\alpha I - A) = \alpha^n - \mathrm{tr}(A)\alpha^{n-1} + \dots$

i.e. α is integral over S. ▨

② Application of $\overline{\mathbb{Z}} \cap \mathbb{Q} = \mathbb{Z}$.

Claim: $\alpha \in \overline{\mathbb{Q}}$ is an algebraic integer ($\in \overline{\mathbb{Z}}$) $\iff$ it's minimal polynomial over $\mathbb{Q}$ has integer coeff.

Pf. ($\Longleftarrow$) is clear by def.

($\Longrightarrow$) Let $\alpha = \alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ be all the conjugates of α over $\mathbb{Q}$

i.e. the roots of the minimal polynomial of α.

α is integral $\Rightarrow \exists\, q \in \mathbb{Z}[x]$ s.t.

$q(\alpha) = 0$.

But then $\forall i = 1, 2, \dots, n$

$q(\alpha_i) = 0$ as well

$\Longrightarrow \{\alpha_i\}$ are all algebraic integers.

Since the algebraic integers are closed under addition and multiplication, all symmetric functions

$s_i(\alpha_1, \dots, \alpha_n)$ are algebraic integers.

But $\beta_i = s_i(\alpha_1, \dots, \alpha_n)$ is the coefficient of x^i in the minimal polynomial of α

$\Longrightarrow \beta_i \in \mathbb{Q}$.

Since $\beta_i \in \overline{\mathbb{Z}} \cap \mathbb{Q} = \mathbb{Z}$ we conclude that $\min_\alpha(x) \in \mathbb{Z}[x]$ a monic integer polynomial.

$\boxtimes$